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Coefficient estimates and Fekete-Szegö coefficient inequality for new subclasses of Bi-univalent functions

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Abstract. In this paper, we investigate on two new subclasses $\mathcal{S}^*_\sigma(a,b)$ and $\nu_\sigma(a,b)$ of σ consisting of analytic and bi-univalent functions satisfying subordinations in the open unit disk $\mathbb{U}$. We consider the Fekete-Szegö inequalities for these new subclasses. Also, we establish estimates for the coefficient for these subclasses.

Keywords:Bi-univalent functions, Coefficient estimates, Fekete-Szegö coefficient inequality.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z|z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in A which are univalent in $\mathbb{U}$. The Koebe one-quarter theorem [3] states that the image of $\mathbb{U}$ under every function f from $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus every

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such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}).$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f), \qquad r_0(f) \geq \frac{1}{4}).$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \quad .$$

A function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$.

Although, the familiar Koebe function is not in the class of σ, there are some examples of functions member of σ, such as [8]

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log(\frac{1+z}{1-z})$$

so on. Other common examples of functions in $\mathcal{S}$ for example

$$z - \frac{z^2}{2} \qquad and \qquad \frac{z}{1-z^2}$$

are also not members of σ.

Let Ω be the family functions $\omega(z)$ in the unit disc $\mathbb{U}$ satisfying the conditions

$\omega(0) = 0, \quad |\omega(z)| < 1$ for $z \in \mathbb{U}$. Note that $f(z) \prec g(z)$ if there is a function $\omega(z) \in \Omega$ such that $f(z) = g(\omega(z))$, (see [3]).

Recently, Srivastava et al.[8] and Frasian and Aouf [4] and Caglar et al. [2] have introduced and have investigated various subclasses of bi-univalent functions and found estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these classes. In this paper we introduce two subclasses $\mathcal{S}^*_\sigma(a,b)$ and $\nu_\sigma(a,b)$ of bi-univalent functions and estimates on the coefficients $|a_2|$, $|a_3|$ and $|a_4|$ and also Fekete-Szegö coefficient inequality for functions in these subclasses are given.

Definition 1.1. A functions f given by 1.1 is said to be in class $\mathcal{S}^*_\sigma(a,b)$, if the following condition are satisfied:

$$f \in \sigma, \qquad |\frac{zf'(z)}{f(z)} - a| < b, \quad |a-1| < b \leq a, \tag{1.2}$$

and

$$|\frac{wg'(w)}{g(w)} - a| < b, \quad |a-1| < b \leq a, \tag{1.3}$$

where $w = f(z)$, $g = f^{-1}$, $w \in \Delta$ and $z \in \Delta$.

Also,

Definition 1.2. A functions f given by 1.1 is said to be in class $\nu_\sigma(a,b)$, if the following condition are satisfied:

$$f \in \sigma, \quad |(\frac{z}{f(z)})^2 f'(z) - a| < b, \quad |a-1| < b \leq a, \tag{1.4}$$

and

$$|(\frac{w}{g(w)})^2 g'(w) - a| < b, \quad |a-1| < b \leq a, \tag{1.5}$$

where $w = f(z)$, $g = f^{-1}$, $w \in \Delta$ and $z \in \Delta$.

Lemma 1.3. *(see[5] or [3]) If $p \in \mathcal{P}$ then $|p_k| \leq 2$ for for each k and $|p_2 - \frac{1}{2}p_1^2| \leq 2 - \frac{1}{2}|p_1|^2$, where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which $Rep(z) > 0$, $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$ for $z \in \mathbb{U}$.*

2. Coefficients Estimates For The Functions in $\mathcal{S}^*_\sigma(a,b)$ and $\nu_\sigma(a,b)$

In the section by using subordination structure and definitions of $\mathcal{S}^*_\sigma(a,b)$ and $\nu_\sigma(a,b)$ the coefficents of functions in these classes are obtained.

