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Cooperative Games with Multiple Scenarios in Intuitionistic Fuzzy Environment

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Abstract. The main aim of this paper is to introduce the core and nucleolus notions of cooperative games with multiple scenarios in uncertain environment. Taking imprecision of information into account, we incorporate fuzzy coalition values, which are represented by intuitionistic fuzzy numbers. They can be applied as an appropriate approach to define a fuzzy set in the case that available information is not sufficient for defining an imprecise concept by means of a conventional fuzzy set. The characteristic function of such games associates a coalition with a vector containing the intuitionistic fuzzy components. The notion of expected interval is defined and computed for the intuitionistic fuzzy numbers. Then, an approach is proposed to transform the problem into a single-objective cooperative game with interval-valued payoffs. The concepts of core and nucleolus are considered. It is shown that the core is nonempty in these games. A method is proposed to compute the nucleolus of such the problems. Finally, the validity and applicability of the approach are illustrated by a numerical example.

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1. Introduction

In many real-world situations, the rational players may cooperate together to obtain more profits. Cooperative game theory provides the mathematical methods for analyzing cooperation and distribution problems. The main part of cooperative game theory deals with how to divide the overall worth among the players in the game. In cooperative games, coalitions are organized by group agreement among some or all of the players and multiple coalitions. For conventional n-person cooperative games, a coalition is defined as any nonempty subset of the set of all players, making the number of possible coalitions at most $2^n - 1$, which includes one-person coalitions. Any player participating in a coalition must accept completely the decisions of the coalition; that is, a coalition behaves like an individual decision maker. In some real-world situations, if we want to use cooperative games, imprecision is oserved inherently in human judgment. One of the methods for studying imprecision is the use of fuzzy set theory. In fuzzy cooperative games, the characteristic function of any coalition corresponds to a fuzzy value.
Let us review some significant works on fuzzy cooperative games in the literature. Aubin and Butnariu studied fuzzy cooperative games. They investigated the notions of core and Shapley value for n-person cooperative games with fuzzy coalitions [1]. Butnariu also performed some similar works in extending the concept of coalitions in n-person cooperative games, and he considered the core and the Shapley value [4, 5]. Moreover, he examined fuzzy games with an infinite number of players [6]. Aubin defined the generalized gradient, which can be regarded as the marginal gains that the players receive when they join the coalition of all players [2].
In n-person cooperative games, lexicographical solutions such as the nucleolus are considered to be as important as the core and the Shapley value. Yu and Zhang [20] introduced the concept of core in the games with fuzzy coalitions. They studied three types of special fuzzy cores in the games with fuzzy coalition and the explicit fuzzy core is represented by the crisp core. Zhang et al. [22] studied the core and nucleolus concepts and bargaining sets of the cooperative games with fuzzy payoffs. Yu et al. [21] considered the cooperative game with fuzzy coalition and payoff value in the generalized integral form. They proposed Shapley

value based on the Hukuhara difference. Zhao and Zhang [23] studied the core solution for the cooperative games with interval payoffs. They showed the nonempty of I-core in interval-valued cooperative games. Nan et al. [13] studied the α-consensus value of a cooperative game with payoffs of triangular intuitionistic fuzzy numbers and gave the formation mechanism of the α-consensus value, as well as some properties. In another work, [14], they presented the definition of the Shapley function for intuitionistic fuzzy cooperative games by extending the fuzzy cooperative games. Based on the extended Hukuhara difference, they obtained the specific expression of the Shapley in the intuitionistic fuzzy cooperative games with multi-linear extension form, and discussed its existence and uniqueness.

In multi-objective n-person cooperative games, it is not assumed that a payoff is the value of a utility, but multiple attribute values are directly dealt with as a vector of payoffs.

Bergstresser and Yu [3] investigated multi-objective cooperative games. They mainly considered the core defined by the domination structures and referred to a couple of solution concepts which yield a unique solution such as the nucleolus in n-person cooperative games. Sakawa and Nishizaki considered the nucleolus in n-person cooperative games with multiple scenarios [18]. Bigdeli and Hassanpour [7] studied a kind of the fuzzy multiobjective cooperative game that called the multiobjective production programming with fuzzy parameters. They presented a method for solving of this kind of cooperative games. Also, the authors considered the types of the fuzzy multi-objective games in other works [8, 9, 10, 11].

In this paper, we consider n-person cooperative games with multiple coalition values that are assumed to be intuitionistic fuzzy numbers. An approach is proposed to transform the problem into a single-objective cooperative game with interval-valued payoffs. It is shown that core sets in these games are nonempty. A method is proposed to compute the nucleolus of such the problems.

The rest of this paper is organized as follows. In Section 2, some preliminaries and necessary definitions about intuitionistic fuzzy sets and n-person cooperative games are presented. In Section 3, intuitionistic fuzzy vector-valued cooperative games are introduced. An approach is proposed to transform the problem into a single-objective cooperative game with interval valued payoffs. The core and nucleolus solution concepts are considered. It is shown that core sets in these games are nonempty. An algorithm is proposed to compute the nucleolus solution of the problem. In Section 4, the validity and applicability of the method is illustrated by a numerical example. Finally, conclusion is made in Section 5.

