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Weighted composition operators between Lipschitz algebras of complex-valued bounded functions

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Abstract. In this paper, we study weighted composition operators between Lipschitz algebras of complex-valued bounded functions on metric spaces, not necessarily compact. We give necessary and sufficient conditions for the injectivity and the surjectivity of these operators. We also obtain sufficient and necessary conditions for a weighted composition operator between these spaces to be compact.

Keywords: Compact linear operator, Lipschitz algebra, Pointed metric space, Weighted composition operator.

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1. Introduction and preliminaries

Let X be a Hausdorff space. We denote by $C(X)$ the set of all complex-valued continuous functions on X. Then $C(X)$ is a commutative complex algebra with unit 1_X, the constant function on X with value 1. The set of all bounded functions in $C(X)$ is denoted by $C^b(X)$. It is known that $C^b(X)$ is a unital commutative complex Banach algebra with unit 1_X when equipped with the uniform norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\} \qquad (f \in C^b(X)).$$

Let X and Y be Hausdorff spaces and let $S(X)$ and $S(Y)$ be complex linear subspaces of $C(X)$ and $C(Y)$, respectively. A map $T : S(X) \to$

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$S(Y)$ is called a *weighted composition operator* if there exist a complex-valued function u on Y, not necessarily continuous, and a map $\varphi : Y \to X$ such that $T(f)(y) = u(y)f(\varphi(y))$ for all $f \in S(X)$ and $y \in Y$. Then T is denoted by uC_φ and called the weighted composition operator induced by u and φ. Clearly, uC_φ is a linear operator. In the case $u = 1_Y$, the weighted composition operator uC_φ reduces to the composition operator C_φ.

Let (X, d) and (Y, ρ) be metric spaces and K be a nonempty subset of Y. A map $\varphi : K \to X$ is called a *Lipschitz mapping* from (K, ρ) to (X, d) if there exists a constant C such that $d(\varphi(x), \varphi(y)) \le C\rho(x, y)$ for all $x, y \in K$. A map $\varphi : K \to X$ is called a *supercontractive mapping* from (K, ρ) to (X, d) if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\varphi(x), \varphi(y))/\rho(x, y) < \varepsilon$ whenever $x, y \in K$ with $0 < \rho(x, y) < \delta$.

Let (X, d) be a metric space. A function $f : X \to \mathbb{C}$ is called a *complex-valued Lipschitz function* on (X, d) if f is a Lipschitz mapping from (X, d) to the Euclidean metric space $\mathbb{C}$. We denote by $p_{(X,d)}(f)$ the constant Lipschitz of f, i.e.,

$$p_{(X,d)}(f) = \sup\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y\}.$$

We denote by $\mathrm{Lip}(X, d)$ the set of all complex-valued bounded Lipschitz functions on (X, d). Then $\mathrm{Lip}(X, d)$ is a complex subalgebra of $C^b(X)$ containing 1_X. Moreover, $\mathrm{Lip}(X, d)$ is a Banach space with the Lipschitz norm

$$\|f\|_M = \max\{\|f\|_X,\ p_{(X,d)}(f)\} \qquad (f \in \mathrm{Lip}(X, d))$$

and a commutative unital complex Banach algebra with the Lipschitz algebra norm

$$\|f\|_{\mathrm{Lip}(X,d)} = \|f\|_X + p_{(X,d)}(f) \qquad (f \in \mathrm{Lip}(X, d)).$$

These algebras were first introduced by Sherbert in [9, 10]. Note that Lipschitz algebras are semisimple.

Let (X, d) be a pointed metric space with a basepoint designated by x_0. We denote by $\mathrm{Lip}_0(X, d)$ the set of all complex-valued Lipschitz functions f on (X, d) for which $f(x_0) = 0$. Then $\mathrm{Lip}_0(X, d)$ is a Banach space with the $p_{(X,d)}(\cdot)$-norm.

Kamowitz and Scheinberg in [7] proved that a composition endomorphism C_φ of $\mathrm{Lip}(X, d)$ is compact if and only if φ is a supercontraction from (X, d) to (X, d) whenever (X, d) is a compact metric space. Jiménez-Vargas and Villegas-Vallecillos in [6] generalized some results of [7] by omitting the compactness condition of considered metric spaces. Chen, Li, R. Wang and Y.-S. Wang in [2] characterized compact weighted composition operators between spaces of scalar-valued Lipschitz functions. Botelho and Jamison in [1] and Esmaeili and Mahyar in [3] studied weighted composition operators between spaces of vector-valued Lipschitz functions. Golbaharan and Mahyar in [4] provided a complete description of weighted composition operators on the Lipschitz algebras $\mathrm{Lip}(X,d)$ when (X,d) is a compact metric space. They also gave necessary and sufficient conditions for the injectivity and the surjectivity of these operators and established a necessary and sufficient condition for a weighted composition operator on $\mathrm{Lip}(X,d)$ to be compact.

In this paper, we provide a complete description of weighted composition operators between Lipschitz algebras of complex-valued bounded functions on metric spaces, not necessarily compact. We generalized some obtained results in [4].

2. Some properties of weighted composition operators

For a complex-valued function u on a nonempty set Y, we denote by $\mathrm{coz}(u)$ the set of all $y \in Y$ for which $u(y) \neq 0$.

Let (X,d) and (Y,ρ) be metric spaces. It is clear that if u belongs to $\mathrm{Lip}(Y,\rho)$ and φ is a Lipschitz mapping from (Y,ρ) to (X,d), then uC_φ is a weighted composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$. The following example shows that there exists a nonzero weighted composition operator uC_φ from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$ where φ is not a Lipschitz mapping from (Y,ρ) to (X,d).

Example 2.1. Let $X=(-\infty,\infty)$, let d be the Euclidean metric on X, let $Y=\{\frac{1}{n} : n \in \mathbb{Z}\setminus\{0\}\}$ and let ρ be the Euclidean metric on Y. Define the map $\varphi : Y \to X$ by

$$\varphi(\frac{1}{n}) = (-1)^n \frac{1}{n} \qquad (n \in \mathbb{Z}\setminus\{0\}).$$

Then φ is not a Lipschitz mapping from (Y,ρ) to (X,d) since

$$\frac{d(\varphi(\frac{1}{n}),\varphi(\frac{1}{n+1}))}{\rho(\frac{1}{n},\frac{1}{n+1})} = 2n+1,$$

for all $n \in \mathbb{N}$. Define the function $u : Y \to \mathbb{C}$ by

$$u(y) = y \qquad (y \in Y).$$

Then u is a bounded complex-valued Lipschitz function on Y and

$$\sup\{|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in Y, x \neq y\} \leq 3.$$

Set $T = uC_\varphi$. We show that $T(f) \in \mathrm{Lip}(Y, \rho)$ for all $f \in \mathrm{Lip}(X, d)$. Let $f \in \mathrm{Lip}(X, d)$. Then for each $x, y \in Y$ with $x \neq y$, we have

$$\begin{aligned}\frac{|T(f)(x) - T(f)(y)|}{\rho(x, y)} &= \frac{|u(x)f(\varphi(x)) - u(y)f(\varphi(y))|}{\rho(x, y)} \\ &\leq |u(x)| \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(\varphi(x), \varphi(y))} \frac{d(\varphi(x), \varphi(y))}{\rho(x, y)} \\ &\quad + |f(\varphi(y))| \frac{|u(x) - u(y)|}{\rho(x, y)} \\ &\leq 3p_{(X,d)}(f) + \|f\|_X \\ &\leq 4\|f\|_{\mathrm{Lip}(X,d)}.\end{aligned}$$

This implies that $T(f)$ is a Lipschitz function on Y. On the other hand,

$$|T(f)(y)| = |u(y)||f(\varphi(y))| \leq \|u\|_Y \|f\|_X,$$

for all $y \in Y$. Therefore, $T(f) \in \mathrm{Lip}(Y, \rho)$ and so $T = uC_\varphi$ is a weighted composition operator from $\mathrm{Lip}(X, d)$ to $\mathrm{Lip}(Y, \rho)$.

Theorem 2.2. *Let (X, d) and (Y, ρ) be metric spaces, let u be a complex-valued function on Y and let φ be a map from Y to X. Suppose that $T = uC_\varphi$ is a weighted composition operator from $\mathrm{Lip}(X, d)$ to $\mathrm{Lip}(Y, \rho)$. Then $u \in \mathrm{Lip}(Y, \rho)$ and T is a bounded linear operator. Moreover, if $1_Y \in T(\mathrm{Lip}(X, d))$, then $u(y) \neq 0$ for all $y \in Y$.*

Proof. Since $1_X \in \mathrm{Lip}(X, d)$ and T is a weighted composition operator, we have $u = T(1_X)$ and so $u \in \mathrm{Lip}(Y, \rho)$. Let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathrm{Lip}(X, d)$ that converges to the function 0 in $(\mathrm{Lip}(X, d), \|\cdot\|_{\mathrm{Lip}(X,d)})$ and $\{T(f_n)\}_{n=1}^\infty$ converges to a function $g \in \mathrm{Lip}(Y, \rho)$ in $(\mathrm{Lip}(Y, \rho), \|\cdot\|_{\mathrm{Lip}(Y,\rho)})$. Since the uniform norm is weaker than the Lipschitz algebra norm, we have $\lim_{n\to\infty} f_n(\varphi(y)) = 0$ and $\lim_{n\to\infty} T(f_n)(y) = g(y)$ for all $y \in Y$. The boundedness of u implies that $\lim_{n\to\infty} u(y)(f_n(\varphi(y))) = 0$ for all $y \in Y$. Therefore, $g = 0$ and by the closed graph theorem T is continuous and so bounded.