Theorem 2.1. *Let $(a-b) = \beta$ and $f(z)$ given by 1.1 be in the class $\mathcal{S}^*_\sigma(a,b)$, $|a-1| < b \leq a$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)}, \tag{2.1}$$

$$|a_3| \leq (1-\beta) + 4(1-\beta)^2 \tag{2.2}$$

and

$$|a_4| \leq \frac{2(1-\beta)}{3}(1 + 4(1-\beta)). \tag{2.3}$$

Proof. It follows from 1.2 and 1.3 that

$$\frac{zf'(z)}{f(z)} \prec \beta + (1-\beta)z$$

and

$$\frac{wg'(w)}{g(w)} \prec \beta + (1-\beta)w$$

then

$$\frac{zf'(z)}{f(z)} = \beta + (1-\beta)p(z) \tag{2.4}$$

and

$$\frac{wg'(w)}{g(w)} = \beta + (1-\beta)q(w) \tag{2.5}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... \tag{2.6}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + ... \tag{2.7}$$

where

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2 a_3 + a_2^3)z^3 + ...$$

and

$$\frac{wg'(w)}{g(w)} = 1 - a_2 w + (3a_2^2 - 2a_3)w^2 - (3a_4 - 12a_2 a_3 + 10a_2^3)w^3 + ...$$

Now, equating the coefficients in 2.4 and 2.5, we get

$$a_2 = p_1(1-\beta), \tag{2.8}$$

$$2a_3 - a_2^2 = (1-\beta)p_2, \tag{2.9}$$

$$2a_3 = (1-\beta)p_2 + (1-\beta)^2 p_1^2, \tag{2.10}$$

$$3a_4 - 3a_2 a_3 + a_2^3 = (1-\beta)p_3, \tag{2.11}$$

$$a_2 = -q_1(1-\beta) \tag{2.12}$$

and

$$3a_2^2 - 2a_3 = (1-\beta)q_2, \tag{2.13}$$

$$4a_2^2 - 2a_3 = (1-\beta)q_2 + (1-\beta)^2 q_1^2 \tag{2.14}$$

and

$$-(10a_2^3 - 12a_2 a_3 + 3a_4) = (1-\beta)q_3. \tag{2.15}$$

From 2.8 and 2.12, we get

$$p_1 = -q_1 \tag{2.16}$$

and

$$2a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{2.17}$$

Now from 2.10, 2.14 and 2.17, we obtain

$$\begin{aligned} 4a_2^2 &= (1-\beta)(p_2+q_2) + (1-\beta)^2(p_1^2+q_1^2) \\ &= (1-\beta)(p_2+q_2) + 2a_2^2 \end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{1}{2}(1-\beta)(p_2+q_2).$$

Applying Lemma 1.3 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leq \sqrt{2(1-\beta)}.$$

This gives the bound on $|a_2|$ as asserted in 2.1. Next, in order to find the bound on $|a_3|$, by subtracting 2.14 from 2.10, we get

$$\begin{aligned} 4a_3 - 4a_2^2 &= (1-\beta)p_2 + (1-\beta)^2 p_1^2 - ((1-\beta)q_2 \\ &\quad +(1-\beta)^2 q_1^2). \end{aligned} \tag{2.18}$$

It follows from 2.14-2.18 that

$$4a_3 = (1-\beta)(p_2 - q_2) + 2(1-\beta)^2(p_1^2 + q_1^2)$$

or, equivalently,

$$a_3 = \frac{(1-\beta)(p_2 - q_2)}{4} + \frac{(1-\beta)^2(p_1^2 + q_1^2)}{2}.$$

Applying Lemma 1.3 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq (1-\beta) + 4(1-\beta)^2.$$

