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Orthogonality of Homogeneous geodesics on the tangent bundle

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Abstract. Let $\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K})$ be the associated bundle and $\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \mathrm{R}^m)$ be the tangent bundle of special examples of odd dimension solvable Lie groups equipped with left invariant Riemannian metric. In this paper we prove some conditions about the existence of homogeneous geodesic on the base space of $\tau_{G/K}$ and homogeneous (geodesic) vectors on the fiber space of ξ .

Keywords: Hadamard matrices, tangent bundle, homogeneous manifold, solvable Lie group, homogeneous (geodesic)vector.

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1. Introduction and preliminaries

Let G be a connected Lie group and K be a closed subgroup of G. The set of left cosets of K in G is denoted by G/K and can be given a unique differentiable structure ([7], vol.II, chap.2), and hence $M = G/K$ is called a homogeneous manifold. When a Lie group G acts transitively isometric on a Riemannian manifold M, we can identify M with the set G/K of left cosets of the isotropy group K of a point $x_0 \in M$. The point x_0 is called the origin of M. Let $\bigtriangledown$ be an affine connection on $M = G/K$ and let $\bigtriangledown$ be invariant under the natural action of $T : G \times M \longrightarrow M$. Then a geodesic $\gamma : I \longrightarrow M$ is called a *homogeneous geodesic* if, there

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exists a 1-parameter subgroup $t \longrightarrow \exp tX$, $t \in \mathbf{R}$, of G with $X \in \mathcal{G}= T_eG$ such that

$$\gamma(t) = T(\exp tX, x_0).$$

Where $\gamma(0) = x_0 \in M$, and $exp : \mathcal{G} \to G$ is the exponential map [8]. A vector $0 \neq X \in \mathcal{G}$ is called a *homogeneous vector* (or *geodesic vector*), if the curve $\gamma(t) = (\exp tX)(x_0)$ is a geodesic on $M = G/K$ [8]. The following result can be found in [8], proposition 1. *Any homogeneous Riemannian manifold G/K has the reductive decomposition of the form*

$$\mathcal{G} = \mathcal{M} + \mathcal{K}$$

where $\mathcal{M} \subset \mathcal{G}$ is a vector subspace, such that $Ad(K)(\mathcal{M}) \subset \mathcal{M}$. Let $M = G/K$ be a Riemannian manifold and $\mathcal{G} = \mathcal{M} + \mathcal{K}$, its reductive decomposition. Then the natural map $\phi : G \longrightarrow G/K = M$ induces a linear epimorphism $(d\phi)_e : T_eG \longrightarrow T_{x_0}M$, and the vector space $\mathcal{M}$ can be identified with $T_{x_0}M$.

Proposition 1.1.[10] *A finite family $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ of homogeneous geodesics through $x_o \in M$ is orthogonal (respectively, linearly independent) if the $\mathcal{M}$-component of the corresponding homogeneous vectors are orthogonal (respectively, linearly independent).*

Let $\Im = (G, \pi, G/K, K)$ be a principal homogeneous fiber bundle with group structure K, and let G be a connected Lie group and K a closed subgroup of G, (see [1], definition 2.2). We take the Lie algebras $\mathcal{G}$ and $\mathcal{K}$ of G and K respectively, in [1] and [2], we proved some relations between the homogeneous vector in the fiber space of the associated bundle, $\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K})$ and the homogeneous geodesic in the base space of a principal homogeneous bundle $\Im = (G, \pi, G/K, K)$.In [3], we consider the homogeneous bundle $\Im = (G, \pi, G/K, K)$ and the tangent bundle $\tau_{G/K}$ of $M = G/K$, and give some results about the existence of homogeneous vectors on the fiber space of $\tau_{G/K}$, for both cases of G semisimple and weakly semisimple Lie group. Homogeneous geodesics on homogeneous Finsler manifolds have been studied by many authors. For results on homogeneous geodesics in homogeneous Finsler manifolds and some properties on symmetric Finsler spaces we refer for example to [6, 11, 12] Now, we investigate the existence of mutually orthogonal linearly independent homogeneous geodesics in the base space of the tangent bundle τ_G of homogeneous Riemannian manifold G given in theorem 2.1.