We now assume that $1_Y \in T(\mathrm{Lip}(X, d))$. Then there exists a function f in $\mathrm{Lip}(X, d)$ such that $T(f) = 1_Y$. Hence, $u(y)f(\varphi(y)) = 1$ for all $y \in Y$ and so $u(y) \neq 0$ for all $y \in Y$. $\square$

Notation 2.3. Let (X, d) and (Y, ρ) be metric spaces, let u be a complex-valued function on Y and let φ be a map from Y to X. We denote

$$C(u, \varphi) = \sup\{|u(x)| \frac{d(\varphi(x), \varphi(y))}{\rho(x, y)} : x, y \in Y, x \neq y\}.$$

Here, we give a sufficient condition for the operator $T = uC_\varphi$ to be a weighted composition operator from $\mathrm{Lip}(X, d)$ to $\mathrm{Lip}(Y, \rho)$.

Theorem 2.4. *Let (X, d) and (Y, ρ) be metric spaces, let u be a complex-valued function on Y and let φ be a map from Y to X. If $u \in \mathrm{Lip}(Y, \rho)$ and $C(u, \varphi) < \infty$, then $T = uC_\varphi$ is a weighted composition operator from $\mathrm{Lip}(X, d)$ to $\mathrm{Lip}(Y, \rho)$ and $\|T\| \leq C(u, \varphi) + \|u\|_{\mathrm{Lip}(Y,\rho)}$.*

Proof. Suppose that $u \in \mathrm{Lip}(Y, \rho)$ and $C(u, \varphi) < \infty$. Let $f \in \mathrm{Lip}(X, d)$. Then for each $x, y \in Y$ with $\varphi(x) \neq \varphi(y)$, we have

$$\begin{aligned}\frac{|T(f)(x) - T(f)(y)|}{\rho(x, y)} &= \frac{|u(x)f(\varphi(x)) - u(y)f(\varphi(y))|}{\rho(x, y)} \\ &\leq |u(x)|\frac{|f(\varphi(x)) - f(\varphi(y))|}{d(\varphi(x), \varphi(y))}\frac{d(\varphi(x), \varphi(y))}{\rho(x, y)} \\ &\quad + |f(\varphi(y))|\frac{|u(x) - u(y)|}{\rho(x, y)} \\ &\leq C(u, \varphi)p_{(X,d)}(f) + \|f\|_X p_{(Y,\rho)}(u).\end{aligned}$$

Moreover, for each $x, y \in Y$ with $x \neq y$ and $\varphi(x) = \varphi(y)$, we have

$$\frac{|T(f)(x) - T(f)(y)|}{\rho(x, y)} = \frac{|u(x) - u(y)|}{\rho(x, y)}|f(\varphi(y))| \leq p_{(Y,\rho)}(u)\|f\|_X.$$

Therefore, $T(f)$ is a Lipschitz function on (Y, ρ).

On the other hand,

$$|T(f)(y)| = |u(y)||f(\varphi(y))| \leq \|u\|_Y\|f\|_X,$$

for all $y \in Y$. Hence, $T(f) \in \mathrm{Lip}(Y, \rho)$ and

$$\begin{aligned}\|T(f)\|_{\mathrm{Lip}(Y,\rho)} &\leq \|u\|_Y\|f\|_X + C(u, \varphi)p_{(X,d)}(f) + \|f\|_X p_{(Y,\rho)}(u) \\ &\leq (C(u, \varphi) + \|u\|_{\mathrm{Lip}(Y,\rho)})\|f\|_{\mathrm{Lip}(X,d)}.\end{aligned}$$

Therefore, T is bounded and $\|T\| \leq C(u, \varphi) + \|u\|_{\mathrm{Lip}(Y,\rho)}$. This completes the proof. □

Theorem 2.5. *Let (X, d) and (Y, ρ) be metric spaces, let u be a complex-valued function on Y and let φ be a map from Y to X. Suppose that $\mathrm{diam}(\varphi(\mathrm{coz}(u))) < \infty$ and $T = uC_\varphi$ is a weighted composition operator from $\mathrm{Lip}(X, d)$ to $\mathrm{Lip}(Y, \rho)$. Then*

$$C(u, \varphi) \leq \|T\|(1 + \mathrm{diam}(\varphi(\mathrm{coz}(u)))).$$

Proof. Let $x, y \in \mathrm{coz}(u)$ with $x \neq y$. Define the function $f_y : \varphi(\mathrm{coz}(u)) \to \mathbb{R}$ by

$$f_y(t) = d(t, \varphi(y)) \qquad (t \in \varphi(\mathrm{coz}(u))). \tag{2.1}$$

Then $\|f_y\|_{\varphi(\mathrm{coz}(u))} \leq \mathrm{diam}(\varphi(\mathrm{coz}(u)))$ and

$$|f_y(s) - f_y(t)| = |d(s, \varphi(y)) - d(t, \varphi(y))| \leq d(s, t),$$

for all $s, t \in \varphi(\mathrm{coz}(u))$. By Sherbert's extension theorem [10, Proposition 1.4], there exists a function $F_y : X \to \mathbb{R}$ with $F_y\,|_{\varphi(\mathrm{coz}(u))}= f_y$, $\|F_y\|_X \leq$

$\operatorname{diam}(\varphi(\operatorname{coz}(u)))$ and $|F_y(s) - F_y(t)| \leq d(s,t)$ for all $s, t \in X$. Hence, $F_y \in \operatorname{Lip}(X,d)$ and

$$\|F_y\|_{\operatorname{Lip}(X,d)} \leq \operatorname{diam}(\varphi(\operatorname{coz}(u))) + 1. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\begin{aligned}
|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} &= \frac{|u(x)f_y(\varphi(x)) - u(y)f_y(\varphi(y))|}{\rho(x,y)} \\
&= \frac{|u(x)F_y(\varphi(x)) - u(y)F_y(\varphi(y))|}{\rho(x,y)} \\
&= \frac{|T(F_y)(x) - T(F_y)(y)|}{\rho(x,y)} \\
&\leq p_{(Y,\rho)}(T(F_y)) \\
&\leq \|T(F_y)\|_{\operatorname{Lip}(Y,\rho)} \\
&\leq \|T\|\|F_y\|_{\operatorname{Lip}(X,d)} \\
&\leq \|T\|(1 + \operatorname{diam}(\varphi(\operatorname{coz}(u)))).
\end{aligned}$$

Therefore,

$$C(u,\varphi) \leq \|T\|(1 + \operatorname{diam}(\varphi(\operatorname{coz}(u)))$$

Hence, the proof is complete. □

Corollary 2.6. *Let (X,d) and (Y,ρ) be metric spaces, let u be a complex-valued function on Y, let φ be a map from Y to X such that* $\operatorname{diam}(\varphi(\operatorname{coz}(u))) < \infty$ *and let $T = uC_\varphi$ be a weighted composition operator from* $\operatorname{Lip}(X,d)$ *to* $\operatorname{Lip}(Y,\rho)$*. Then φ is a Lipschitz mapping from (K,ρ) to (X,d) for each nonempty compact subset K of* $\operatorname{coz}(u)$*.*

Proof. Let K be a nonempty compact subset of $\operatorname{coz}(u)$. Take $C = \inf\{|u(y)| : y \in K\}$. The continuity of u on $\operatorname{coz}(u)$ implies that $C > 0$. Suppose that $x, y \in K$ with $x \neq y$. By Theorem 2.5, we deduce that

$$\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} \leq \frac{\|T\|(1 + \operatorname{diam}(\varphi(\operatorname{coz}(u))))}{C}.$$

Hence, φ is a Lipschitz mapping from (K,ρ) to (X,d). □

Theorem 2.7. *Let (X,d) and (Y,ρ) be metric spaces, let u be a complex-valued function on Y, let φ be a map from Y to X and let $T = uC_\varphi$ be a weighted composition operator from* $\operatorname{Lip}(X,d)$ *to* $\operatorname{Lip}(Y,\rho)$*. Then φ is continuous on* $\operatorname{coz}(u)$*.*

Proof. Suppose that there exists $y \in \operatorname{coz}(u)$ such that φ is not continuous at y. Then there exist a positive number ε and a sequence $\{y_n\}_{n=1}^{\infty}$ in

Y such that $\rho(y_n, y) < \frac{1}{n}$ and $d(\varphi(y_n), \varphi(y)) \geq \varepsilon$ for all $n \in \mathbb{N}$. Define the function $h : X \to \mathbb{C}$ by

$$h(x) = \max\{0, 1 - \frac{d(\varphi(y), x)}{\varepsilon}\} \qquad (x \in X).$$

Clearly, $h \in \mathrm{Lip}(X, d)$. Since $\lim_{n\to\infty} y_n = y$ in (Y, ρ) and $T(h) \in \mathrm{Lip}(Y, \rho)$, we deduce that

$$\lim_{n\to\infty} T(h)(y_n) = T(h)(y),$$

that is

$$\lim_{n\to\infty} u(y_n)h(\varphi(y_n)) = u(y)h(\varphi(y)). \tag{2.3}$$

Since $h(\varphi(y_n)) = 0$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} u(y_n)h(\varphi(y_n)) = 0. \tag{2.4}$$

By (2.3) and (2.4), we get $u(y)h(\varphi(y)) = 0$ which is a contradiction since $u(y) \neq 0$ and $h(\varphi(y)) = 1$. Therefore, φ is continuous at every $y \in \mathrm{coz}(u)$ and the proof is complete. $\square$

3. Injectivity and surjectivity of weighted composition operators

In this section, we give necessary and sufficient conditions for the injectivity and the surjectivity of weighted composition operators between Lipschitz algebras. We first obtain a generalization of [4, Theorem 3.2] as the following.