Addition of 2.9 with 2.13 yields:

$$2a_2^2 = (1-\beta)(p_2 + q_2). \tag{2.19}$$

Putting $a_2 = (1-\beta)p_1$ from 2.8 we have after simplification:

$$p_1^2 = \frac{p_2 + q_2}{2(1-\beta)}. \tag{2.20}$$

Next, we subtract 2.13 from 2.9, add the equations 2.11 and 2.15 and get respectively:

$$4a_3 = 4a_2^2 + (1-\beta)(p_2 + q_2) \tag{2.21}$$

and

$$-9a_2^3 + 9a_2a_3 = (1-\beta)(p_3 + q_3) \tag{2.22}$$

We shall now find an estimate on $|a_4|$. We wish express a_4 in terms of the first three coefficients of $p(z)$ and $q(w)$. For this we subtract 2.15 from 2.11, and get

$$\begin{aligned} 6a_4 &= -11a_2^3 + 15a_2a_3 + (1-\beta)(p_3 - q_3) \\ &= -9a_2^3 + 9a_2a_3 - 2a_2^3 + 6a_2a_3 + (1-\beta)(p_3 - q_3). \end{aligned}$$

We replace $-9a_2^3 + 9a_2a_3$ by the right hand side of 2.22, put $a_3 = (1-\beta)^2 p_1^2 + \frac{(1-\beta)}{4}(p_2 - q_2)$(see 2.21) and $a_2 = (1-\beta)p_1$. Thus, we have:

$$\begin{aligned}6a_4 &= (1-\beta)(p_3+q_3) - 2(1-\beta)^3 p_1^3 \\ &\quad +6(1-\beta)p_1((1-\beta)^2 p_1^2 + \frac{(1-\beta)}{4}(p_2-q_2)) \\ &\quad +(1-\beta)(p_3-q_3) \\ &= 2(1-\beta)p_3 + 4(1-\beta)^3 p_1^3 + \frac{6(1-\beta)^2}{4} p_1(p_2-q_2).\end{aligned}$$

Next replacing p_1^2 by (2.20) we finally get

$$\begin{aligned}6a_4 &= 2(1-\beta)p_3 + 4(1-\beta)^3 p_1 \frac{p_2+q_2}{2(1-\beta)} + \frac{6(1-\beta)^2}{4} p_1(p_2-q_2) \\ &= 2(1-\beta)p_3 + 2(1-\beta)^2 p_1(p_2+q_2) + \frac{3(1-\beta)^2}{2} p_1(p_2-q_2) \\ &= 2(1-\beta)p_3 + \frac{7(1-\beta)^2}{2} p_1 p_2 + \frac{(1-\beta)^2}{2} p_1 q_2.\end{aligned}$$

By applying the inequalities $|p_1| \leq 2$, $|p_2| \leq 2$, $|p_3| \leq 2$ and $|q_2| \leq 2$ we have

$$\begin{aligned}6|a_4| &\leq 2(1-\beta)|p_3| + \frac{7(1-\beta)^2}{2}|p_1||p_2| + \frac{(1-\beta)^2}{2}|p_1||q_2|. \\ &\leq 4(1-\beta) + 16(1-\beta)^2.\end{aligned}$$

Or equivalently:

$$|a_4| \leq \frac{2(1-\beta)}{3}(1+4(1-\beta)).$$

□

Theorem 2.2. *Let* $(a-b) = \beta$ *and* $f(z)$ *given by 1.1 be in the class* $\nu_\sigma(a,b)$, $|a-1| < b \leq a$. *Then*

$$|a_2| \leq 1, \tag{2.23}$$

$$|a_3| \leq 3 - 2\beta \tag{2.24}$$

and

$$|a_4| \leq \frac{9}{2} - 4\beta. \tag{2.25}$$

Proof. It follows from 1.4 and 1.5 that

$$(\frac{z}{f(z)})^2 f'(z) \prec \beta + (1-\beta)z$$

and

$$(\frac{w}{g(w)})^2 g'(w) \prec \beta + (1-\beta)w$$

then

$$\begin{aligned}(\frac{z}{f(z)})^2 f'(z) &= \frac{z}{f(z)} - z(\frac{z}{f(z)})' \\ &= \beta + (1-\beta)p(z) \end{aligned} \tag{2.26}$$