2. Main result

Let $\Im = (G, \pi, G/K, K)$ be a principal homogeneous bundle, with the associated bundle $\xi = (G \times_K \mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K})$. Let G be the matrix

group of all matrices of the form

$$\begin{pmatrix} e^{z_0} & 0 & \dots & 0 & x_0 \\ 0 & e^{z_1} & \dots & 0 & x_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{z_n} & x_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

where, $(x_0, x_1, \cdots, x_n, z_1 \cdots, z_n) \in \mathbf{R}^{2n+1}$. The Lie group G is unimodular and solvable (see [9], pp.134-136), with the left invariant Riemannian metric

$$g = \sum_{i=0}^{n} e^{-2z_i} dx_i^2 + \lambda^2 \sum_{k,j=0}^{n} dz_k dz_j.$$

Where $\lambda \neq 0$ is a constant. Then G is a homogeneous Riemannian manifold with the origin at $(0, 0, \cdots, 0)$ ([9], p.134). Let $\mathcal{G} = \mathcal{M} + \mathcal{K}$ be the reductive decomposition of $\mathcal{G}$, then $\mathcal{K}$= 0, and hence $\mathcal{G} = \mathcal{M}$. In [3], we prove that $\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K})$, is the associated bundle of $\Im = (G, \pi, G/K, K)$. Then we can take $\xi = (G \times \mathcal{M}, \rho_\xi, G, \mathcal{M})$, be the associated bundle of $\Im = (G, \pi, G/K, K)$,(see[5]).
In [4], we let G be a 3-dimensional solvable Lie group, given in [9], pp.134, and prove some results about the existence of homogeneous vectors on the fiber space of $\tau_{G/K}$ and ξ.
In [5], we extend theorems 5.6 and 5.7 in [4], and give the following theorem, for the odd dimensional solvable Lie group.
Theorem 2.1.[5]. *Let $\Im = (G, \pi, G/K, K)$, be a principal homogeneous bundle and $\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K})$, be the associated bundle of $\Im = (G, \pi, G/K, K)$. If G is the matrix group of all matrices of the form*

$$\begin{pmatrix} e^{z_0} & 0 & \dots & 0 & x_0 \\ 0 & e^{z_1} & \dots & 0 & x_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{z_n} & x_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

where $(x_0, x_1, \cdots, x_n, z_1 \cdots, z_n) \in \boldsymbol{R}^{2n+1}$ and $z_0 = -(z_1 + z_2 + \cdots + z_n)$. Then a vector V in the fiber space of ξ is a homogenous (geodesic) if and only if its components

$$(x_0, x_1, \cdots, x_n, z_1, \cdots, z_n)$$

satisfy the following conditions

$$x_0(z_1 + z_2 + \cdots + z_n) = 0 \qquad x_1 z_1 = 0, \cdots, x_n z_1 = 0$$

$$x_0^2 - x_1^2 = 0, \cdots, x_0^2 - x_n^2 = 0.$$

In the proof of the Theorem 5.3 in [3], we give a strong isomorphism between the tangent bundle

$$\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \mathbf{R}^m)$$

and the associated bundle

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K}),$$

then under hypothesis of theorem 2.2 there is a strong isomorphism between,the associated bundle

$$\xi = (G \times \mathcal{M}, \rho_\xi, G, \mathcal{M})$$

and the tangent bundle

$$\tau_G = (T_G, \pi_G, G, \mathbf{R}^{2n+1})$$

so we have,

Corollary 2.2.[5] *With hypothesis of theorem 2.2, let*

$$\tau_G = (T_G, \pi_G, G, \mathbf{R}^{2n+1})$$

be the tangent bundle of the homogeneous Riemannian manifold G. Then a vector W in $\boldsymbol{R}^{2n+1}$ is a homogeneous vector (under isomorphism), if and only if its component $(x_0, x_1, \cdots, x_n, z_1 \cdots, z_n)$ satisfy the following conditions

$$x_0(z_1 + z_2 + \cdots + z_n) = 0 \qquad x_1 z_1 = 0, \cdots, x_n z_1 = 0$$

$$x_0^2 - x_1^2 = 0, \cdots, x_0^2 - x_n^2 = 0.$$

In [3], theorem 5.4. we give a subspace of $\mathcal{G}'$ such that all member of this subspace are homogeneous vectors, and by strong isomorphism between $\tau_{G/K}$ and ξ we can find a subspace of $\mathbf{R}^m$ (under isomorphism) such that all members of this subspace are homogeneous vectors, In the following theorem, we consider the tangent bundle

$$\tau_G = (T_G, \pi_G, G, \mathbf{R}^{2n+1})$$

of the homogeneous Riemannian manifold G in theorem 2.2, and give structure of all subspaces of $\mathbf{R}^{2n+1}$ such that all their members are homogeneous vectors.

Theorem 2.3.([5]). *Let*

$$\tau_G = (T_G, \pi_G, G, \mathbf{R}^{2n+1})$$

be the tangent bundle of the homogeneous Riemannian manifold G, (given in theorem 2.1). Then all homogeneous vectors are decomposed into an n-dimension vector subspace W in $\boldsymbol{R}^{2n+1}$ and 2^n, one-dimension vector subspace in $\boldsymbol{R}^{2n+1}$ generated by all vectors of the form $X_0 \pm X_1 \pm \cdots \pm X_n$.