Theorem 3.1. *Let (X, d) and (Y, ρ) be metric spaces, let u be a complex-valued function on Y, let φ be a map from Y to X and let $T = uC_\varphi$ be a weighted composition operator from* $\mathrm{Lip}(X, d)$ *to* $\mathrm{Lip}(Y, \rho)$*. Then T is injective if and only if $\varphi(\mathrm{coz}(u))$ is dense in X.*

Proof. Suppose that $\varphi(\mathrm{coz}(u))$ is not dense in X. Choose $x_1 \in X$ such that $\mathrm{dist}(x_1, \varphi(\mathrm{coz}(u))) > 0$. Take $\delta = \mathrm{dist}(x_1, \varphi(\mathrm{coz}(u)))$. Then $\delta > 0$. Define the function $h_{x_1,\delta} : X \to \mathbb{C}$ by

$$h_{x_1,\delta}(x) = \max\{0, 1 - \frac{d(x_1, x)}{\delta}\} \qquad (x \in X).$$

Clearly $h_{x_1,\delta} \in \mathrm{Lip}(X, d)$. On the other hand, $T(h_{x_1,\delta}) = 0$ and $h_{x_1,\delta}(x_1) = 1$. Hence, T is not injective.

Conversely, suppose that $\varphi(\mathrm{coz}(u))$ is dense in X. Let $f \in \mathrm{Lip}(X, d)$ with $T(f) = 0$. Assume that $x \in \varphi(\mathrm{coz}(u))$ and choose $y \in \mathrm{coz}(u)$ such that $x = \varphi(y)$. Since $u(y) \neq 0$ and $0 = T(f)(y) = u(y)f(\varphi(y)) = u(y)f(x)$, we deduce that $f(x) = 0$. Hence, the continuous complex-valued function f on X vanishes on the dense subset $\varphi(\mathrm{coz}(u))$ of X. This implies that $f = 0$ on X. Therefore, T is injective. $\square$

We now give an extension of the sufficiency part of [4, Theorem 3.5].

Theorem 3.2. *Let (X,d) and (Y,ρ) be metric spaces. Suppose that u is a complex-valued function on Y such that $u(y) \neq 0$ for all $y \in Y$ and $\frac{1}{u} \in$ $\mathrm{Lip}(Y,\rho)$. Let φ be a map from Y to X and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$. If $\inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in Y, x \neq y\} > 0$, then T is surjective.*

Proof. Suppose that

$$\inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x,y \in Y, x \neq y\} > 0. \tag{3.1}$$

We can consider $(\varphi(Y),d)$ as a metric space. Define the map $\psi : \varphi(Y) \to Y$ by

$$\psi(\varphi(y)) = y \qquad (y \in Y).$$

Then ψ is well-defined since φ is injective. Moreover, (3.1) implies that ψ is a Lipschitz mapping from $(\varphi(Y),d)$ to (Y,ρ). Let $g \in \mathrm{Lip}(Y,\rho)$. Then $\frac{g}{u} \circ \psi \in \mathrm{Lip}(\varphi(Y),d)$ since $\frac{1}{u} \in \mathrm{Lip}(Y,\rho)$. By [11, Theorem 1.5.6], there exists a function $f \in \mathrm{Lip}(X,d)$ such that $f = \frac{g}{u} \circ \psi$ on $\varphi(Y)$. Hence,

$$T(f)(y) = u(y)f(\varphi(y)) = u(y)(\frac{g}{u} \circ \psi)(\varphi(y)) = g(y)$$

for all $y \in Y$. Therefore, $T(f) = g$ and so T is surjective. $\square$

Here, we obtain a generalization of the necessity part of [4, Theorem 3.5]. For this purpose, we need the following lemma.

Lemma 3.3. *Let (X,d) and (Y,ρ) be metric spaces, let $\mathrm{diam}(Y) < \infty$, let φ be a map from Y to X and let $S = C_\varphi$ be a composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$. If S is surjective, then φ is injective and $\inf\{\frac{d(\varphi(x),\varphi(y)}{\rho(x,y)} : x,y \in Y, x \neq y\} > 0$.*

Proof. Suppose that S is surjective. Let $y \in Y$ and define the function $g_y : Y \to \mathbb{C}$ by

$$g_y(z) = \rho(y,z) \qquad (z \in Y).$$

Since $\mathrm{diam}(Y) < \infty$, we deduce that $|g_y(z)| \leq \mathrm{diam}(Y)$ for all $z \in Y$, g_y is a complex-valued Lipschitz function on (Y,ρ) and $p_{(Y,\rho)}(g_y) \leq 1$. Hence, $g_y \in \mathrm{Lip}(Y,\rho)$ and $\|g_y\|_{\mathrm{Lip}(Y,\rho)} \leq \mathrm{diam}(Y) + 1$.

To prove the injectivity of φ, we assume that $x,y \in Y$ with $\varphi(x) = \varphi(y)$. Since $g_y \in \mathrm{Lip}(Y,\rho)$ and S is surjective, there exists a function $f_y \in \mathrm{Lip}(X,d)$ such that $g_y = S(f_y) = C_\varphi(f_y) = f_y \circ \varphi$. This implies that

$$\rho(x,y) = g_y(x) = S(f_y)(x) = f_y(\varphi(x)) = f_y(\varphi(y)) = S(f_y)(y) = g_y(y) = 0,$$

and so $x = y$. Hence, φ is injective.

Define the map $\rho' : Y \times Y \to \mathbb{R}$ by

$$\rho'(x,y) = d(\varphi(x), \varphi(y)) \qquad (x, y \in Y).$$

Since $d : X \times X \to \mathbb{R}$ is a metric on X and $\varphi : Y \to X$ is injective, we conclude that ρ' is a metric on Y. We claim that $\mathrm{Lip}(Y,\rho)$ is a subset of $\mathrm{Lip}(Y,\rho')$. Suppose that $g \in \mathrm{Lip}(Y,\rho)$. Then g is a bounded complex-valued function on Y. The surjectivity of S implies that there exists a function $f \in \mathrm{Lip}(X,d)$ such that $g = C_\varphi(f)$. Let $x, y \in Y$ with $x \neq y$. Then

$$\begin{aligned}\frac{|g(x)-g(y)|}{\rho'(x,y)} &= \frac{|C_\varphi(f)(x) - C_\varphi(f)(y)|}{d(\varphi(x),\varphi(y))} = \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(\varphi(x),\varphi(y))} \\ &\le p_{(X,d)}(f) \le \|f\|_{\mathrm{Lip}(X,d)}.\end{aligned}$$

This implies that g is a Lipschitz function on (Y,ρ'). Hence, $g \in \mathrm{Lip}(Y,\rho')$ and so our claim is justified. Therefore, the map $g \mapsto g : \mathrm{Lip}(Y,\rho) \to \mathrm{Lip}(Y,\rho')$ is an algebra homomorphism. Since $\mathrm{Lip}(Y,\rho)$ and $\mathrm{Lip}(Y,\rho')$ are unital semisimple commutative Banach algebras, we deduce that the map mentioned is continuous linear mapping. Hence, there exists a positive constant M such that

$$\|g\|_{\mathrm{Lip}(Y,\rho')} \le M\|g\|_{\mathrm{Lip}(Y,\rho)},$$

for all $g \in \mathrm{Lip}(Y,\rho)$.

Let $x, y \in Y$ with $x \neq y$. Since $g_y \in \mathrm{Lip}(Y,\rho)$ and $\|g_y\|_{\mathrm{Lip}(Y,\rho)} \le \mathrm{diam}(Y) + 1$, we deduce that

$$\begin{aligned}\frac{|g_y(x)-g_y(y)|}{\rho'(x,y)} &\le p_{(Y,\rho')}(g_y) \le \|g_y\|_{\mathrm{Lip}(Y,\rho')} \\ &\le M\|g_y\|_{\mathrm{Lip}(Y,\rho)} \le M(\mathrm{diam}(Y)+1).\end{aligned}$$

Take $M' = \frac{1}{M(\mathrm{diam}(Y)+1)}$. Then $M' > 0$ and

$$\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = \frac{\rho'(x,y)}{|g_y(x)-g_y(y)|} \ge M'.$$

Hence,

$$\inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x, y \in Y, x \neq y\} \ge M' > 0,$$

and so the proof is complete. □

Theorem 3.4. *Let (X,d) and (Y,ρ) be metric spaces, let $\mathrm{diam}(Y) < \infty$, let u be a complex-valued function on Y, let φ be a map from Y to X and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$. Suppose that T is surjective. If φ is Lipschitz mapping or $\frac{1}{u} \in \mathrm{Lip}(Y,\rho)$, then $\inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x, y \in Y, x \neq y\} > 0$ and*

$\inf\{|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x \in K, y \in Y, x \neq y\} > 0$, *where* K *is a nonempty compact subset of* Y.

Proof. The surjectivity of T implies that $u(y) \neq 0$ for all $y \in Y$ since $1_Y \in \mathrm{Lip}(Y,\rho)$ and $1_Y = T(f_1) = u\cdot(f_1 \circ \varphi)$ for some $f_1 \in \mathrm{Lip}(X,d)$. We first assume that φ is a Lipschitz mapping from (Y,ρ) to (X,d). Then $f \circ \varphi \in \mathrm{Lip}(Y,\rho)$ for all $f \in \mathrm{Lip}(X,d)$ and so C_φ is composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$. We now assume that $\frac{1}{u} \in \mathrm{Lip}(Y,\rho)$. Then $f \circ \varphi = \frac{1}{u}T(f) \in \mathrm{Lip}(Y,\rho)$ for all $f \in \mathrm{Lip}(X,d)$ and so C_φ is composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$.

We claim that C_φ is surjective. Suppose that $g \in \mathrm{Lip}(Y,\rho)$. Then $ug \in \mathrm{Lip}(Y,\rho)$. The surjectivity of T implies that $ug = T(f)$ for some $f \in \mathrm{Lip}(X,d)$ and so $g = f \circ \varphi = C_\varphi(f)$ for some $f \in \mathrm{Lip}(X,d)$. Hence, our claim is justified.