and

$$\begin{aligned}(\frac{w}{g(w)})^2 g'(w) &= \frac{w}{g(w)} - w(\frac{w}{g(w)})' \\ &= \beta + (1-\beta)q(w) \end{aligned} \tag{2.27}$$

and

$$\frac{z}{f(z)} - z(\frac{z}{f(z)})' = 1 + (a_3 - a_2^2)z^2 + 2(a_4 - 2a_2a_3 + a_2^3)z^3 + ...$$

and

$$\frac{w}{g(w)} - w(\frac{w}{g(w)})' = 1 - (a_3 - a_2^2)w^2 - 2(a_4 - 3a_2a_3 + 2a_2^3)w^3 + ...$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + ... \tag{2.28}$$

and

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + ... \tag{2.29}$$

Now, equating the coefficients in 2.26 and 2.27, we get

$$0 = p_1(1-\beta), \tag{2.30}$$

$$(a_3 - a_2^2) = p_2(1-\beta), \tag{2.31}$$

$$2(a_4 - 2a_2a_3 + a_2^3) = p_3(1-\beta) \tag{2.32}$$

and

$$0 = q_1(1-\beta), \tag{2.33}$$

$$-(a_3 - a_2^2) = q_2(1-\beta), \tag{2.34}$$

$$-2(a_4 - 3a_2a_3 + 2a_2^3) = q_3(1-\beta). \tag{2.35}$$

From 2.31 from 2.34, we have

$$p_2 = -q_2 \tag{2.36}$$

By subtracting 2.34 from 2.31, we get

$$2(a_3 - a_2^2) = (1-\beta)(p_2 - q_2). \tag{2.37}$$

Addition of 2.32 with 2.35 yields:

$$-2a_2^3 + 2a_2a_3 = (1-\beta)(p_3 + q_3) \tag{2.38}$$

substituting from 2.37 into 2.38 we get

$$a_2(1-\beta)(p_2-q_2)=(1-\beta)(p_3+q_3).$$

Applying Lemma 1.3 for the coefficients p_2, q_2, p_3 and q_3, we immediately have

$$|a_2|\leq\frac{|p_3|+|q_3|}{|p_2|+|q_2|}$$

we obtain

$$|a_2|\leq 1.$$

which is the bound on $|a_2|$ as given in 2.23. Next, in order to find the bound on $|a_3|$, by following from 2.37 we have

$$a_3=\frac{1}{2}((1-\beta)(p_2-q_2))+a_2^2$$

Applying Lemma 1.3 for the coefficients p_2 and q_2, so for the inequality $|a_2|\leq 1$ we get:

$$\begin{aligned}|a_3| &\leq \frac{1}{2}((1-\beta)(|p_2|+|q_2|))+|a_2|^2\\ &\leq 2(1-\beta)+1.\end{aligned}$$

Therefore, we have

$$|a_3|\leq 3-2\beta.$$

This is precisely the assertion of 2.24. We shall next find an estimate on $|a_4|$.
By subtracting 2.35 from 2.32, we get

$$4a_4+6a_2^3-8a_2a_3=(1-\beta)(p_3-q_3)$$

now after simplification and from 2.37 we have

$$4a_4-2a_2^3-4(1-\beta)(p_3+q_3)=(1-\beta)(p_3-q_3)$$

Therefore, we get

$$a_4=(1-\beta)(p_3+q_3+\frac{1}{4}p_3-\frac{1}{4}q_3)+\frac{1}{2}a_2^3$$

Applying Lemma 1.3 for the coefficients p_3 and q_3, so for the inequality $|a_2|\leq 1$ we get:

$$\begin{aligned}|a_4| &\leq \frac{1}{4}((1-\beta)(5|p_3|+3|q_3|))+\frac{1}{2}|a_2|^3\\ &\leq 4(1-\beta)+\frac{1}{2}.\end{aligned}$$