2. Preliminaries

In this section, we provide some definitions and notions of intuitionistic fuzzy sets and cooperative games according to [7, 15, 19].

Definition 2.1. Let $X = \{x_1, \cdots, x_n\}$ be a finite universal set. An intuitionistic fuzzy set $\tilde{A}$ in X is mathematically expressed as $\tilde{A} = \{\langle x_i, \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i)\rangle \mid x_i \in X\}$, in which $\mu_{\tilde{A}} : X \to [0,1]$ and $\vartheta_{\tilde{A}} : X \to [0,1]$ are respectively the membership degree and the non-membership degree of an element $x_i \in X$ to $\tilde{A}$ such that they satisfy the inequality $\mu_{\tilde{A}}(x_i) + \vartheta_{\tilde{A}}(x_i) \leq 1$ for all $x_i \in X$.

Definition 2.2. ($(\alpha,\beta) - cuts$), Let $\alpha, \beta \in [0,1]$ be fixed numbers such that $\alpha + \beta \leq 1$. An (α,β)-cut generated by an intuitionistic fuzzy set $\tilde{A}$ is defined as

$$\tilde{A}_{\alpha,\beta} = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha, \vartheta_{\tilde{A}}(x) \leq \beta\}.$$

So, $\tilde{A}_{\alpha,\beta}$ is a crisp set of elements $x \in X$, which belong to $\tilde{A}$ at least with the degree α and which do not belong to $\tilde{A}$ at most with the degree β.

In the following, a special type of intuitionistic fuzzy numbers is introduced.

Definition 2.3. A triangular intuitionistic fuzzy number (TIFN) $\tilde{a} = (a, l_a, r_a; w_a, u_a)$ is a special intuitionistic fuzzy number, whose membership and non-membership functions are defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x-a+l}{l} w_a & a-l \leq x < a, \\ \dfrac{a+r-x}{r} w_a & a \leq x < a+r, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\vartheta_{\tilde{a}}(x) = \begin{cases} \dfrac{(a-x)+u_a(x-a+l)}{l} & a-l \leq x < a, \\ \\ \dfrac{(x-a)+u_a(a+r-x)}{r} & a \leq x < a+r, \\ \\ 1 & \text{otherwise,} \end{cases}$$

where l_a, r_a are respectively the left and right spreads and a is the mean value. w_a and u_a represent the maximum degree of membership and minimum degree of non-membership, respectively. Moreover, they satisfy the following conditions

$$0 \leq w_a \leq 1,\ 0 \leq u_a \leq 1,\ 0 \leq w_a + u_a \leq 1.$$

It can be seen that for a TIFN $\tilde{a} = (a, l_a, r_a; w_a, u_a)$ and for $0 \leq \alpha \leq w_a$, $u_a \leq \beta \leq 1$ where $0 \leq \alpha + \beta \leq 1$, we have $\tilde{a}_{\alpha,\beta} = \tilde{a}_\alpha \wedge \tilde{a}_\beta$, where symbol "$\wedge$" denotes the minimum operator between $\tilde{a}_\alpha$ and $\tilde{a}_\beta$. Thus, (α, β)-cut of a TIFN $\tilde{a} = (a, l_a, r_a; w_a, u_a)$ is as follows [19]:

$$\tilde{a}_{\alpha,\beta} = \begin{cases} \hat{a}_\beta = [a^L(\beta), a^R(\beta)] & \alpha < \dfrac{1-\beta}{1-u_a} w_a, \\ \\ \hat{a}_\alpha = [a^L(\alpha), a^R(\alpha)] & \alpha > \dfrac{1-\beta}{1-u_a} w_a, \\ \\ \hat{a}_\alpha \ or \ \hat{a}_\beta & \alpha = \dfrac{1-\beta}{1-u_a} w_a, \end{cases}$$

where

$$a^L(\alpha) = (a - l_a) + \frac{l_a \alpha}{w_a}, \quad a^R(\alpha) = (a + r_a) - \frac{r_a \alpha}{w_a},$$

$$a^L(\beta) = (a - l_a) + \frac{(1-\beta) l_a}{1-u_a}, \quad a^R(\alpha) = (a + r_a) - \frac{(1-\beta) r_a}{1-u_a}.$$

The arithmetic on intervals can be explicitly expressed in the following manner. Let $a = [a^L, a^R]$ and $b = [b^L, b^R]$ be two intervals. Then, we have

$$[a^L, a^R] + [b^L, b^R] = [a^L + b^L, a^R + b^R],$$
$$[a^L, a^R] - [b^L, b^R] = [a^L - b^R, a^R - b^L],$$
$$\lambda A = \begin{cases} [\lambda a^L, \lambda a^R] & \lambda \geq 0 \\ [\lambda a^R, \lambda a^L] & \lambda < 0 \end{cases}$$

where λ is a real scalar.
Different order relations between intervals are presented in the literature. In this paper, we use one of the most well-known them.