By Lemma 3.3, $\varphi : Y \to \varphi(Y)$ is injective and

$$\inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x, y \in Y, x \neq y\} > 0. \tag{3.2}$$

We now assume that K is a nonempty compact subset of Y. Then $\inf\{|u(x)| : x \in K\} = |u(x_1)|$ for some $x_1 \in K$. This implies that $|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} \geq |u(x_1)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}$ for all $x \in K$, $y \in Y$ with $x \neq y$. Hence, by (3.2) and $|u(x_1)| > 0$, we have

$$\begin{aligned}\inf\{|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} :&x \in K, y \in Y, x \neq y\} \\ &\geq |u(x_1)| \inf\{\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} : x, y \in Y, x \neq y\} \\ &> 0.\end{aligned}$$

Therefore, the proof is complete. □

4. Compactness of weighted composition operators

Let (X,d) be a metric space, let $(\widetilde{X},\widetilde{d})$ be the completion of (X,d) and let (Y,ρ) be a complete metric space. By [11, Proposition 1.7.1], every Lipschitz mapping φ from (X,d) to (Y,ρ) has a Lipschitz extension $\widetilde{\varphi}$ from $(\widetilde{X},\widetilde{d})$ to (Y,ρ) such that

$$\sup\{\frac{\rho(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))}{\widetilde{d}(\widetilde{x},\widetilde{y})} : \widetilde{x}, \widetilde{y} \in \widetilde{X}, \widetilde{x} \neq \widetilde{y}\} = \sup\{\frac{\rho(\varphi(x),\varphi(y))}{d(x,y)} : x, y \in X, x \neq y\}.$$

In fact $\widetilde{\varphi}(\widetilde{x}) = \lim_{n\to\infty} \varphi(x_n)$, where $\widetilde{x} \in \widetilde{X}$ and $\{x_n\}_{n=1}^\infty$ is a sequence in X such that $\lim_{n\to\infty} x_n = \widetilde{x}$ in $(\widetilde{X},\widetilde{d})$. We assume that $A = \mathrm{Lip}(X,d)$ and $\widetilde{A} = \mathrm{Lip}(\widetilde{X},\widetilde{d})$. By [8, Lemma 2.8], $\widetilde{A} = \{\widetilde{f} : f \in A\}$ and the map

$f \rightsquigarrow \widetilde{f} : A \to \widetilde{A}$ is an isometrical isomorphism from $(A, \|\cdot\|_{\mathrm{Lip}(X,d)})$ onto $(\widetilde{A}, \|\cdot\|_{\mathrm{Lip}(\widetilde{X},\widetilde{d})})$.

Here we obtain an extension of [4, Theorem 4.3], whenever φ is a Lipschitz mapping from (Y,ρ) to (X,d). For this purpose we need the following lemma.

Lemma 4.1. *Let (X,d) and (Y,ρ) be metric spaces, let $A = \mathrm{Lip}(X,d)$ and $B = \mathrm{Lip}(Y,\rho)$ and let $T : A \to B$ be a linear mapping. Suppose that $(\widetilde{X},\widetilde{d})$ and $(\widetilde{Y},\widetilde{\rho})$ are the completions of (X,d) and (Y,ρ), respectively, $\widetilde{A} = \mathrm{Lip}(\widetilde{X},\widetilde{d})$ and $\widetilde{B} = \mathrm{Lip}(\widetilde{Y},\widetilde{\rho})$. Define the map $\widetilde{T} : \widetilde{A} \to \widetilde{B}$ by $\widetilde{T}(\widetilde{f}) = \widetilde{T(f)} \quad (f \in A)$. Then the following statements hold.*

(1) *$\widetilde{T}$ is a linear mapping.*
(2) *$\widetilde{T}$ is bounded if and only if T is bounded. Moreover, $\|\widetilde{T}\| = \|T\|$.*
(3) *$\widetilde{T}$ is compact if and only if T is compact.*

Proof. Define the maps $\Phi : A \to \widetilde{A}$ and $\Psi : B \to \widetilde{B}$ by

$$\Phi(f) = \widetilde{f} \qquad (f \in A), \qquad \Psi(g) = \widetilde{g} \qquad (g \in B).$$

Then Φ is an isometrical isomorphism from $(A, \|\cdot\|_{\mathrm{Lip}(X,d)})$ onto $(\widetilde{A}, \|\cdot\|_{\mathrm{Lip}(\widetilde{X},\widetilde{d})})$ and Ψ is an isometrical isomorphism from $(B, \|\cdot\|_{\mathrm{Lip}(Y,\rho)})$ onto $(\widetilde{B}, \|\cdot\|_{\mathrm{Lip}(\widetilde{Y},\widetilde{\rho})})$. It is clear that

$$\widetilde{T} = \Psi \circ T \circ \Phi^{-1}.$$

This implies that (1)-(3) hold. □

Theorem 4.2. *Let (X,d) and (Y,ρ) be metric spaces such that $(\widetilde{X},\widetilde{d})$ and $(\widetilde{Y},\widetilde{\rho})$ are compact. Let φ be a Lipschitz mapping from (Y,ρ) to (X,d), let u be a complex-valued function on Y and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$. Then T is compact if and only if $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0.*

Proof. Define the map $\widetilde{T} : \mathrm{Lip}(\widetilde{X},\widetilde{d}) \to \mathrm{Lip}(\widetilde{Y},\widetilde{\rho})$ by

$$\widetilde{T}(\widetilde{f}) = \widetilde{T(f)} \qquad (f \in \mathrm{Lip}(X,d)).$$

By Lemma 4.1, $\widetilde{T}$ is a bounded linear mapping. Since $u \in \mathrm{Lip}(Y,\rho)$, we have $\widetilde{u} \in \mathrm{Lip}(\widetilde{Y},\widetilde{\rho})$. This implies that $\widetilde{u}\cdot(g \circ \widetilde{\varphi}) \in \mathrm{Lip}(\widetilde{Y},\widetilde{\rho})$ for all $g \in \mathrm{Lip}(\widetilde{X},\widetilde{d})$ since $\widetilde{\varphi} : \widetilde{Y} \to \widetilde{X}$ is a Lipschitz mapping from $(\widetilde{Y},\widetilde{\rho})$ to $(\widetilde{X},\widetilde{d})$. Therefore, $\widetilde{u}C_{\widetilde{\varphi}}$ is a weighted composition operator from $\mathrm{Lip}(\widetilde{X},\widetilde{d})$ to $\mathrm{Lip}(\widetilde{Y},\widetilde{\rho})$. We claim that $\widetilde{u}C_{\widetilde{\varphi}} = \widetilde{T}$. Let $g \in \mathrm{Lip}(\widetilde{X},\widetilde{d})$. Then there exists a function $f \in \mathrm{Lip}(X,d)$ such that $g = \widetilde{f}$. Let $\widetilde{y} \in \widetilde{Y}$

and $\{y_n\}_{n=1}^{\infty}$ be a sequence in Y with $\lim_{n\to\infty} y_n = \widetilde{y}$ in $(\widetilde{Y}, \widetilde{\rho})$. Then $\lim_{n\to\infty} \varphi(y_n) = \widetilde{\varphi}(\widetilde{y})$ in $(\widetilde{X}, \widetilde{d})$ and so $\lim_{n\to\infty} f(\varphi(y_n)) = \widetilde{f}(\widetilde{\varphi}(\widetilde{y}))$. Since $\lim_{n\to\infty} u(y_n) = \widetilde{u}(\widetilde{y})$, hence, $\lim_{n\to\infty}(u \cdot (f \circ \varphi))(y_n) = (\widetilde{u} \cdot (\widetilde{f} \circ \widetilde{\varphi}))(\widetilde{y})$ and so $\lim_{n\to\infty} T(f)(y_n) = (\widetilde{u}C_{\widetilde{\varphi}})(\widetilde{y})$. Therefore,

$$\widetilde{T(f)}(\widetilde{y}) = (\widetilde{u}C_{\widetilde{\varphi}})(\widetilde{f})(\widetilde{y}). \tag{4.1}$$

Since (4.1) holds for all $\widetilde{y} \in \widetilde{Y}$, we deduce that $\widetilde{T(f)} = \widetilde{u}C_{\widetilde{\varphi}}(\widetilde{f})$. Hence,

$$\widetilde{T}(g) = \widetilde{T}(\widetilde{f}) = \widetilde{T(f)} = (\widetilde{u}C_{\widetilde{\varphi}})(g). \tag{4.2}$$

Since (4.2) holds for all $g \in \mathrm{Lip}(\widetilde{X}, \widetilde{d})$, we have $\widetilde{T} = \widetilde{u}C_{\widetilde{\varphi}}$. Hence, our claim is justified.

We first assume that T is compact. By Lemma 4.1, $\widetilde{T}$ is compact. According to [4, Theorem 4.3], we deduce that $\lim \widetilde{u}(\widetilde{x})\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))}{\widetilde{\rho}(\widetilde{x},\widetilde{y})} = 0$ when $\widetilde{d}(\widetilde{\varphi}(\widetilde{x}), \widetilde{\varphi}(\widetilde{y}))$ tends to 0. Since

$$u(x)\frac{d(\varphi(x), \varphi(y))}{\rho(x, y)} = \widetilde{u}(\widetilde{x})\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}), \widetilde{\varphi}(\widetilde{y}))}{\widetilde{\rho}(\widetilde{x}, \widetilde{y})},$$

for all $x, y \in Y$ with $x \neq y$, we conclude that

$$\lim u(x)\frac{d(\varphi(x), \varphi(y))}{\rho(x, y)} = 0,$$

when $d(\varphi(x), \varphi(y))$ tends to 0.