Therefore, we have

$$|a_4|\leq\frac{9}{2}-4\beta.$$

□

3. Fekete-Szegö Coefficient Inequality For The Function Classe $\mathcal{S}_\sigma^*(a,b)$ And The Function Class $\nu_\sigma(a,b)$

In the last section by using the analytic functions r and s and definitions of $\mathcal{S}_\sigma^*(a,b)$ and $\nu_\sigma(a,b)$ the Fekete-Szegö inequality on functions in these classes are investigated.

Theorem 3.1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + ...,\quad (z \in \mathbb{U})$, where B_n are real with $B_1 > 0$ and $B_2 \geq 0$. If $(a-b) = \beta$ and $f(z)$ given by 1.1 belongs to $\mathcal{S}_\sigma^*(a,b)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}[2|B_2|(1-\beta) + (7-4\mu)B_1^2(1-\beta)^2]. \tag{3.1}$$

Proof. If $f(z) \in \mathcal{S}_\sigma^*(a,b)$, there exist two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with
$r(0) = 0 = s(0)$ such that,

$$\frac{zf'(z)}{f(z)} = \beta + (1-\beta)\varphi(r(z)). \tag{3.2}$$

$$\frac{wg'(w)}{g(w)} = \beta + (1-\beta)\varphi(s(w)). \tag{3.3}$$

Define the functions p and q by

$$p(z) = \frac{1+r(z)}{1-r(z)} = 1 + p_1 z + p_2 z^2 + ...$$

and

$$q(w) = \frac{1+s(w)}{1-s(w)} = 1 + q_1 w + q_2 w^2 + ...$$

or equivalently,

$$r(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}(p_1 z + (p_2 - \frac{p_1^2}{2})z^2 + (p_3 - \frac{p_1^3}{4} - p_1 p_2)z^3 + ...) \tag{3.4}$$

and

$$s(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}(q_1 w + (q_2 - \frac{q_1^2}{2})w^2 + (q_3 - \frac{q_1^3}{4} - q_1 q_2)w^3 + ...) \tag{3.5}$$

Using 3.4 and 3.5 in 3.2 and 3.3, we have

$$\frac{zf'(z)}{f(z)} = \beta + (1-\beta)\varphi(\frac{p(z)-1}{p(z)+1}). \tag{3.6}$$

and

$$\frac{wg'(w)}{g(w)} = \beta + (1-\beta)\varphi(\frac{q(w)-1}{q(w)+1}). \tag{3.7}$$

Again using 3.4 and 3.5 along with $\varphi(z) = 1 + B_1z + B_2z^2 + ...$, it is evident that

$$\begin{aligned}\varphi(\frac{p(z)-1}{p(z)+1}) &= 1 + \frac{1}{2}B_1p_1z + (\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) \\ &+ \frac{B_2p_1^2}{4})z^2 + ... \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\varphi(\frac{q(w)-1}{q(w)+1}) &= 1 + \frac{1}{2}B_1q_1w + (\frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) \\ &+ \frac{B_2q_1^2}{4})w^2 + ... \end{aligned} \tag{3.9}$$

It follows from 3.6, 3.7, 3.8 and 3.9 that

$$a_2 = \frac{B_1p_1}{2}(1-\beta) \tag{3.10}$$

$$\begin{aligned}a_3 &= \frac{1}{2}[(\frac{1}{2}B_1(p_2 - \frac{1}{2}p_1^2) + \frac{1}{4}B_2p_1^2)(1-\beta) \\ &+ \frac{1}{4}B_1^2p_1^2(1-\beta)^2]\end{aligned} \tag{3.11}$$

$$a_2 = -\frac{B_1q_1}{2}(1-\beta) \tag{3.12}$$

$$\begin{aligned}a_3 &= \frac{1}{2}[\frac{3}{2}B_1^2q_1^2(1-\beta)^2 - (\frac{1}{2}B_1(q_2 - \frac{1}{2}q_1^2) \\ &+ \frac{1}{4}B_2q_1^2)(1-\beta)]\end{aligned} \tag{3.13}$$