Definition 2.4. For each two intervals $a = [a^L, a^R]$ and $b = [b^L, b^R]$, the order relation is defined as follows,

$$1)\ a \succcurlyeq b \ \textit{if}\ a^L \geq b^L \ \textit{and}\ a^R \geq b^R;$$
$$2)\ a \preccurlyeq b \ \textit{if}\ a^L \leq b^L \ \textit{and}\ a^R \leq b^R;$$
$$3)\ a = b \ \textit{if}\ a^L = b^L \ \textit{and}\ a^R = b^R.$$

Now, we briefly review some fundamental results about cooperative games. A cooperative game (transferable utility game) is a pair (N, ν), where $N = \{1, 2, \cdots, n\}$ is a finite set of players and ν is a real-valued function defined on the power set of N, i.e., $\nu : 2^N \to \mathbb{R}$ satisfying $\nu(\emptyset) = 0$. Each subset S of N is called a coalition and the value $\nu(S)$ is referred to as the worth of S . Throughout this paper, we assume that the player set N is fixed. So we can regard a function ν , called a characteristic function, as a game. We denote by G^N the set of all games on N . We use some abbreviated notations such as $\nu(\{i\}) = \nu(i)$, $S \cup \{i\} = S \cup i$, and so on.

Definition 2.5. A game $\nu \in G^N$ is said to be

- *monotonic* if $\nu(S) \leq \nu(T), \quad \forall S, T \subseteq N : S \subseteq T;$
- *additive* if $\nu(S \cup T) = \nu(S) + \nu(T), \quad \forall S, T \subseteq N : S \cap T = \emptyset;$
- *superadditive* if $\nu(S \cup T) \geq \nu(S) + \nu(T), \quad \forall S, T \subseteq N : S \cap T = \emptyset;$
- *convex* if $\nu(S \cup T) + \nu(S \cap T) \geq \nu(S) + \nu(T), \quad \forall S, T \subseteq N.$

In cooperative games, the most important topic is to find an appropriate rule to allocate the worth of the grand coalition among the players. Such a rule is usually called a solution of the cooperative game. The allocated profit vector is denoted by $x = (x_1, x_2, \cdots, x_n)$, where x_i is the profit of the ith player. The set of all imputations of the game $\nu \in G^N$ is denoted by

$$I(\nu) = \{x \in \mathbb{R}^n \Big| \sum_{i \in N} x_i = \nu(N), x_i \geq \nu(i), \forall i \in N\}.$$

Definition 2.6. For a game $\nu \in G^N$, the core of the game is a set-valued solution, which is defined by

$$C(\nu) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = \nu(N), \sum_{i \in S} x_i \geq \nu(S), \forall S \subseteq N\}.$$

If we define the excess of the coalition S with respect to x by

$$e(S,x) = \nu(S) - \sum_{i\in S} x_i,$$

then the core can be rewritten as follows:

$$C(\nu) = \{x \in \mathbb{R}^n \mid e(N,x) = 0, e(S,x) \leq 0 \ \forall S \subseteq N\}.$$

It is clear that the core is a convex polyhedron, since it is represented by a linear equation and $2^n - 2$ linear inequalities. Unfortunately, the core of a game may be empty in general. Hence some extended solution concepts, such as the ε-core and the least core, were proposed. Moreover, the lexicographic minimization of the excesses leads to the concept of "nucleolus" [15].

Definition 2.7. A game which has a nonempty core is called balanced.

Actually balancedness is specified by introducing an optimization problem for checking the emptiness of the core and its dual problem [15].

3. Cooperative games with multiple scenarios and intuitionistic fuzzy coalition values

A single objective n-person cooperative game is represented by a characteristic function ν. It associates a coalition S with the value $\nu(S)$, that is interpreted as the payoff which the coalition S can acquire only through S . We assume that p kinds of different scenarios are expected in the game. Thus, we can consider the coalition value as an p-dimensional vector. Taking imprecision of information into account, we incorporate intuitionistic fuzzy coalition values, which are represented by p-dimensional vector with the components of triangular intuitionistic fuzzy numbers. Therefore, in an n-person cooperative game with scenarios, a characteristic function $\tilde{\nu}$ associates any coalition S with its fuzzy vector $\tilde{v}(S) = (\tilde{v}^1(S), \tilde{v}^2(S), \cdots, \tilde{v}^p(S))$. Then, we represent the game with multiple scenarios by $(N, \tilde{v})$. Let $\mu_{\tilde{v}^k(S)}(v)$ and $\vartheta_{\tilde{v}^k(S)}(v)$ respectively denote the membership and non-membership functions of a triangular intuitionistic fuzzy number $\tilde{v}^k(S), k = 1, \cdots, p$ representing the intuitionistic fuzzy value that the players of the coalition for the k-th scenario could earn without any help from the players outside of S . We consider the (α, β)-level cut of any triangular intuitionistic fuzzy number $\tilde{v}^k(S) = (v^k, l_v^k, r_v^k, w_v^k, u_v^k)$ defined as the following ordinary set over $\mathbb{R}$ as follows:

$$\tilde{v}^k_{\alpha,\beta}(S) = \{v^k \in \mathbb{R} \mid \mu_{\tilde{v}^k(S)}(v) \geq \alpha, \vartheta_{\tilde{v}^k(S)}(v) \leq \beta\}.$$