By proposition 1.3 and Theorem 2.3 we have, the following result about

linearly independence of homogeneous geodesics on the base space of τ_G

Corollary 2.4.[5] *With hypothesis of theorem 2.2, the tangent bundle*

$$\tau_G = (T_G, \pi_G, G, \mathbf{R}^{2n+1})$$

admits $2n+1$ *linearly independent homogeneous geodesics through the origin* $\{e\}$ *of the base space of* τ_G.

Now, we investigate orthogonality of homogeneous vectors on the fiber space of tangent bundle,

$$\tau_G = (T_G, \pi_G, G, \mathbf{R}^{2n+1}).$$

In [3] we prove some conditions about existence and orthogonality of homogeneous vectors for both cases of G semisimple and weakly semisimple. For example in theorem 5.3 in [3], we prove that if G is a semisimple Lie group then there are m orthogonal homogeneous vectors on the fiber space of the tangent bundle,

$$\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \mathrm{R}^m)$$

In the follow, we want to get some conditions about linearly independent and orthogonality of homogeneous vectors on the fiber space and homogeneous geodesics on the base space of the tangent bundle of the homogeneous Riemannian manifold G (given in theorem 2.1). For this we need to considering to relations between orthogonality of homogeneous vectors and the Hadamard matrices.

Definition 2.5. A *Hadamard matrix of order* k is $k \times k$ square matrix whose entries are all equal to ± 1, and such that $A.A^t = kI_k$, where I_k is the unit matrix.

The condition $A.A^t = kI_k$, in definition 2.5, implies that the k rows or columns of a Hadamard matrix represent orthogonal k-tuples, with all entries equal to +1 or -1, we can use this fact for considering to structure of Hadamard matrices and orthogonality of homogeneous (geodesic) vectors.

Lemma 2.6. *Let* $\tau_G = (T_G, \pi_G, G, \boldsymbol{R}^{2n+1})$ *be the tangent bundle of the homogeneous Riemannian Lie group* G *of all matrices of the form*

$$\begin{pmatrix} e^{z_0} & 0 & \dots & 0 & x_0 \\ 0 & e^{z_1} & \dots & 0 & x_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{z_n} & x_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

where $(x_0, x_1, \cdots, x_n, z_1 \cdots, z_n) \in \boldsymbol{R}^{2n+1}$ *and* $z_0 = -(z_1 + z_2 + \cdots + z_n)$,

then;

(i) *If* $(n+1)$ *is odd, then there are not any two mutually orthogonal*

$(n+1)$-tuples with all entries equal to ± 1.
(ii)*If $(n+1)$ is even and not divisible by 4, then there are exactly two mutually orthogonal $(n+1)$-tuples with all entries equal to ± 1.*
Proof. Let $\tau_G = (T_G, \pi_G, G, \mathbf{R}^{2n+1})$ be the tangent bundle of the homogeneous Riemannian manifold G (given in theorem 2.2). By Corollary 2.3, and theorem 2.4 a vector w in $\mathbf{R}^{2n+1}$ is a homogeneous (geodesics) vector (under isomorphism), if and only if

$$A) \qquad w \in W = span(Z_1, Z_2, \cdots, Z_n)$$

$$B) \qquad w = \sum_{i=0}^{i=n} x_i X_i \quad and \quad x_0^2 - x_1^2 = 0, \cdots, x_0^2 - x_n^2 = 0.$$