Therefore from 3.10 and 3.12 we have,

$$a_2^2 = \frac{1}{8}B_1^2(1-\beta)^2(p_1^2 + q_1^2)$$

Adding 3.11 and 3.13, we get

$$\begin{aligned}a_3 &= \frac{1}{16}[B_1^2(1-\beta)^2[p_1^2 + 6q_1^2] + (1-\beta)[2B_1((p_2 \\ &- \frac{1}{2}p_1^2) - (q_2 - \frac{1}{2}q_1^2)) + B_2(p_1^2 + q_1^2)]]\end{aligned} \tag{3.14}$$

Therefore

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{16}[B_1^2(1-\beta)^2[p_1^2+6q_1^2] + (1-\beta)[2B_1((p_2 - \frac{1}{2}p_1^2) \\ &\quad -(q_2 - \frac{1}{2}q_1^2)) + B_2(p_1^2+q_1^2]] \\ &\quad -\frac{1}{8}\mu B_1^2(1-\beta)^2(p_1^2+q_1^2). \end{aligned}$$

Taking the absolute values we obtain:

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{16}[B_1^2(1-\beta)^2[|p_1|^2+6|q_1|^2] + (1-\beta)[2B_1(|p_2 - \frac{1}{2}p_1^2| + |q_2 - \frac{1}{2}q_1^2|) \\ &\quad +B_2(|p_1|^2+|q_1|^2)]] - \frac{1}{8}\mu B_1^2(1-\beta)^2(|p_1|^2+|q_1|^2). \end{aligned}$$

Applying Lemma 1.3 for the coefficients p_1, q_1 and $|p_2 - \frac{1}{2}p_1^2| \leq 2 - \frac{1}{2}|p_1|^2$, $|q_2 - \frac{1}{2}q_1^2| \leq 2 - \frac{1}{2}|q_1|^2$ we have:

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{16}[B_1^2(1-\beta)^2[|p_1|^2+6|q_1|^2] + (1-\beta)[2B_1(2 \\ &\quad -\frac{1}{2}|p_1|^2 + 2 - \frac{1}{2}|q_1|^2) + B_2(|p_1|^2+|q_1|^2)]] \\ &\quad -\frac{1}{8}\mu B_1^2(1-\beta)^2(|p_1|^2+|q_1|^2). \end{aligned}$$

Upon simplification we obtain:

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}[2|B_2|(1-\beta) + (7-4\mu)B_1^2(1-\beta)^2].$$

□

Theorem 3.2. *Let* $(a-b)=\beta$. *further let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots, \quad (z \in \mathbb{U})$, *where* B_n *are real with* $B_1 > 0$ *and* $B_2 \geq 0$. *If* $f(z)$ *given by* (1.1) *belongs to* $\nu_\sigma(a,b)$,

$$(\frac{z}{f(z)})^2 f'(z) = \beta + (1-\beta)\varphi(r(z)).$$

$$(\frac{w}{g(w)})^2 g'(w) = \beta + (1-\beta)\varphi(s(w)).$$

then

$$|a_3 - a_2^2| \leq |B_2|(1-\beta).$$

4. Conclusion

The Fekete-Szegö problem have always been the main interest of researchers in Univalent and bi-Univalent classes. Many studies related to this problem are around analytic normalized functions. Here the the Fekete-Szegö inequality is obtained for functions in $\mathcal{S}^*_\sigma(a,b)$ and $\nu_\sigma(a,b)$.

Using subordination structure. Also by using the integral and differential operators we may obtain the bounds of coefficients and Fekete-Szegö problem in future.

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