The (α, β)-level cut of $\tilde{v}(S)$ is presented as

$$\tilde{v}^k_{\alpha,\beta}(S) = \begin{cases} \hat{v}^k_\beta = [v^{kL}(\beta), v^{kR}(\beta)], & \alpha < \dfrac{1-\beta}{1-u^k_v} w^k_v, \\ \\ \hat{v}^k_\alpha = [v^{kL}(\alpha), v^{kR}(\alpha)], & \alpha > \dfrac{1-\beta}{1-u^k_v} w^k_v, \\ \\ \hat{v}^k_\alpha \ or \ \hat{v}^k_\beta, & \alpha = \dfrac{1-\beta}{1-u^k_v} w^k_v. \end{cases}$$

Using the (α,β)-level cut of triangular intuitionistic fuzzy numbers, a characteristic function $\tilde{v}_{\alpha,\beta}(S) = (\tilde{v}^1_{\alpha,\beta}(S), \tilde{v}^2_{\alpha,\beta}(S), \cdots, \tilde{v}^p_{\alpha,\beta}(S))$ associates any coalition. In other words, fuzzy-valued cooperative game is transformed into an interval-valued cooperative game for any $\alpha, \beta \in [0,1]$.

In the following, we illustrate the core and nucleolus concepts for the cooperative game $(N, \tilde{v}_{\alpha,\beta}(S))$. We use the following steps to get the core and nucleolus solutions for the cooperative game $(N, \tilde{v}_{\alpha,\beta}(S))$.

Step 1. (Computing weights for the scenarios):

Before we consider solution concepts in cooperative game $(N, \tilde{v}_{\alpha,\beta}(S))$, we state how to reduce it to single-objective game. For this purpose, we consider the probabilities of encountering scenarios. Moreover, we can use the well-known group AHP method [12] to obtain the preference degrees of different scenarios.

Assume that the probability of encountering the k-th scenario is represented by λ_k . Thus, the coalition value S is $v^\lambda_{\alpha,\beta}(S) = \sum_{k=1}^p \lambda_k \tilde{v}^k_{\alpha,\beta}(S) = (v^\lambda, l^\lambda_v, r^\lambda_v, w^\lambda_v, u^\lambda_v)$ where $\lambda_k \geq 0$, for $k = 1, \cdots, p$, and $\sum_{k=1}^p \lambda_k = 1$. Also, $v^\lambda = \sum_{k=1}^p \lambda_k v^k, l^\lambda_v = \sum_{k=1}^p \lambda_k l^k_v, r^\lambda_v = \sum_{k=1}^p \lambda_k r^k_v, w^k_v = \min_{k=1,\cdots,p}\{w^k_v\}, u^\lambda_v = \max_{k=1,\cdots,p}\{u^k_v\}$.

Since that for any $\alpha, \beta \in [0,1]$, we obtain a value for coalition S , we should search a way to obtain appropriate α, β or compute the expected value for the coalition S. In this paper, we apply a method based on the expected value of coalition values.

Step 2. (Computing the expected interval):

We compute the expected interval of the coalition S, as follows:

$$E(v^\lambda_{\alpha,\beta}(S)) = \begin{cases} E(v^\lambda_\beta(S)) = [E^L(v^\lambda_\beta(S)), E^R(v^\lambda_\beta(S))] & \alpha < \dfrac{1-\beta}{1-u^\lambda_v} w^\lambda_v, \\ E(v^\lambda_\alpha(S)) = [E^L(v^\lambda_\alpha(S)), E^R(v^\lambda_\alpha(S))] & \alpha > \dfrac{1-\beta}{1-u^\lambda_v} w^\lambda_v, \\ E(v^\lambda_\alpha(S)) \ or \ E(v^\lambda_\beta(S)) & \alpha = \dfrac{1-\beta}{1-u^\lambda_v} w^\lambda_v, \end{cases}$$

where

$$[E^L(v_\beta^\lambda(S)), E^R(v_\beta^\lambda(S))] = \int_{u_v^\lambda}^{1} \left[\sum_{k=1}^{l} \lambda_k \tilde{v}_\beta^{KL}(S), \sum_{k=1}^{l} \lambda_k \tilde{v}_\beta^{KR}(S) \right] d\beta,$$

$$[E^L(v_\alpha^\lambda(S)), E^R(v_\alpha^\lambda(S))] = \int_{0}^{w_v^\lambda} \left[\sum_{k=1}^{l} \lambda_k \tilde{v}_\alpha^{KL}(S), \sum_{k=1}^{l} \lambda_k \tilde{v}_\alpha^{KR}(S) \right] d\alpha.$$