As concerns homogeneous (geodesics) vectors of type (B), they are all generated by the vectors of the form $X_0 + \epsilon_1 X_1 + \cdots + \epsilon_n X_n$, where $\epsilon_i \in \{1, -1\}$. Therefore, the problem of finding mutually orthogonal geodesics vectors of type (B) is equivalent to the algebraic problem of finding $(n+1)$-tuples, with all entries equal to ± 1, which are mutually orthogonal with respect to the standard scaler product in $\mathbf{R}^{n+1}$.
Let $(n+1)$ be odd number and W_1 and W_2 be two $(n+1)$-tuples with all entries equal to ± 1. The scaler product of W_1 and W_2 is the sum of the products of their entries and all such products are equal to ± 1. By hypotheses, $(n+1)$ is odd, then sum of the products of their entries dose not vanish, so W_1 and W_2 can not be orthogonal, so we obtain (i). For the second statement of the lemma, we spouse that $(n+1) = 2m$, where m is odd, let V_1 and V_2 be two $(n+1)$-tuples with all entries equal to ± 1, such that $V_1 = (1, 1, \cdots, 1)$ and $V_2 = (-1, 1, \cdots - 1, 1)$, then V_1 and V_2 are orthogonal. Now, we spouse that V, W, Z, are three mutually orthogonal $(n+1)$-tuples with all entries equal to ± 1. Then, we compute the scaler product of V, W and Z by V. In this way, we can obtain three mutually orthogonal $(n+1)$-tuples vectors V', W', Z' such that all entries equal to ± 1. If we take $V' = (-1, -1, \cdots, -1)$, then by orthogonality of V' and W', W' has exactly m entries equal to -1 and exactly m entries equal to 1. We then multiply, component by component, and applying a fixed permutation of the all entries for mutually orthogonal $(n+1)$-tuples vectors V', W', Z', such that this applications will preserve the orthogonality of V', W', Z'. By this way, we can obtain $W' = (1, 1, \cdots, 1, -1, -1, \cdots, -1)$, but m is odd and the orthogonality of V', W', Z' is imposable, this gives a contradiction, and the proof of the lemma is complete. □
Before starting some additional results, we recall the fact that $A.A^t = kI_k$, in definition 2.5 implies that the k rows or columns of a Hadamard matrix represent orthogonal k-tuples, with all entries equal to $+1$ or

-1, for the case $n+1$ be divisible by 4, the problem related to algebraic problem of the existence of Hadamard matrices of order $n+1$. Therefore, we get at once the following proposition.

Proposition 2.7. *With hypothesis of lemma 2.7, let $n+1$ be divisible by 4, then $\boldsymbol{R}^{2n+1}$ admits $n+1$ mutually orthogonal $(n+1)$-tuples vectors with all entries equal to ± 1, if and only if, there exists a Hadamard matrices of order $n+1$.*

Now, we can prove the following theorem about the linearly independent and the maximum number of the orthogonal homogeneous (geodesic) vectors on the fiber space of

$$\tau_G = (T_G, \pi_G, G, \mathbf{R}^{2n+1}).$$

Theorem 2.8. *Let $\tau_G = (T_G, \pi_G, G, \boldsymbol{R}^{2n+1})$ be the tangent bundle of the homogeneous Riemannian Lie group G, (given in theorem 2.1 and lemma 2.6) then;*

(i) *There are $2n+1$ linearly independent homogeneous (geodesics) vectors in the fiber space of through the τ_G.*

(ii) *The maximum number of the orthogonal homogeneous (geodesic) vectors on the fiber space of τ_G is $n+1$, in the case that $n+1$ is odd.*

(ii) *The maximum number of the orthogonal homogeneous (geodesic) vectors on the fiber space of τ_G, is $n+2$, in the case that $n+1$ is even and not divisible by 4.*

(iv) *The maximum number of the orthogonal homogeneous (geodesic) vectors on the fiber space of τ_G, is $2n+1$, in the case that $n+1$ is even and divisible by 4 and there exists a Hadamard matrices of order $n+1$.*

Proof. Theorem 2.3 and corollary 2.2, conclude the fist part of theorem, it is easy to see that, there exist $n+1$ linearly independent homogeneous (geodesics) vectors of type (B), (see proof of lemma 2.6), then there are $2n+1$ linearly independent homogeneous (geodesics) vectors in the fiber space of τ_G.

The second and the third part of the theorem follows from (i) and (ii), in lemma 2.6. Finally, as an immediate consequence from proposition 2.7, we obtain (iv).□

By proposition 1.3 and theorem 2.8, we complete corollary 2.5 in [5], about the number of linearly independent homogeneous geodesics through origin of the base space of τ_G.

Corollary 2.9. *Let $\tau_G = (T_G, \pi_G, G, \boldsymbol{R}^{2n+1})$ be the tangent bundle of the homogeneous Riemannian Lie group G, (given in theorem 2.1 and lemma 2.6) then;*

(i) *There are $2n+1$ linearly independent homogeneous geodesics vectors through the origin $\{e\}$ of the base space of τ_G.*

(ii) *The maximum number of the orthogonal homogeneous geodesic through*

the origin $\{e\}$ *of the base space of* τ_G*, is* $n+1$*, in the case that* $n+1$ *is odd.*
(ii) *The maximum number of the orthogonal homogeneous geodesic through the origin* $\{e\}$ *of the base space of* τ_G*, is* $n+2$*, in the case that* $n+1$ *is even and not divisible by* 4*.*
(iv) *The maximum number of the orthogonal homogeneous geodesic through the origin* $\{e\}$ *of the base space of* τ_G*, is* $2n+1$*, in the case that* $n+1$ *is even and divisible by* 4 *and there exists a Hadamard matrices of order* $n+1$*.*

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