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Spherical Indicatrices of Admissible Curves in pseudo-Galilean Space G_3^1

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Abstract. In this study, position vectors of admissible curve in the natural representation are researched and spherical indicatrices for tangent, normal and binormal vectors of a general helix in the pseudo-Galilean space are investigated. Moreover some examples are given.

Keywords: Pseudo-Galilean space, spherical indicatrices.

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1. Introduction

In the local differential geometry, the curves can be considered a geometric set of points. For identfy the behavior of the curve, position vectors of this curves can be investigated.

In general case, to solve the problem of parametric representation of the position vector for an arbitrary space curve with respect to the intrinsic equations is not easy in the Euclidean space E^3 [13, 16]. But, in some special case this problem can be solved . Recently Ali [3, 4] adapted fundamental existence and uniqueness theorem for space curves in the Euclidean space E_1^3 and to solve the problem in the case of a general helix and a slant helix he constructed a vector differential equation.

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Helices have important properties and application in science and nature. Helices can be seen in DNA double, nano-springs, bacterial flagella, carbon nano-tubes etc.[17, 1, 10]. In Euclidean 3-space a helix can be defined a geometric curve with non-zero constant curvature κ and torsion τ. In [11, 12] spherical images of tangent indicatrix and binormal indicatrix of a slant helix and characterizations of slant helices in Euclidean 3-space are given. Besides the Euclidean geometry, many researchers investigated helices in non-Euclidean geometry. For example in [14, 15, 2] characterizations of helices and general helices are expressed in Galilean and pseudo-Galilean space. In these space the generalization of helices are researched in [18]. Also, in [5] position vectors of curves in Galilean space and in [9] spherical indicatrices of special curves in the Galilean space are studied.

In this study position vectors of admissible curve in the natural representation and spherical indicatrices for tangent, normal and binormal vectors of a general helix in the pseudo-Galilean space are researched. Moreover some examples are given.

2. Preliminaries

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries of projective signature (0, 0, +,-). As in [6], pseudo-Galilean inner product can be written as

$$\langle v_1, v_2\rangle = \begin{cases} x_1x_2 & , if\ x_1 \neq 0 \vee x_2 \neq 0 \\ y_1y_2 - z_1z_2 & , if\ x_1 = 0 \wedge x_2 = 0 \end{cases} \tag{2.1}$$

where $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$. The pseudo-Galilean norm of the vector $v = (x, y, z)$ is given by

$$\|v\| = \begin{cases} x & , if\ x \neq 0 \\ \sqrt{|y^2 - z^2|} & , if\ x = 0 \end{cases} \tag{2.2}$$

and the pseudo-Galilean cross product can be given as follows

$$v_1 \wedge v_2 = \begin{cases} \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} & , if\ x_1 \neq 0 \vee x_2 \neq 0 \\ \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} & , if\ x_1 = 0 \wedge x_2 = 0 \end{cases} \tag{2.3}$$

where $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$. In pseudo-Galilean space a curve is defined by $\alpha : I \to G_3^1$

$$\alpha(t) = (x(t), y(t), z(t)) \tag{2.4}$$

where $I \subseteq \mathrm{R}$ and $x(t), y(t), z(t) \in C^3$. A curve α given by (2.4) is admissible if $x'(t) \neq 0$ [6].

The curves in pseudo-Galilean space are characterized as follows [7]:

An admissible curve in G_3^1 can be parametrized by arc length $t = s$, given in coordinate form

$$\alpha(s) = (s, y(s), z(s)) \tag{2.5}$$

For an admissible curve $\alpha : I \subseteq \mathrm{R} \to G_3^1$, the curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\begin{aligned} \kappa(x) &= \sqrt{\left|y''^2 - z''^2\right|} && (2.6) \\ \tau(s) &= \frac{1}{\kappa^2(s)} \det\left(\alpha'(s), \alpha''(s), \alpha'''(s)\right) && (2.7) \end{aligned}$$