We now assume that

$$\lim u(x)\frac{d(\varphi(x), \varphi(y))}{\rho(x, y)} = 0, \tag{4.3}$$

when $d(\varphi(x), \varphi(y))$ tends to 0. To prove compactness of T, by Lemma 4.1, it is enough to show that $\widetilde{T}$ is compact. Clearly, $\widetilde{u} \in \mathrm{Lip}(\widetilde{Y}, \widetilde{\rho})$. To prove the compactness of $\widetilde{T}$, by [4, Theorem 4.3], it is enough to show that $\lim \widetilde{u}(\widetilde{x})\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))}{\widetilde{\rho}(\widetilde{x},\widetilde{y})} = 0$ when $\widetilde{d}(\widetilde{\varphi}(\widetilde{x}), \widetilde{\varphi}(\widetilde{y}))$ tends to 0.

Let $\varepsilon > 0$ be given. By(4.3), there exists a $\delta_1 > 0$ such that

$$|u(x)|\frac{d(\varphi(x), \varphi(y))}{\rho(x, y)} < \frac{\varepsilon}{2}, \tag{4.4}$$

for all $x, y \in Y$ with $0 < d(\varphi(x), \varphi(y)) < \delta_1$. Choose $\delta = \frac{\delta_1}{2}$. Let $\widetilde{x}, \widetilde{y} \in \widetilde{Y}$ with $0 < \widetilde{d}(\widetilde{\varphi}(\widetilde{x}), \widetilde{\varphi}(\widetilde{y})) < \delta$. There exist two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in Y such that $\lim_{n\to\infty} \widetilde{\rho}(x_n, \widetilde{x}) = 0$ and $\lim_{n\to\infty} \widetilde{\rho}(y_n, \widetilde{y}) = 0$ and so $\lim_{n\to\infty} \widetilde{d}(\varphi(x_n), \widetilde{\varphi}(\widetilde{x})) = 0$ and $\lim_{n\to\infty} \widetilde{d}(\varphi(y_n), \widetilde{\varphi}(\widetilde{y})) = 0$. Hence, there exists $N_1 \in \mathbb{N}$ such that

$$|d(\varphi(x_n), \varphi(y_n)) - \widetilde{d}(\widetilde{\varphi}(\widetilde{x}), \widetilde{\varphi}(\widetilde{y}))| < \delta,$$

and $\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))}{2} < d(\varphi(x_n),\varphi(y_n))$ for all $n \in \mathbb{N}$ with $n \geq N_1$. Therefore, $0 < d(\varphi(x_n),\varphi(y_n)) < 2\delta = \delta_1$ for all $n \in \mathbb{N}$ with $n \geq N_1$. Hence, by (4.4), we have

$$|u(x_n)|\frac{d(\varphi(x_n),\varphi(y_n))}{\rho(x_n,y_n)} < \frac{\varepsilon}{2}, \tag{4.5}$$

for all $n \in \mathbb{N}$ with $n \geq N_1$. Since

$$|\widetilde{u}(\widetilde{x})|\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))}{\widetilde{\rho}(\widetilde{x},\widetilde{y})} = \lim_{n\to\infty}|u(x_n)|\frac{d(\varphi(x_n),\varphi(y_n))}{\rho(x_n,y_n)},$$

there exists $N_2 \in \mathbb{N}$ such that

$$||u(x_n)|\frac{d(\varphi(x_n),\varphi(y_n))}{\rho(x_n,y_n)} - |\widetilde{u}(\widetilde{x})|\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))}{\widetilde{\rho}(\widetilde{x},\widetilde{y})}| < \frac{\varepsilon}{2}, \tag{4.6}$$

for all $n \in \mathbb{N}$ with $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then (4.5) and (4.6) hold for $n = N$ and so

$$|\widetilde{u}(\widetilde{x})|\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))}{\widetilde{\rho}(\widetilde{x},\widetilde{y})} < \varepsilon.$$

This implies that $\lim \widetilde{u}(\widetilde{x})\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))}{\widetilde{\rho}(\widetilde{x},\widetilde{y})} = 0$ when $\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))$ tends to 0. Therefore, $\widetilde{T}$ is compact and the proof is complete. □

We recall that the essential norm of a bounded linear operator T on a Banach space $(E, \|\cdot\|)$ is denoted by $\|T\|_e$ and defined by

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is a compact linear operator on } E\}.$$

For each $\alpha \in (0, 1]$, the map $d^\alpha : X \times X \to \mathbb{R}$ defined by

$$d^\alpha(x,y) = (d(x,y))^\alpha, \quad ((x,y) \in X \times X)$$

is a metric on X and the induced topology on X by d^α coincides with the induced topology on X by d.

For a weighted composition operator $T = uC_\varphi$ from $\mathrm{Lip}(X, d^\alpha)$ to $\mathrm{Lip}(X, d^\alpha)$, we obtain a lower bound for the essential norm $\|T\|_e$ of T, whenever $(\widetilde{X}, \widetilde{d})$ is a compact metric space, $0 < \alpha < 1$ and φ is a Lipschitz mapping from (X, d) to (X, d).

Theorem 4.3. *Let (X, d) be a metric space such that $(\widetilde{X}, \widetilde{d})$, the completion of (X, d), is compact. Let $\alpha \in (0, 1)$, let φ be a Lipschitz mapping from (X, d) to (X, d), let u be a complex-valued function on X and let $T = uC_\varphi$ be a weighted composition operator from* $\mathrm{Lip}(X, d^\alpha)$ *to* $\mathrm{Lip}(X, d^\alpha)$. *Then*

$$\limsup_{d(\varphi(x),\varphi(y))\to 0}|u(x)|\left(\frac{d(\varphi(x),\varphi(y))}{d(x,y)}\right)^\alpha \leq \|T\|_e.$$

Proof. Define the map $\widetilde{T} : \mathrm{Lip}(\widetilde{X}, \widetilde{d}^\alpha) \to \mathrm{Lip}(\widetilde{X}, \widetilde{d}^\alpha)$ by

$$\widetilde{T}(\widetilde{f}) = \widetilde{T(f)} \qquad (f \in \mathrm{Lip}(X, d^\alpha)).$$

By the argument given in the proof of Theorem 4.2, we deduce that $\widetilde{T}$ is a weighted composition operator from $\mathrm{Lip}(\widetilde{X}, \widetilde{d}^\alpha)$ to $\mathrm{Lip}(\widetilde{X}, \widetilde{d}^\alpha)$ induced by $\widetilde{u}$ and $\widetilde{\varphi}$. By [4, Theorem 5.1], we have

$$\limsup_{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))\to 0} |\widetilde{u}(\widetilde{x})|(\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}), \widetilde{\varphi}(\widetilde{y}))}{\widetilde{d}(\widetilde{x}, \widetilde{y})})^\alpha \leq \|\widetilde{T}\|_e. \tag{4.7}$$

On the other hand,

$$\|\widetilde{T}\|_e = \|T\|_e, \tag{4.8}$$

by [8, Proposition 2.13]. Since $(\widetilde{X}, \widetilde{d}^\alpha)$ is a compact metric space and $\widetilde{T} = \widetilde{u}C_{\widetilde{\varphi}}$ is a weighted composition operator from $\mathrm{Lip}(\widetilde{X}, \widetilde{d}^\alpha)$ to $\mathrm{Lip}(\widetilde{X}, \widetilde{d}^\alpha)$, we deduce that

$$\sup\{|\widetilde{u}(\widetilde{x})|\frac{\widetilde{d}^\alpha(\widetilde{\varphi}(\widetilde{x}), \widetilde{\varphi}(\widetilde{y}))}{\widetilde{d}^\alpha(\widetilde{x}, \widetilde{y})} : \widetilde{x}, \widetilde{y} \in \widetilde{X}, \widetilde{x} \neq \widetilde{y}\} < \infty,$$

by [4, Theorem 2.1]. This implies that

$$\sup\{|u(x)|\frac{d^\alpha(\varphi(x), \varphi(y))}{d^\alpha(x, y)} : x, y \in X, x \neq y\} < \infty.$$

For each $t > 0$, set

$$E_t = \{|u(x)|(\frac{d(\varphi(x), \varphi(y))}{d(x, y)})^\alpha : x, y \in X, x \neq y, 0 < d(\varphi(x), \varphi(y)) < t\},$$

and

$$\widetilde{E}_t = \{|\widetilde{u}(\widetilde{x})|(\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}), \widetilde{\varphi}(\widetilde{y}))}{\widetilde{d}(\widetilde{x}, \widetilde{y})})^\alpha : \widetilde{x}, \widetilde{y} \in \widetilde{X}, \widetilde{x} \neq \widetilde{y}, 0 < \widetilde{d}(\widetilde{\varphi}(\widetilde{x}), \widetilde{\varphi}(\widetilde{y})) < t\}.$$

By the argument above, we have $\sup E_t < \infty$ and $\sup \widetilde{E}_t < \infty$ for all $t > 0$.

Let $t > 0$ be given. If $x, y \in X$ with $0 < d(\varphi(x), \varphi(y)) < t$, then

$$|u(x)|(\frac{d(\varphi(x), \varphi(y))}{d(x, y)})^\alpha = |\widetilde{u}(x)|(\frac{\widetilde{d}(\widetilde{\varphi}(x), \widetilde{\varphi}(y))}{\widetilde{d}(x, y)})^\alpha \leq \sup \widetilde{E}_t.$$

Hence,

$$\sup E_t \leq \sup \widetilde{E}_t. \tag{4.9}$$

Since (4.9) holds for each $t > 0$, we deduce that

$$\inf\{\sup E_t : t > 0\} \leq \inf\{\sup \widetilde{E}_t : t > 0\}.$$

This implies that

$$\limsup_{d(\varphi(x),\varphi(y))\to 0} |u(x)|(\frac{d(\varphi(x),\varphi(y))}{d(x,y)})^{\alpha} \le \limsup_{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))\to 0} |\widetilde{u}(\widetilde{x})|(\frac{\widetilde{d}(\widetilde{\varphi}(\widetilde{x}),\widetilde{\varphi}(\widetilde{y}))}{\widetilde{d}(\widetilde{x},\widetilde{y})})^{\alpha}. \tag{4.10}$$

From (4.10), (4.7) and (4.8), we conclude that

$$\limsup_{d(\varphi(x),\varphi(y))\to 0} |u(x)|(\frac{d(\varphi(x),\varphi(y))}{d(x,y)})^{\alpha} \le \|T\|_e.$$

Hence, the proof is complete. □

We now give a generalization of [4, Corollary 4.2]. To this purpose we need the following lemma.