Therefore,

$$E^L(v_\beta^\lambda(S)) = \sum_{k=1}^{p} \lambda_k((v^k - \frac{l_v^k}{2}(1-u_v^k)),$$

$$E^R(v_\beta^\lambda(S)) = \sum_{k=1}^{p} \lambda_k((v^k + \frac{r_v^k}{2}(1-u_v^k)),$$

$$E^L(v_\alpha^\lambda(S)) = \sum_{k=1}^{p} \lambda_k(v^k - \frac{l_v^k}{2} w_v^\lambda),$$

$$E^R(v_\alpha^\lambda(S)) = \sum_{k=1}^{p} \lambda_k(v^k - \frac{r_v^k}{2} w_v^\lambda).$$

It is notable that the coalition value of S is an interval.

3.1. **Core.** So far, we transformed a cooperative game with multiple scenarios and intuitionistic fuzzy coalition values into a single-objective cooperative game with interval coalition values. According to the following definition, we consider the core concepts for the current game.

Definition 3.1. A cooperative game $(N, E(v_{\alpha,\beta}^\lambda(S))$ with interval coalition values is said to be $L-R$-supperadditive if

$$E\bigg(v_{\alpha,\beta}^\lambda(S \cup T)\bigg) \succcurlyeq E\bigg(v_{\alpha,\beta}^\lambda(S)\bigg) + E\bigg(v_{\alpha,\beta}^\lambda(T)\bigg).$$

Definition 3.2. For a cooperative game $(N, E(v_{\alpha,\beta}^\lambda(S))$, an $L-R$-imputation is a vector $x = (x_1, \cdots, x_n)$ where $x_i = [x_i^L, x_i^R]$, and it satisfies the relations

$$x_i \succcurlyeq E\bigg(v_{\alpha,\beta}^\lambda(i)\bigg) \ \forall i \in N,$$

$$\sum_{i \in N} x_i = E\bigg(v_{\alpha,\beta}^\lambda(N)\bigg).$$

Definition 3.3. For a cooperative game $(N, E(v_{\alpha,\beta}^\lambda(S))$, $L-R$ core solution is defined as follows

$$C(E(v^{\lambda}_{\alpha,\beta}(S)) = \Bigg\{(x_1, \cdots, x_n) \mid \sum_{i=1}^{n} x_i = E(v^{\lambda}_{\alpha,\beta}(N)) \ and \ \sum_{i\in S} x_i \succcurlyeq E(v^{\lambda}_{\alpha,\beta}(S)), \ \forall S \subset N\Bigg\}.$$

Definition 3.4. Let $B = \{S_1, \cdots, S_m\}$ be a collection of non-empty subsets of N. B is called a balanced collection if there exists a vector of positive numbers, the balancing vector $y = (y_1, \cdots, y_m)$, such that

$$\sum_{\substack{S_j\in B\\ i\in S_j}} y_s = 1, \ \forall i \in N.$$

Definition 3.5. A cooperative game $(N, E(v^{\lambda}_{\alpha,\beta}(S))$ is $L-R$ balanced if for any balanced vector $y = (y_1, \cdots, y_m)$,

$$\sum_{j=1}^{m} y_j E(v^{\lambda}_{\alpha,\beta}(S_j)) \preccurlyeq E(v^{\lambda}_{\alpha,\beta}(N)).$$

Theorem 3.6. *The $L-R$ core $C(E(v^{\lambda}_{\alpha,\beta}))$ is nonempty if and only if the game $E(v^{\lambda}_{\alpha,\beta})$ is $L-R$ balanced.*

Proof. By Definition 3.3, $C(E(v^{\lambda}_{\alpha,\beta})) \neq \emptyset$ if and only if

$$C^L(E(v^{\lambda}_{\alpha,\beta})) = \Bigg\{(x_1, \cdots, x_n) \mid \sum_{i=1}^{n} x_i^L = E^L(v^{\lambda}_{\alpha,\beta}(N)) \ and \sum_{i\in S} x_i^L \geq E^L(v^{\lambda}_{\alpha,\beta}(S)), \forall S \subseteq N\Bigg\} \tag{3.1}$$

and

$$C^R(E(v^{\lambda}_{\alpha,\beta})) = \Bigg\{(x_1, \cdots, x_n) \mid \sum_{i=1}^{n} x_i^R = E^R(v^{\lambda}_{\alpha,\beta}(N)) \ and \sum_{i\in S} x_i^R \geq E^R(v^{\lambda}_{\alpha,\beta}(S)), \forall S \subseteq N\Bigg\} \tag{3.2}$$