The associated trihedron is given by

$$\begin{aligned} t(s) &= \left(1, y'(s), z'(s)\right) \\ n(s) &= \frac{1}{\kappa(s)}\left(0, y''(s), z''(s)\right) \\ b(s) &= \frac{1}{\kappa(s)}\left(0, z''(s), y''(s)\right) \end{aligned} \tag{2.8}$$

The vectors $t(s)$,$n(s)$ and $b(s)$ are called the vectors of tangent, principal normal and binormal line of α, respectively. The curve α given by (2.5) is timelike if $n(s)$ is spacelike vector. For derivatives of tangent vector $t(s)$, principal normal vector $n(s)$and binormal vector $b(s)$, respectively, the following Frenet formulas hold

$$\begin{aligned} t'(s) &= \kappa(s)\, n(s) \\ n'(s) &= \tau(s)\, b(s) \\ b'(s) &= \tau(s)\, n(s) \end{aligned} \tag{2.9}$$

3. Position vectors of admissible curve with respect to standart frame of G_3^1

In this section, we obtain the position vectors of admissible curve with respect to standart frame in G_3^1.

Theorem 3.1. *Let $\alpha(s) = (s, y(s), z(s))$ be an admissible curve with curvature $\kappa(s)$ and torsion $\tau(s)$ in the pseudo-Galilean space G_3^1.*

i) *if α is an admissible curve with spacelike normal, then the position vector of α is computed from the natural representation form*

$$\alpha(s) = (s, y(s), z(s)) \tag{3.1}$$

where $y(s)=\int\left(\int\kappa(s)\cosh\left(\int\tau(s)\,ds\right)ds\right)ds$ *and*
$z(s)=\int\left(\int\kappa(s)\sinh\left(\int\tau(s)\,ds\right)ds\right)ds.$

ii) *if α is an admissible curve with timelike normal, then the position vector of α is computed from the natural representation form*

$$\alpha(s)=(s,y(s),z(s)) \tag{3.2}$$

where $y(s)=\int\left(\int\kappa(s)\sinh\left(\int\tau(s)\,ds\right)ds\right)ds$ *and*
$z(s)=\int\left(\int\kappa(s)\cosh\left(\int\tau(s)\,ds\right)ds\right)ds.$

Proof. if α is an admissible curve in G_3^1, then using equations (2.9) we have

$$b(s)=\frac{1}{\tau}n'(s). \tag{3.3}$$

Considering equation (3.3) in the third equation of (2.9) we get the following equation,

$$\left(\frac{1}{\tau}n'(s)\right)'-\tau(s)\,n(s)=0. \tag{3.4}$$

The equation (3.4) can be written as

$$\frac{d^2n}{dt^2}-n=0, \tag{3.5}$$

where t is $t=\int\tau(s)\,ds$.

i) Let α be an admissible curve with spacelike normal. The principal normal vector can be written

$$n=(0,\cosh\theta(t),\sinh\theta(t)). \tag{3.6}$$

Substiuting the components of the vector n in the equation (3.5) we get

$$\left(\dot{\theta}^2(t)-1\right)\cosh\theta(t)+\ddot{\theta}(t)\sinh\theta(t) = 0, \tag{3.7}$$

$$\left(\dot{\theta}^2(t)-1\right)\sinh\theta(t)+\ddot{\theta}(t)\cosh\theta(t) = 0. \tag{3.8}$$

Using the above equation we have

$$\dot{\theta}(t)=\pm1,\ddot{\theta}(t)=0,$$

which lead to $\theta(t)=\pm t=\pm\int\tau(s)\,ds$. If the positive sign for $\theta(t)$ is taken, then the principal normal vector becomes

$$n(s)=\left(0,\cosh\left(\int\tau(s)\,ds\right),\sinh\left(\int\tau(s)\,ds\right)\right) \tag{3.9}$$