Lemma 4.4. *Let (X, d) be a metric space. Then every bounded sequence $\{f_n\}_{n=1}^{\infty}$ in $(\mathrm{Lip}(X, d), \|\cdot\|_{\mathrm{Lip}(X,d)})$ has a subsequence that converges pointwise on X to a function $f \in \mathrm{Lip}(X, d)$. Moreover, this convergence is uniform on each totally bounded subset of X.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $(\mathrm{Lip}(X, d), \|\cdot\|_{\mathrm{Lip}(X,d)})$. Since the norms $\|\cdot\|_{\mathrm{Lip}(X,d)}$ and $\|\cdot\|_M$ are equivalent on complex linear space $\mathrm{Lip}(X, d)$, $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathrm{Lip}(X, d)$ with norm $\|\cdot\|_M$. Let $x_0 \notin X$ and $X_0 = X \cup \{x_0\}$. Define the map $d_0 : X_0 \times X_0 \to \mathbb{R}$ by

$$\begin{aligned} &d_0(x, y) = \min\{d(x, y), 2\} \qquad (x, y \in X),\\ &d_0(x, x_0) = d_0(x_0, y) = 1 \qquad (x, y \in X),\\ &d_0(x_0, x_0) = 0. \end{aligned}$$

Then d_0 is a metric on X_0 and $\mathrm{Lip}_0(X_0, d_0)$ is a complex Banach space with the norm $p_{(X_0,d_0)}(\cdot)$. Define the map $\Phi : \mathrm{Lip}(X, d) \to \mathrm{Lip}_0(X_0, d_0)$ by

$$\Phi(f)(x) = f(x) \quad (x \in X), \qquad \Phi(f)(x_0) = 0.$$

By [11, Proposition 1.7.1 and Theorem 1.7.2], Φ is a complex linear isometry from $(\mathrm{Lip}(X, d), \|\cdot\|_M)$ onto $(\mathrm{Lip}_0(X_0, d_0), p_{(X_0,d_0)}(\cdot))$. Hence, $\{\Phi(f_n)\}_{n=1}^{\infty}$ is a bounded sequence in $\mathrm{Lip}_0(X_0, d_0)$. By [6, Lemma 2.5], there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\{\Phi(f_{n_k})\}_{k=1}^{\infty}$ converges pointwise on X_0 and this convergence is uniform on all totally bounded sets in (X_0, d_0). Hence, there exists a function g in $\mathrm{Lip}_0(X_0, d_0)$ such that

$$g(y) = \lim_{k\to\infty} \Phi(f_{n_k})(y),$$

for all $y \in X_0$ and $\{\Phi(f_{n_k})\}_{k=1}^{\infty}$ converges to the function g uniformly on all totally bounded sets in (X_0, d_0). The surjectivity of Φ implies that there exists a function f in $\mathrm{Lip}(X,d)$ with $\Phi(f)=g$. Since $\Phi(h)(x)=h(x)$ for all $h\in \mathrm{Lip}(X,d)$ and $x\in X$, we deduce that

$$f(x)=\lim_{k\to\infty} f_{n_k}(x),$$

for all $x\in X$. Let $E\subseteq X$ be a totally bounded set in (X,d) and let $\varepsilon>0$ be given. Take $\varepsilon'=\min\{\varepsilon,1\}$. Then there exist $x_1,\dots,x_n\in E$ such that

$$E\subseteq \cup_{j=1}^{n} B_d(x_j,\varepsilon').$$

It is easy to see that

$$E\subseteq \cup_{j=1}^{n} B_{d_0}(x_j,\varepsilon).$$

Hence, E is a totally bounded set in (X_0,d_0). By the argument above, $\{\Phi(f_{n_k})\}_{k=1}^{\infty}$ converges uniformly on E to the function g. This implies that $\{f_{n_k}\}_{k=1}^{\infty}$ converges uniformly on E to the function f. Hence, the proof is complete. □

Theorem 4.5. *Let (X,d) and (Y,ρ) be metric spaces, let u be a complex-valued function on Y, let φ be a map from Y to X and let $T=uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$. Then T is compact if and only if for each bounded sequence $\{f_n\}_{n=1}^{\infty}$ in $(\mathrm{Lip}(X,d),\|\cdot\|_{\mathrm{Lip}(X,d)})$ which converges to the function 0 uniformly on totally bounded subsets of X, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\{T(f_{n_k})\}_{k=1}^{\infty}$ converges to the function 0 in $(\mathrm{Lip}(Y,\rho),\|\cdot\|_{\mathrm{Lip}(Y,\rho)})$.*

Proof. Suppose that $T=uC_\varphi$ is a compact operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$ and $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $(\mathrm{Lip}(X,d),\|\cdot\|_{\mathrm{Lip}(X,d)})$ that converges uniformly to the function 0 on totally bounded subsets of X. By the compactness of T, there exist a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ and a function $g\in\mathrm{Lip}(Y,\rho)$ such that $\{T(f_{n_k})\}_{k=1}^{\infty}$ converges to the function g in $(\mathrm{Lip}(Y,\rho),\|\cdot\|_{\mathrm{Lip}(Y,\rho)})$. Since $\|h\|_Y\le\|h\|_{\mathrm{Lip}(Y,\rho)}$ for all $h\in\mathrm{Lip}(Y,\rho)$, the sequence $\{u(y)f_{n_k}(\varphi(y))\}_{k=1}^{\infty}$ converges to $g(y)$ for all $y\in Y$. On the other hand, for each $y\in Y$ the set $\{\varphi(y)\}$ is totally bounded in (X,d). Hence, $\lim_{k\to\infty} f_{n_k}(\varphi(y))=0$ for all $y\in Y$. This implies that $\lim_{k\to\infty} u(y)f_{n_k}(\varphi(y))=0$ for all $y\in Y$ since u is a complex-valued bounded function on Y. Therefore, $g(y)=0$ for all $y\in Y$ and so $g=0$.

Conversely, assume that every bounded sequence $\{f_n\}_{n=1}^{\infty}$ in the Banach algebra $(\mathrm{Lip}(X,d),\|\cdot\|_{\mathrm{Lip}(X,d)})$ which converges to the function 0 uniformly on totally bounded subsets of X has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\{T(f_{n_k})\}_{k=1}^{\infty}$ converges to the function 0 in $(\mathrm{Lip}(Y,\rho),\|\cdot\|_{\mathrm{Lip}(Y,\rho)})$. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $(\mathrm{Lip}(X,d),\|\cdot\|_{\mathrm{Lip}(X,d)})$. By Lemma 4.4, there exist a strictly increasing function $\gamma:\mathbb{N}\to\mathbb{N}$ and a function $f\in\mathrm{Lip}(X,d)$ such that $\{f_{\gamma(k)}\}_{k=1}^{\infty}$ converges to the function f uniformly on totally bounded subsets of X. Hence, $\{f_{\gamma(k)}-f\}_{k=1}^{\infty}$

converges to the function 0 uniformly on totally bounded subsets of X. Thus, there exists a strictly increasing function $\eta : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{k\to\infty} \|T(f_{\eta(\gamma(k))} - f)\|_{\mathrm{Lip}(Y,\rho)} = 0.$$

For each $k \in \mathbb{N}$, set $n_k = (\eta \circ \gamma)(k)$. Then $\{f_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{f_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} \|T(f_{n_k}) - T(f)\|_{\mathrm{Lip}(Y,\rho)} = 0.$$

Therefore, $T = uC_\varphi$ is compact. □

Now we are ready to obtain an another generalization of [4, Theorem 4.3] as follows.

Theorem 4.6. *Let (X, d) and (Y, ρ) be metric spaces, let u be a complex-valued function on Y, let φ be a map from Y to X, let $\varphi(\mathrm{coz}(u))$ be totally bounded in (X, d) and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}(X, d)$ to $\mathrm{Lip}(Y, \rho)$. Then T is compact if and only if $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x), \varphi(y))$ tends to 0.*

Proof. We first assume that $T = uC_\varphi$ is compact. Suppose that there exist $\varepsilon > 0$ and two sequence $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in Y with $x_n \neq y_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(\varphi(x_n), \varphi(y_n)) = 0$, but $|u(x_n)|\frac{d(\varphi(x_n),\varphi(y_n))}{\rho(x_n,y_n)} \geq \varepsilon$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define the function $f_n : X \to \mathbb{C}$ by

$$f_n(t) = \begin{cases} d(t, \varphi(y_n)) & d(t, \varphi(y_n)) \leq d(\varphi(x_n), \varphi(y_n)), \\ d(\varphi(x_n), \varphi(y_n)) & d(t, \varphi(y_n)) \geq d(\varphi(x_n), \varphi(y_n)), \end{cases}$$

for all $t \in X$. It is easy to see that $\|f_n\|_X \leq d(\varphi(x_n), \varphi(y_n))$ and $p_{(X,d)}(f_n) \leq 1$ for all $n \in \mathbb{N}$. Then $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathrm{Lip}(X, d)$ which converges to the function 0 uniformly on X and so converges to the function 0 uniformly on totally bounded subsets of X. By Theorem 4.5 and the compactness of T, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\{T(f_{n_k})\}_{k=1}^{\infty}$ converges to the function 0 in $(\mathrm{Lip}(Y, \rho), \|\cdot\|_{\mathrm{Lip}(Y,\rho)})$. Hence, there exists a positive integer N such that

$$p_{(Y,\rho)}(T(f_{n_N})) + \|T(f_{n_N})\|_Y = \|T(f_{n_N})\|_{\mathrm{Lip}(Y,\rho)} < \frac{\varepsilon}{2}.$$

This implies that

$$\begin{aligned} |u(x_{n_N})|\frac{d(\varphi(x_{n_N}), \varphi(y_{n_N}))}{\rho(x_{n_N}, y_{n_N})} &= \frac{|T(f_{n_N})(x_{n_N}) - T(f_{n_N})(y_{n_N})|}{\rho(x_{n_N}, y_{n_N})} \\ &\leq p_{(Y,\rho)}(T(f_{n_N})) \\ &< \frac{\varepsilon}{2}, \end{aligned}$$

which is a contradiction.