are nonempty. From the classical cooperative game theory [15], we know

$$E^L\Big(v^{\lambda}_{\alpha,\beta}(N)\Big) = \min\Bigg\{\sum_{i\in N} x_i^L \mid \sum_{i\in S} x_i^L \geq E^L\Big(v^{\lambda}_{\alpha,\beta}(S)\Big), \forall S \subseteq N\Bigg\} \tag{3.3}$$

and

$$E^R\Big(v^{\lambda}_{\alpha,\beta}(N)\Big) = \min\Bigg\{\sum_{i\in N} x_i^R \mid \sum_{i\in S} x_i^L \geq E^R\Big(v^{\lambda}_{\alpha,\beta}(S)\Big), \forall S \subseteq N\Bigg\}. \tag{3.4}$$

Note that the relations 3.3 and 3.4 must be hold simultaneously. Now, considering the dual problems of 3.3 and 3.4, we have

$$E^L\left(v^{\lambda}_{\alpha,\beta}(N)\right) = \max\left\{\sum_{S\subseteq N} y_S E^L\left(v^{\lambda}_{\alpha,\beta}(S)\right) \mid \sum_{S\in N_i} y_S = 1, \forall i\in N, y_S \geq 0, \forall S\subseteq N\right\} \tag{3.5}$$

and

$$E^R\left(v^{\lambda}_{\alpha,\beta}(N)\right) = \max\left\{\sum_{S\subseteq N} y_S E^R\left(v^{\lambda}_{\alpha,\beta}(S)\right) \mid \sum_{S\in N_i} y_S = 1, \forall i\in N, y_S \geq 0, \forall S\subseteq N\right\}. \tag{3.6}$$

These problems are satisfied if and only if

$$\sum_{S\subseteq N} y_S E^L\left(v^{\lambda}_{\alpha,\beta}(S)\right) \leq E^L\left(v^{\lambda}_{\alpha,\beta}(N)\right), \qquad \forall y_S \geq 0, \sum_{S\in N_i} y_S = 1$$

and

$$\sum_{S\subseteq N} y_S E^R\left(v^{\lambda}_{\alpha,\beta}(S)\right) \leq E^R\left(v^{\lambda}_{\alpha,\beta}(N)\right), \qquad \forall y_S \geq 0, \sum_{S\in N_i} y_S = 1.$$

By Definition 3.5, these relations assure that $E(v^{\lambda}_{\alpha,\beta})$ is $L-R$ balanced. □

3.2. **Nucleolus.** After performing Steps 1 and 2, we consider the intuitionistic fuzzy cooperative game with multiple scenarios $(N, \tilde{v}(S))$ as two cooperative games $(N, E^L(v^{\lambda}_{\alpha,\beta}(S)))$ and $(N, E^R(v^{\lambda}_{\alpha,\beta}(S)))$. The players obtain a pessimistic nucleolus by solving the cooperative game $(N, E^L(v^{\lambda}_{\alpha,\beta}(S)))$ and an optimistic nucleolus by solving the cooperative game $(N, E^R(v^{\lambda}_{\alpha,\beta}(S)))$. For the former, the players consider the least value obtained from the coalition and the latter the highest value.
We assume that X^L and X^R are the sets of all imputations, which are defined as

$$X^L = X^L(N, E^L(v^{\lambda}_{\alpha,\beta})) = \left\{x^L \in \mathbb{R}^n \mid \sum_{i\in N} x^L_i = 1, x^L_i \geq 0, i = 1,\cdots,n\right\},$$

and

$$X^R = X^R(N, E^R(v^{\lambda}_{\alpha,\beta})) = \left\{x^R \in \mathbb{R}^n \mid \sum_{i\in N} x^R_i = 1, x^R_i \geq 0, i = 1,\cdots,n\right\},$$

The excesses of a coalition with respect to x^L and x^L of a coalition S are as follows:

$$e^L(S,x^L) = E^L(v^{\lambda}_{\alpha,\beta}(S)) - \sum_{i\in S} x_i^L \qquad k=1,\cdots,p,$$

and

$$e^R(S,x^R) = E^R(v^{\lambda}_{\alpha,\beta}(S)) - \sum_{i\in S} x_i^R \qquad k=1,\cdots,p.$$

Definition 3.7. For a vector x , let $q(x)$ be a vector arranged in non-decreasing order, i.e., if $i < j$, then $q_i(x) \geq q_j(x)$. So x is less than y with respect to the lexicographical order if $x = y$ or, $q_l(x) < q_l(y)$ for the first nonequal component l.