Multiplying the above equation by $\kappa(s)$ and integrating it according to s we obtain

$$t(s)=\int \kappa(s)\left(0,\cosh\left(\int \tau(s)\,ds\right),\sinh\left(\int \tau(s)\,ds\right)\right)ds+c$$

where c is a constant vector. Since the first component of tangent vector equal to one, so we can take $c=(1,0,0)$ and

$$t(s)=\left(1,\int \kappa(s)\cosh\left(\int \tau(s)\,ds\right)ds,\int \kappa(s)\sinh\left(\int \tau(s)\,ds\right)ds\right). \tag{3.10}$$

Integrating equation (3.10) according to s we have

$$\alpha(s)=\int\left(1,\int \kappa(s)\cosh\left(\int \tau(s)\,ds\right)ds,\int \kappa(s)\sinh\left(\int \tau(s)\,ds\right)ds\right) \tag{3.11}$$

which leads to the equation (3.1).

ii) In an analogous way, one can obtain equation (3.2). □

4. Spherical Indicatrices of a General and Circular Helix in G_3^1

Definition 4.1. Let α be a unit speed regular curve in Euclidean 3-space with Frenet vectors t, n and b. The unit tangent vectors along the curve α generate a curve α_t on the sphere of radius 1 about the origin. The curve α_t is called the spherical indicatrix of the curve α. If $\alpha=\alpha(s)$ is a natural representation of α, then $\alpha_t(s_t)=t(s)$ will be a representation of α_t. Similarly one considers the principal normal indicatrix $\alpha_n(s_n)=n(s)$ and binormal indicatrix $\alpha_b(s_b)=b(s)$ [8].

For an admissible curve $\alpha=\alpha(s)$ with curvature $\kappa(s)$ and torsion $\tau(s)$ in the pseudo-Galilean space G_3^1 its position vector in the natural representation is given in the equations (3.1) and (3.2).

i) if α is an admissible curve with spacelike normal, then the position vector of α is computed from the natural representation form

$$\alpha(s)=(s,y(s),z(s)) \tag{4.1}$$

where $y(s)=\int\left(\int \kappa(s)\cosh\left(\int \tau(s)\,ds\right)ds\right)ds$ and
$z(s)=\int\left(\int \kappa(s)\sinh\left(\int \tau(s)\,ds\right)ds\right)ds$.
Suppose that the admissible curve α is a general helix that is $\tau(s)=m\kappa(s)$, m is a constant then the position vector given above becomes

$$\alpha(s)=(s,y(s),z(s)) \tag{4.2}$$

where $y(s)=\int\left(\int\kappa(s)\cosh\left(m\int\kappa(s)\,ds\right)ds\right)ds$ and $z(s)=\int\left(\int\kappa(s)\sinh\left(m\int\kappa(s)\,ds\right)ds\right)ds$. If we differentiate equation (4.1) respect to s we obtain

$$t(s)=\left(1,\int\kappa(s)\cosh\left(m\int\kappa(s)\,ds\right)ds,\int\kappa(s)\sinh\left(m\int\kappa(s)\,ds\right)ds\right)$$

Now, let

$$\Gamma_t=\alpha_t(s_t)=t(s)$$

be the tangent spherical indicatrix of α with the curvature $\kappa_t(s_t)$ and torsion $\tau_t(s_t)$, then it can be written

$$\alpha_t(s_t)=(1,y(s),z(s)) \tag{4.3}$$

where $y(s)=\int\kappa(s)\cosh\left(m\int\kappa(s)\,ds\right)ds$ and $z(s)=\int\kappa(s)\sinh\left(m\int\kappa(s)\,ds\right)ds$. By differentiating equation (4.3) respect to s we obtain

$$\frac{d\alpha_t}{ds_t}\frac{ds_t}{ds}=\left(0,\kappa(s)\cosh\left(m\int\kappa(s)\,ds\right),\kappa(s)\sinh\left(m\int\kappa(s)\,ds\right)\right), \tag{4.4}$$

then

$$\frac{ds_t}{ds}=\kappa(s), \tag{4.5}$$

and

$$t_t(s_t)=\left(0,\cosh\left(m\int\kappa(s)\,ds\right),\sinh\left(m\int\kappa(s)\,ds\right)\right). \tag{4.6}$$