Conversely, suppose that $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $(\mathrm{Lip}(X,d), \|\cdot\|_{\mathrm{Lip}(X,d)})$ that converges uniformly to the function 0 on totally bounded subsets of X. Note that the existence of such sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathrm{Lip}(X,d)$ is guaranteed by Lemma 4.4. Let $M > 0$ with $\|f_n\|_{\mathrm{Lip}(X,d)} < M$ for all $n \in \mathbb{N}$. Take

$$C = C(u,\varphi). \tag{4.11}$$

Since $\varphi(\mathrm{coz}(u))$ is totally bounded in (X,d) and $T = uC_\varphi$ is a weighted composition operator, we deduce that $\mathrm{diam}(\varphi(\mathrm{coz}(u))) < \infty$, T is a bounded linear operator and $C \leq \|T\|(1+\mathrm{diam}(\varphi(\mathrm{coz}(u))))$ by Theorem 2.5. Let $\varepsilon > 0$ be given.Then there exists $\delta > 0$ such that

$$|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \frac{\varepsilon}{2M}, \tag{4.12}$$

whenever $x, y \in Y$ with $0 < d(\varphi(x),\varphi(y)) < \delta$. Since $\varphi(\mathrm{coz}(u))$ is totally bounded in (X,d), the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function 0 on $\varphi(\mathrm{coz}(u))$. This implies that $\{f_n \circ \varphi\}_{n=1}^{\infty}$ converges uniformly to the function 0 on $\mathrm{coz}(u)$. Since u is bounded complex-valued function on Y, we deduce that $\{Tf_n\}_{n=1}^{\infty}$ converges uniformly to the function 0 on $\mathrm{coz}(u)$ and so on Y. Hence, there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq N$, we have

$$|f_n(\varphi(y))| < \frac{\varepsilon}{A} \tag{4.13}$$

for all $y \in \mathrm{coz}(u)$, where $A = 6(1 + \frac{2C}{\delta} + p_{(Y,\rho)}(u))$ and

$$\|T(f_n)\|_Y < \frac{\varepsilon}{3}. \tag{4.14}$$

Let $n \in \mathbb{N}$ with $n \geq N$. Suppose that $x, y \in \mathrm{coz}(u)$ with $\varphi(x) \neq \varphi(y)$. Then we have

$$\begin{aligned}
\frac{|T(f_n)(x) - T(f_n)(y)|}{\rho(x,y)} &= \frac{|u(x)f_n(\varphi(x)) - u(y)f_n(\varphi(y))|}{\rho(x,y)} \\
&\leq \frac{|f_n(\varphi(x)) - f_n(\varphi(y))|}{\rho(x,y)}|u(x)| \\
&\quad + \frac{|u(x) - u(y)|}{\rho(x,y)}|f_n(\varphi(y))| \\
&< \frac{|f_n(\varphi(x)) - f_n(\varphi(y))|}{d(\varphi(x),\varphi(y))}\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}|u(x)| \\
&\quad + \frac{\varepsilon}{6},
\end{aligned}$$

by (4.13). If $0 < d(\varphi(x), \varphi(y)) < \delta$, then

$$\begin{aligned}\frac{|T(f_n)(x) - T(f_n)(y)|}{\rho(x,y)} &\leq p_{(X,d)}(f_n)\frac{\varepsilon}{2M} + \frac{\varepsilon}{6} \\ &\leq \|f_n\|_{\mathrm{Lip}(X,d)}\frac{\varepsilon}{2M} + \frac{\varepsilon}{6} \\ &< \frac{2\varepsilon}{3},\end{aligned}$$

by (4.12). If $d(\varphi(x), \varphi(y)) \geq \delta$, then

$$\begin{aligned}\frac{|T(f_n)(x) - T(f_n)(y)|}{\rho(x,y)} &\leq \frac{|f_n(\varphi(x))| + |f_n(\varphi(y))|}{\delta}C + \frac{\varepsilon}{6} \\ &< \frac{2C\varepsilon}{\delta A} + \frac{\varepsilon}{6} \\ &< \frac{2\varepsilon}{3},\end{aligned}$$

by (4.11).

Suppose that $x, y \in \mathrm{coz}(u)$ with $x \neq y$ and $\varphi(x) = \varphi(y)$. Then

$$\begin{aligned}\frac{|T(f_n)(x) - T(f_n)(y)|}{\rho(x,y)} &\leq \frac{|u(x) - u(y)|}{\rho(x,y)}|f_n(\varphi(y))| \\ &< \frac{2\varepsilon}{3},\end{aligned}$$

by (4.13).

Suppose that $x \in \mathrm{coz}(u)$ and $u(y) = 0$. Then

$$\begin{aligned}\frac{|T(f_n)(x) - T(f_n)(y)|}{\rho(x,y)} &= \frac{|u(x)f_n(\varphi(x))|}{\rho(x,y)} \\ &= \frac{|u(x) - u(y)|}{\rho(x,y)}|f_n(\varphi(x))| \\ &< \frac{2\varepsilon}{3},\end{aligned}$$

by (4.13).

Suppose that $u(x) = 0$ and $y \in \mathrm{coz}(u)$. By similar to the argument above, we have

$$\frac{|T(f_n)(x) - T(f_n)(y)|}{\rho(x,y)} < \frac{2\varepsilon}{3}.$$

Suppose that $x, y \in Y$ with $x \neq y$ and $u(x) = u(y) = 0$. Then

$$\frac{|T(f_n)(x) - T(f_n)(y)|}{\rho(x,y)} = 0.$$

Hence, $\dfrac{|T(f_n)(x)-T(f_n)(y)|}{\rho(x,y)} < \dfrac{2\varepsilon}{3}$ for all $x, y \in Y$ with $x \neq y$.This implies that

$$p_{(Y,\rho)}(T(f_n)) < \frac{2\varepsilon}{3}. \tag{4.15}$$

From (4.14) and (4.15), we have

$$\|T(f_n)\|_{\mathrm{Lip}(Y,\rho)} < \varepsilon,$$

for all $n \in \mathbb{N}$ with $n \geq N$. Hence, $\lim_{n\to\infty} \|T(f_n)\|_{\mathrm{Lip}(Y,\rho)} = 0$. Therefore, T is compact by Theorem 4.5. $\square$

Note that in the sufficiency part of Theorem 4.6, we can not remove the totally boundedness of $\varphi(\mathrm{coz}(u))$ in (X,d) in general. To show this assertion we need the following lemmas.

Lemma 4.7. *Let (X,d) and (Y,ρ) be metric spaces and let φ be a uniformly continuous mapping from (Y,ρ) to (X,d). Then* $\lim \frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ *when $d(\varphi(x),\varphi(y))$ tends to 0 if and only if φ is supercontractive from (Y,ρ) to (X,d).*

Proof. We first assume that $\lim \frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0. Let $\varepsilon > 0$ be given. Then there exists $\delta_1 > 0$ such that $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \varepsilon$, when $x, y \in Y$ with $0 < d(\varphi(x),\varphi(y)) < \delta_1$. Since φ is a uniformly continuous mapping from (Y,ρ) to (X,d), we deduce that there exists $\delta > 0$ such that $d(\varphi(s),\varphi(t)) < \delta_1$, when $s, t \in Y$ with $\rho(s,t) < \delta$. Suppose that $x, y \in Y$ with $0 < \rho(x,y) < \delta$. Then $d(\varphi(x),\varphi(y)) < \delta_1$. If $\varphi(x) = \varphi(y)$, then $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0 < \varepsilon$. If $0 < d(\varphi(x),\varphi(y)) < \delta_1$, then by the argument above, we have $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \varepsilon$. Therefore, φ is supercontractive from (Y,ρ) to (X,d).

We now assume that φ is supercontractive. Let $\varepsilon > 0$ be given. Then there exists $\delta_0 > 0$ such that $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \varepsilon$ when $x, y \in Y$ with $0 < \rho(x,y) < \delta_0$. Take $\delta = \varepsilon\delta_0$ and assume that $0 < d(\varphi(x),\varphi(y)) < \delta$ when $x, y \in Y$. If $0 < \rho(x,y) < \delta_0$, then $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < \varepsilon$. If $\rho(x,y) \geq \delta_0$, then $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} \leq \frac{d(\varphi(x),\varphi(y))}{\delta_0} < \frac{\delta}{\delta_0} = \varepsilon$. Therefore, $\lim \frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0. Hence, the proof is complete. $\square$

Lemma 4.8. *Let (X,d) and (Y,ρ) be metric spaces, let φ be a Lipschitz mapping from (Y,ρ) to (X,d) and let $u \in \mathrm{Lip}(Y,\rho)$ with $|u(y)| = 1$ for all $y \in Y$. Then $C_\varphi : \mathrm{Lip}(X,d) \to \mathrm{Lip}(Y,\rho)$ is compact if and only if $uC_\varphi : \mathrm{Lip}(X,d) \to \mathrm{Lip}(Y,\rho)$ is compact.*