Based on the excesses $e^L(S, x^L)$ and $e^R(S, x^R)$ over X^L and X^R, the nucleolus is defined as

$$\begin{aligned} N^L(N, E^L(v^{\lambda}_{\alpha,\beta}), X^L) = \{x \in X^L \mid\ & H_{2^n-2}(e^L(S_1,x^L),\cdots,e^L(S_{2^n-2},x^L)) \\ & \leq_L H_{2^n-2}(e^L(S_1,y^L),\cdots,e^L(S_{2^n-2},y^L)), \forall y^L \in X^L\} \end{aligned}$$

and

$$\begin{aligned} N^R(N, E^R(v^{\lambda}_{\alpha,\beta}), X^R) = \{x \in X^R \mid\ & H_{2^n-2}(e^R(S_1,x^R),\cdots,e^R(S_{2^n-2},x^R)) \\ & \leq_L H_{2^n-2}(e^R(S_1,y^R),\cdots,e^R(S_{2^n-2},y^R)), \forall y^R \in X^R\}, \end{aligned}$$

where $H_{2^n-2} : \mathbb{R}^{2^n-2} \to \mathbb{R}^{2^n-2}$ is a mapping which arranges elements of a $(2^n - 2)$-dimensional vector in nonincreasing order, and $\leq_L$ means "less than or equal to" in the lexicographical order [15].

3.3. **An algorithm for obtaining the nucleolus.** We now present an algorithm to compute the nucleolus solutions in the game with multiple scenarios $(N, v^L_{\alpha,\beta})$ and $(N, v^R_{\alpha,\beta})$. For the nucleoluses $N^L(N, E^L(v^{\lambda}_{\alpha,\beta}), X^L)$ and $N^R(N, E^R(v^{\lambda}_{\alpha,\beta}), X^R)$, we have the following linear programming problems, respectively:

$$\begin{aligned} \min\ & \varepsilon^L \\ s.t.\ & E^L(v^{\lambda}_{\alpha,\beta}(S)) - \sum_{i\in S} x_i^L \leq \varepsilon^L, \forall S \subset N, \\ & x_1^L + \cdots + x_n^L = 1, \\ & x_i^L \geq 0, \qquad i = 1,\cdots,n, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \min \ & \varepsilon^R \\ s.t. \ & E^R(v^\lambda_{\alpha,\beta}(S)) - \sum_{i \in S} x_i^R \leq \varepsilon^R, \forall S \subset N, \\ & x_1^R + \cdots + x_n^R = 1, \\ & x_i^R \geq 0, \qquad i = 1, \cdots, n. \end{aligned} \tag{3.8}$$

The nucleolus $N^L(N, E^L(v^\lambda_{\alpha,\beta}), X^L)$ and $N^R(N, E^R(v^\lambda_{\alpha,\beta}), X^R)$ can be obtained by solving problems (3.7) and (3.8) and the updated problems as the following algorithm.

Algorithm

Step 1. Set $t = 1$.

Step 2. Formulate problems (3.7) and (3.8), and solve them.

Step 3. If we obtain a unique optimal solution, stop. Otherwise, update $t = t + 1$ and go to Step 4.

Step 4. Let ε^L_{t-1} and ε^R_{t-1} be the optimal values in the $(t-1)th$ iteration. Moreover, let Γ^L_{t-1} and Γ^R_{t-1} be the set of coalitions with active inequality constraints for any optimal solution, respectively. Solve the following problems and then, go to Step 3.

$$\begin{aligned} \min \ & \varepsilon^L \\ S.t. \ & E^L(v^\lambda_{\alpha,\beta}(S)) - \sum_{i \in S} x_i^L = \varepsilon^L_k, \forall S \in \Gamma^L_k, k = 1, \cdots, t-1, \\ & E^L(v^\lambda_{\alpha,\beta}(S)) - \sum_{i \in S} x_i^L \leq \varepsilon^L, \forall S \notin \Gamma^L_1, \cdots \cup \Gamma^L_{t-1}, S \subset N, \\ & x_1^L + \cdots + x_n^L = E^L(v^\lambda_{\alpha,\beta}(N)), \\ & x_i^L \geq 0, i = 1, \cdots, n. \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \min \ & \varepsilon^R \\ S.t. \ & E^R(v^\lambda_{\alpha,\beta}(S)) - \sum_{i \in S} x_i^R = \varepsilon^R_k, \forall S \in \Gamma^R_k, k = 1, \cdots, t-1, \\ & E^R(v^\lambda_{\alpha,\beta}(S)) - \sum_{i \in S} x_i^R \leq \varepsilon^R, \forall S \notin \Gamma^R_1, \cdots \cup \Gamma^R_{t-1}, S \subset N, \\ & x_1^R + \cdots + x_n^R = E^R(v^\lambda_{\alpha,\beta}(N)), \\ & x_i^R \geq 0, i = 1, \cdots, n. \end{aligned} \tag{3.10}$$

The following lemma shows that the above algorithm is convergent.

Lemma 3.8. *The unique payoff vectors x^L and x^R minimizing ε^L and ε^R, respectively, can always be determined by at most n steps.*

Proof. With a little change, the proof is similar to the crisp case (see [15]). □

This lemma allows us to prove the following theorem.