Also from equation (4.6) we have

$$\frac{dt_t}{ds_t}\frac{ds_t}{ds}=\left(0,m\kappa(s)\sinh\left(m\int\kappa(s)\,ds\right),m\kappa(s)\cosh\left(m\int\kappa(s)\,ds\right)\right).$$

If we use Frenet equations defined by (2.9)

$$\kappa_t(s_t)\,n_t(s_t)=\left(0,m\sinh\left(m\int\kappa(s)\,ds\right),m\cosh\left(m\int\kappa(s)\,ds\right)\right)$$

Then it follows that

$$\begin{aligned}\kappa_t(s_t)&=m\\ n_t(s_t)&=\left(0,\sinh\left(m\int\kappa(s)\,ds\right),\cosh\left(m\int\kappa(s)\,ds\right)\right).\end{aligned}$$

Since $b_t(s_t)=t_t(s_t)\times n_t(s_t)$ we have

$$\begin{aligned}b_t(s_t)&=(1,0,0),\\ \tau_t(s_t)&=0.\end{aligned}$$

ii) if α is an admissible curve with timelike normal, then the position vector of α is computed from the natural representation form

$$\alpha(s) = (s, y(s), z(s)) \tag{4.7}$$

where $y(s) = \int \left(\int \kappa(s) \sinh \left(\int \tau(s)\, ds \right) ds \right) ds$ and
$z(s) = \int \left(\int \kappa(s) \cosh \left(\int \tau(s)\, ds \right) ds \right) ds$. In analogous way, one can obtain similarly equations for above form of α.

Theorem 4.2. *Suppose that the admissible curve α is a general helix in the pseudo-Galilean space G_3^1 with $\kappa(s) \neq 0$. The curvatures of the tangent spherical curve of α satisfy the following equalities*

$$\langle t_t, \alpha_t \rangle = 0, \tag{4.8}$$

$$\langle n_t, \alpha_t \rangle = \frac{-1}{\kappa_t}, \tag{4.9}$$

$$\langle b_t, \alpha_t \rangle = \frac{\kappa_t'}{\kappa_t^2 \kappa \tau_t}. \tag{4.10}$$

Proof. By assumption we have

$$\langle \alpha_t, \alpha_t \rangle = \pm r^2,$$

where r is the radius of the pseudo-Galilean sphere. If we differentiate above equation with respect to s we obtain

$$\left\langle \frac{d\alpha_t}{ds_t} \frac{ds_t}{ds}, \alpha_t \right\rangle = 0$$

where s_t is the arc length of the tangent spherical curve in G_3^1. Since

$$\frac{ds_t}{ds} = \kappa(s)$$

then

$$\langle t_t, \alpha_t \rangle = 0.$$

Also,

$$\left\langle \frac{dt_t}{ds_t} \frac{ds_t}{ds}, \alpha_t \right\rangle + \left\langle t_t, \frac{d\alpha_t}{ds_t} \frac{ds_t}{ds} \right\rangle = 0.$$

Because of $\langle t_t, t_t \rangle = 1$ and κ_t is the curvature of the tangent spherical curve α_t, we get

$$\langle n_t, \alpha_t \rangle = \frac{-1}{\kappa_t}.$$

By a new differentiation of above equation gives

$$\left\langle \frac{dn_t}{ds_t} \frac{ds_t}{ds}, \alpha_t \right\rangle + \left\langle n_t, \frac{d\alpha_t}{ds_t} \frac{ds_t}{ds} \right\rangle = \frac{\kappa_t'}{\kappa_t^2}.$$

Because of $\langle n_t, t_t\rangle = 0$ and $\dfrac{dn_t}{ds_t} = \tau_t b_t$ we have

$$\langle b_t, \alpha_t\rangle = \frac{\kappa_t^{'}}{\kappa_t^2 \kappa \tau_t}.$$