Proof. Since $u \in \mathrm{Lip}(Y,\rho)$ and $|u(y)| = 1$ for all $y \in Y$, we deduce that $\frac{1}{u} \in \mathrm{Lip}(Y,\rho)$ and $|\frac{1}{u}(y)| = 1$ for all $y \in Y$. It is easy to see that if $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathrm{Lip}(X,d)$, then $\{f_n \circ \varphi\}_{n=1}^{\infty}$ converges in $(\mathrm{Lip}(Y,\rho), \|\cdot\|_{\mathrm{Lip}(Y,\rho)})$ if and only if $\{u\cdot(f_n\circ\varphi)\}_{n=1}^{\infty}$ converges in $(\mathrm{Lip}(Y,\rho), \|\cdot\|_{\mathrm{Lip}(Y,\rho)})$. This implies that C_φ is compact if and only if uC_φ is compact. $\square$

Theorem 4.9. *Let (X,d) be a metric space, let φ be a supercontractive Lipschitz mapping from (X,d) to (X,d) such that $\varphi(X)$ is not totally bounded in (X,d) and let $u \in \mathrm{Lip}(X,d)$ with $|u(x)| = 1$ for all $x \in X$. Then $T = uC_\varphi$ is a weighted composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(X,d)$ which is not compact.*

Proof. By Lemma 4.7, $\lim \frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0. This implies that $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0 since $|u(x)| = 1$ for all $x \in X$. Since $\varphi(X)$ is not totally bounded in (X,d), C_φ is not compact operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(X,d)$ by [6, Theorem 1.1]. Hence, $T = uC_\varphi$ is not compact by Lemma 4.8. $\square$

In the following examples we give a metric space (X,d), a supercontractive Lipschitz mapping φ from (X,d) to (X,d) and a complex-valued function u on X satisfying the conditions of Theorem 4.9.

Example 4.10. Let $\{z_n\}_{n\in\mathbb{Z}}$ be an unbounded sequence in $\mathbb{C}\setminus\{0\}$ that $|z_m - z_n| \geq 1$ for all $m,n \in \mathbb{Z}$ with $m \neq n$. Let $X = \{z_n : n \in \mathbb{Z}\}$ and d be the Euclidean metric on X. Define the map $\varphi : X \to X$ by

$$\varphi(z) = z \qquad (z \in X).$$

It is easy to see that φ is a supercontractive Lipschitz mapping from (X,d) to (X,d) and $\varphi(X)$ is not totally bounded in (X,d). Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$ and let $\lambda \in \mathbb{T}$. Define the function $u_\lambda : X \to \mathbb{C}$ by

$$u_\lambda(z) = \frac{\lambda z}{|z|} \qquad (z \in X).$$

Then for each $z,w \in X$ with $z \neq w$, we have

$$\frac{|u_\lambda(z) - u_\lambda(w)|}{d(z,w)} \leq |\frac{\lambda z}{|z|} - \frac{\lambda w}{|w|}| \leq 2.$$

Hence, u_λ is a Lipschitz function on (X,d). Moreover, $|u_\lambda(z)| = 1$ for all $z \in X$. It is clear that $T_\lambda = u_\lambda C_\varphi$ is a weighted composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(X,d)$.

Example 4.11. Let $X = \{\frac{1}{n} : n \in \mathbb{Z}\setminus\{0\}\}$ and d be the discrete metric on X. Define the map $\varphi : X \to X$ by

$$\varphi(x) = x \qquad (x \in X).$$

Then φ is a supercontractive Lipschitz mapping from (X,d) to (X,d) and $\varphi(X)$ is not totally bounded in (X,d). Let $\lambda\in\mathbb{T}$ and define the function $u_\lambda : X\to\mathbb{C}$ by

$$u_\lambda(x)=\lambda\,\mathrm{sgn}(x)\qquad (x\in X).$$

Then u_λ is a complex-valued Lipschitz function on (X,d) and $|u_\lambda(x)|=1$ for all $x\in X$. It is clear that $T_\lambda=u_\lambda C_\varphi$ is a weighted composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(X,d)$.

As a consequence of Theorem 4.6, we obtain the following result which is generalization of [4, Theorem 4.5(ii)].

Theorem 4.12. *Let (X,d) and (Y,ρ) be metric spaces, let u be a complex-valued function on Y, let φ be a map from Y to X, let $\varphi(\mathrm{coz}(u))$ be totally bounded in (X,d) and let $T=uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$. If φ is supercontractive on $\mathrm{coz}(u)$, then T is compact.*

Proof. Assume that φ is supercontractive on $\mathrm{coz}(u)$. Let $\varepsilon>0$ be given. Then there exists a positive number δ_0 with $\delta_0<1$ such that $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}<\frac{\varepsilon}{1+\|u\|_{\mathrm{Lip}(Y,\rho)}}$ when $x,y\in\mathrm{coz}(u)$ with $0<\rho(x,y)<\delta_0$. Take $\delta=\frac{\varepsilon\delta_0}{1+\|u\|_{\mathrm{Lip}(Y,\rho)}}$ and assume that $x,y\in Y$ with $0<d(\varphi(x),\varphi(y))<\delta$. If $x,y\in\mathrm{coz}(u)$ with $0<\rho(x,y)<\delta_0$, then

$$\begin{aligned}|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}&\le\|u\|_Y\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}\\&<\|u\|_Y\frac{\varepsilon}{1+\|u\|_{\mathrm{Lip}(Y,\rho)}}\\&<\varepsilon.\end{aligned}$$

If $x,y\in\mathrm{coz}(u)$ with $\rho(x,y)\ge\delta_0$, then

$$\begin{aligned}|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)}&\le\|u\|_Y\frac{d(\varphi(x),\varphi(y))}{\delta_0}\\&<\frac{\|u\|_Y\varepsilon\delta_0}{\delta_0(1+\|u\|_{\mathrm{Lip}(Y,\rho)})}\\&<\varepsilon.\end{aligned}$$

If $x \in \mathrm{coz}(u)$ and $u(y) = 0$, then

$$\begin{aligned}|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} &= \frac{|u(x)-u(y)|}{\rho(x,y)}d(\varphi(x),\varphi(y))\\ &< p_{(Y,\rho)}(u)\delta\\ &= \frac{p_{(Y,\rho)}(u)\varepsilon\delta_0}{1+\|u\|_{\mathrm{Lip}(Y,\rho)}}\\ &< \varepsilon.\end{aligned}$$

If $u(x) = 0$ and $y \in \mathrm{coz}(u)$, then

$$|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0 < \varepsilon.$$

Hence, $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0. Therefore, T is compact by Theorem 4.6. □

The following example shows that the converse of Theorem 4.12, is not valid and Theorem 4.6 is an extension of [2, Theorem 11] for $\alpha = 1$.

Example 4.13. Let $X = (-2, 2)$ and let d be the Euclidean metric on X. Define the function $u : X \to \mathbb{C}$ by

$$u(x) = x \qquad (x \in X).$$

Then $u \in \mathrm{Lip}(X, d)$. Define the map $\varphi : X \to X$ by

$$\varphi(x) = \mathrm{sgn}(x) \qquad (x \in X).$$

It is easy to see that $C(u, \varphi) < 2$. Hence, $T = uC_\varphi$ is a weighted composition operator on $\mathrm{Lip}(X, d)$, by Theorem 2.4. Moreover, it is clear that $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0. Since $\varphi(\mathrm{coz}(u)) = \{-1, 1\}$, we deduce that $\varphi(\mathrm{coz}(u))$ is a totally bounded set in (X, d). Therefore, T is compact by Theorem 4.6.

On the other hand,

$$\frac{d(\varphi(\frac{1}{n}),\varphi(\frac{-1}{n}))}{d(\frac{1}{n},\frac{-1}{n})} = \frac{2}{\frac{2}{n}} = n,$$

for all $n \in \mathbb{N}$ with $n \geq 2$. Hence, φ is not supercontractive on $\mathrm{coz}(u)$.

We now generalize [4, Theorem 4.5(i)] as the following.

Theorem 4.14. *Let (X, d) and (Y, ρ) be metric spaces, let u be a complex-valued function on Y, let φ be a map from Y to X, let $\varphi(\mathrm{coz}(u))$ be totally bounded in (X, d) and let $T = uC_\varphi$ be a weighted composition operator from $\mathrm{Lip}(X, d)$ to $\mathrm{Lip}(Y, \rho)$. If T is compact, then φ is supercontractive on compact subsets of $\mathrm{coz}(u)$.*

Proof. Suppose that T is compact. By Theorem 4.6, $\lim u(x)\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0$ when $d(\varphi(x),\varphi(y))$ tends to 0. Let K be a nonempty compact subset of $\operatorname{coz}(u)$. Let $\varepsilon > 0$ be given. Take $C = \inf\{|u(y)| : y \in K\}$. The continuity of u on $\operatorname{coz}(u)$ implies that $C > 0$. Then there exists $\delta_1 > 0$ such that

$$|u(x)|\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} < C\varepsilon, \tag{4.16}$$

when $x, y \in Y$ with $0 < d(\varphi(x),\varphi(y)) < \delta_1$. By Corollary 2.6, φ is a Lipschitz mapping from (K,ρ) to (X,d) and so φ is a uniformly continuous mapping from (K,ρ) to (X,d). This implies that there exists $\delta > 0$ such that $d(\varphi(s),\varphi(t)) < \delta_1$ when $s, t \in K$ with $\rho(s,t) < \delta$. Suppose that $x, y \in K$ with $0 < \rho(x,y) < \delta$. Then $d(\varphi(x),\varphi(y)) < \delta_1$. If $\varphi(x) = \varphi(y)$, then $\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} = 0 < \varepsilon$. If $0 < d(\varphi(x),\varphi(y)) < \delta_1$, then we have

$$\frac{d(\varphi(x),\varphi(y))}{\rho(x,y)} \leq \frac{|u(x)|d(\varphi(x),\varphi(y))}{C\rho(x,y)} < \frac{C\varepsilon}{C} = \varepsilon,$$

by (4.16). Therefore, φ is supercontractive from (K,ρ) to (X,d) and the proof is complete. □

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