Theorem 3.9. *The solutions obtained by the algorithm are the nucleolus for the games $(N, E^L(v^\lambda_{\alpha,\beta}))$ and $(N, E^R(v^\lambda_{\alpha,\beta}))$.*

Proof. Assume that the solutions obtained from the algorithm are x^{*L} and x^{*R}. We must show that these solutions are the pessimistic and optimistic nucleolus solutions. For this purpose, suppose that x^{*L} and x^{*R} are not the nucleolus solutions for the games $(N, E^L(v^\lambda_{\alpha,\beta}))$ and $(N, E^R(v^\lambda_{\alpha,\beta}))$, respectively. Let $q(x^{*L})$ and $q(x^{*R})$ be the vectors of all excesses of x^{*L} and x^{*R}, respectively, whose elements are arranged in descending order. Then, there are the imputations such as y^L and y^R, for a certain l,

$$q_i(x^{*L}) = q_i(y^L), i = 1, \cdots, l-1; \quad q_l(x^{*L}) < q_l(y^L),$$

or

$$q_i(x^{*R}) = q_i(y^R), i = 1, \cdots, l-1; \quad q_l(x^{*R}) < q_l(y^R).$$

This proves that for an iteration k with $q_l(x^{*L}) = \varepsilon^L_k$ and $q_l(x^{*R}) = \varepsilon^R_k$, the objective function values y^L and y^R are less than the minimum values, which is a contradiction. □

4. Numerical Example

We consider a three-person cooperative game with three different scenarios. For all scenarios, the values of the one-person coalitions are $\widetilde{zero} = (0, 0, 0; 1, 0)$, and the value of the grand coalition is $\widetilde{one} = (1, 0, 0; 1, 0)$. The values of two-person coalitions are shown in Table 1.

Table 1. The values of two-person coalitions

S	Scenario 1	Scenario 2	Scenario 3
{ 1,2}	(0.3, 0.1, 0.2;0.6,0.3)	(0.5, 0.1, 0.2;1,0)	(0.4, 0.1, 0.1;1,0)
{ 1,3}	(0.6, 0.2, 0.2;1,0)	(0.2, 0.1, 0.1;1,0)	(0.5, 0.2, 0.1;0.6,0.4)
{ 2,3}	(0.5, 0.2, 0.1;0.6,0.4)	(0.7, 0.2, 0.1;0.6,0.4)	(0.7, 0.2, 0.1;1,0)

Table 2. The weighting expected intervals

S	Aggregated scenario
{ 1,2}	[0.192, 0.273]
{ 1,3}	[0.15, 0.249]
{ 2,3}	[0.3, 0.39]

Using Step 1, let weighting coefficients of scenarios be $(0.5, 0.2, 0.3)$. By Step 2, we have Table 2. So the following linear programming problems are formulated:

$$\begin{aligned}
&\min\ \varepsilon^L \\
&\text{s.t. } x_1^L + \varepsilon^L \geq 0, \quad x_2^L + \varepsilon^L \geq 0, \quad x_3^L + \varepsilon^L \geq 0, \\
&\quad x_1^L + x_2^L + \varepsilon^L \geq 0.192, \\
&\quad x_1^L + x_3^L + \varepsilon^L \geq 0.15, \\
&\quad x_2^L + x_3^L + \varepsilon^L \geq 0.30, \\
&\quad x_1^L + x_2^L + x_3^L = 1, \\
&\quad x_i^L \geq 0, i = 1, 2, 3,
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\min\ \varepsilon^R \\
&\text{s.t. } x_1^R + \varepsilon^R \geq 0, \quad x_2^R + \varepsilon^R \geq 0, \quad x_3^R + \varepsilon^R \geq 0, \\
&\quad x_1^R + x_2^R + \varepsilon^R \geq 0.273, \\
&\quad x_1^R + x_3^R + \varepsilon^R \geq 0.249, \\
&\quad x_2^R + x_3^R + \varepsilon^R \geq 0.39, \\
&\quad x_1^R + x_2^R + x_3^R = 1, \\
&\quad x_i^R \geq 0, i = 1, 2, 3.
\end{aligned} \tag{4.2}$$

The results of solving the above problems are shown in Table 3. More-

Table 3.

	Player 1	Player 2	Player 3
Solutions of Problem 4.1	0.70	0	0.30
Solutions of Problem 4.2	0.61	0	0.39

over, the values of ε^L and ε^R are zero.

5. Conclusion

In this paper, we considered intuitionistic fuzzy-valued cooperative games with multiple scenarios. The concepts of core and nucleus are introduced for the obtained interval-valued cooperative games. Then, an algorithm is presented to compute the nucleolus. Finally, a numerical example is presented to examine the proposed algorithm.

The proposed method of this paper can be applied to interval-valued cooperative games with multiple scenarios. The main advantage of this method is that it does not require any defuzzification method. Note that, when a defuzzification method is used, the intuitionistic fuzzy aspect of payoffs are actually lost, which is not desirable.

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