□

Suppose that the admissible curve α is a circular helix and we denote it by $\alpha^* = \alpha^*(s)$ with non-zero constant curvature $\kappa^*(s^*)$ and torsion $\tau^*(s^*)$. The position vector of α in equation (3.1) becomes for $\alpha^* = \alpha^*(s)$ as follows

$$\alpha^*(s^*) = \left(s*, \frac{\kappa^*}{\tau^{*2}} \cosh(\tau^* s^*), \frac{\kappa^*}{\tau^{*2}} \sinh(\tau^* s^*)\right) \tag{4.11}$$

if we differentiate equation (4.11) with respect to s^* we obtain the representation of the tangent spherical image of α^* in the form

$$\alpha^*_{t^*}(s^*_{t^*}) = t^*(s^*) = \left(1, \frac{\kappa^*}{\tau^*} \sinh(\tau^* s^*), \frac{\kappa^*}{\tau^*} \cosh(\tau^* s^*)\right). \tag{4.12}$$

By differentiating equation (4.12) with respect to s^*

$$\frac{d\alpha^*_{t^*}(s^*_{t^*})}{ds^*_{t^*}} \frac{ds^*_{t^*}}{ds^*} = (0, \kappa^* \cosh(\tau^* s^*), \kappa^* \sinh(\tau^* s^*)) \tag{4.13}$$

From equation (4.13)

$$\begin{aligned} \frac{ds^*_{t^*}}{ds^*} &= \kappa^*(s^*), \\ t^*_{t^*}(s^*_{t^*}) &= (0, \cosh(\tau^* s^*), \sinh(\tau^* s^*)). \end{aligned}$$

Then, we get

$$\frac{dt^*_{t^*}}{ds^*_{t^*}} \frac{ds^*_{t^*}}{ds^*} = (0, \tau^* \sinh(\tau^* s^*), \tau^* \cosh(\tau^* s^*)),$$

it follows that

$$\kappa^*_{t^*}(s^*_{t^*})\, n^*_{t^*}(s^*_{t^*}) = \frac{\tau^*}{\kappa^*}(0, \sinh(\tau^* s^*), \cosh(\tau^* s^*)).$$

So we obtain

$$\begin{aligned} \kappa^*_{t^*} &= \frac{\tau^*}{\kappa^*} \\ n^*_{t^*}(s^*_{t^*}) &= (0, \sinh(\tau^* s^*), \cosh(\tau^* s^*)) \end{aligned}$$

The binormal vector is given by

$$b^*_{t^*} = (1, 0, 0),$$

it yields

$$\tau^*_{t^*} = 0.$$

In analogous way the position vector of α in equation (3.2) can obtain for $\alpha^* = \alpha^*(s)$.

Proposition 4.3. The pseudo-Galilean spherical images of a general helix (or circular helix) in the three-dimensional pseudo-Galilean space are circles on the unit pseudo-Galilean sphere.

Remark 4.4. In similar way we can calculate the other spherical images (the principal normal and the binormal image).

Example 4.3. Let $\alpha(u)$ be a general helix in three-dimensional pseudo-Galilean space G_3^1 with $\kappa = \dfrac{1}{2\sqrt{u}}$ and $\tau = \dfrac{1}{2\sqrt{u}}$ given by

$$\alpha(u) = \left(u, -2\left(\sinh\sqrt{u} - \sqrt{u}\cosh\sqrt{u}\right), -2\left(\cosh\sqrt{u} - \sqrt{u}\sinh\sqrt{u}\right)\right).$$

The tangent spherical image of α is defined by

$$\alpha_t = \left(1, \sinh\sqrt{u}, \cosh\sqrt{u}\right) \tag{4.14}$$

Then from equation (4.14) we can obtain the following vectors of α_t

$$\begin{aligned} t_t &= \left(0, \cosh\sqrt{u}, \sinh\sqrt{u}\right), \\ n_t &= \left(0, \sinh\sqrt{u}, \cosh\sqrt{u}\right), \\ b_t &= (1, 0, 0). \end{aligned}$$

Moreover the curvature and the torsion of α_t are given by

$$\begin{aligned} \kappa_t &= 1, \\ \tau_t &= 0. \end{aligned}$$

In an analogous way we can obtain the normal and binormal spherical images of α and their curvatures.

Example 4.4. For a given circular helix $\alpha^*(u^*)$ in G_3^1 with non-zero constant curvature and torsion $\kappa^* = 1$ and $\tau^* = 1$ respectively

$$\alpha^*(u^*) = (u^*, \cosh u^*, \sinh u^*).$$

The tangent spherical image of α^* is defined by

$$\alpha^*_{t^*} = (1, \sinh u^*, \cosh u^*).$$

Then simple calculations give us the following vectors

$$\begin{aligned} t^*_{t^*} &= (0, \cosh u^*, \sinh u^*), \\ n^*_{t^*} &= (0, \sinh u^*, \cosh u^*), \\ b^*_{t^*} &= (1, 0, 0). \end{aligned}$$

Moreover the curvature and the torsion of $\alpha^*_{t^*}$ are given by

$$\kappa^*_{t^*} = 1,$$

$$\tau^*_{t^*} = 0.$$

In an analogous way we can obtain the normal and binormal spherical images of α^* and their curvatures.

References

[1] A. A. Lucas, P. Lambin, Diffraction by DNA, carbon nanotubes and other helical nanostructures, *Rep. Prog. Phys.,* **68***(2005), 1181-1249.*

[2] A. O. Ogrenmis, M. Ergut and M. Bektas, On the helices in the Galilean space G_3, *Iranian J. of Science and Technology,* **31***(2007), 177-181.*

[3] A. T. Ali, M. Turgut, Position Vector of a timelike slant helix in Minkowski 3-Space, *J. Math. Anal. Appl.* **365***(2010), 559-569.*

[4] A. T. Ali, Determination of the Position Vector of General Helices From Intrinisic Equations in E_1^3 *Bull. Math. Anal. Appl.* **3***(2011), 198-205.*

[5] A. T. Ali, Position vectors of curves in the Galilean space G_3, *Matematicki Vesnik* **64***(2012), 198-205.*

[6] B. Divjak, Curves in Pseudo-Galilean Geometry, *Annales Univ. Sci. Budapest.* **41***(1998), 117-128.*

[7] B. Divjak and Z. Milin Spus, Special curves on ruled surfaces in Galilean and pseudo-Galilean spaces, *Acta Math. Hungar.* **98** *(2003), 203-215.*

[8] D. J. Struik, Lectures on Classical Differential Geometry, Dover, New-York, 1988.

[9] H. S. Abdel-Aziz, M.K. Saad, D.M. Farghal, Spherical indicatrices of special curves in the Galilean space G_3, *The Egyptian Mathematical Society, (2013), 1-10.*

[10] J. D. Watson, F.H.C. Crick, Generic implications of the structure of deoxyribonucleic acid, *Nature,* **171***(1953), 964-967.*

[11] L. Kula, and Y. Yayli, On slant helix and its spherical indicatrix, *Appl. Math. Comput.* **169** *(2005), 600-607.*

[12] L. Kula, N. Ekmekçi, Y.Yayli and K. İlarslan, Characterizations of slant helices in Euclidean 3-space, *Turk. J. Math.* **34***(2010), 261-273.*

[13] L. P. Eisenhart, A Treatside on the Differential Geometry of Curves and Surfaces, Ginn and Co., 1909.

[14] M. Bektas, On characterizations of general helices for ruled surfaces in the pseudo-Galilean space G_3^1, *J. Math. Kyoto Univ.* **44***(2004), 523-528.*

[15] M. Bektas, The characterizations of general helices in the 3-dimensional pseudo-Galilean space, *Soochow J. Math.* **31***(2005), 441-447.*

[16] M. M. Lipschitz, Schum's Outline of Theory and Problems of Differential Geometry, McGraw-Hill Book Company, New York, 1969.

[17] N. Chouaieb, A. Goriely, J.H. Maddocks, Helices, *PANS* **103***(2006), 9398-9403.*

[18] Z. Erjavec, On generalization of helices in the Galilean and the pseudo-Galilean space, *Journal of Mathematics Research,* **6***(2014), 39-50.*