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The Generalized difference of $d\left(\chi^{3I}\right)$ of fuzzy real numbers over p metric spaces defined by Musielak Orlicz function

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Abstract. In this article we introduce the sequence spaces

$$\left[\chi^{3q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$$

and $\left[\Lambda^{3q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$, associated with the differential operator of sequence space defined by Musielak. We study some basic topological and algebraic properties of these spaces. We also investigate some inclusion relations related to these spaces.

Keywords: analytic sequence, triple sequences, χ^3 space, difference sequence space,Musielak - Orlicz function, $p-$ metric space, Ideal; ideal convergent; fuzzy number; multiplier,differential operator .

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A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \rightarrow \mathbb{R}\left(\mathbb{C}\right)$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by "Duden et al [3], Sahiner et al. [15], Esi et al. [4-7], Datta et al. [1], Debnath et al. [2]" and many others.
A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} \left|x_{mnk}\right|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3.
A triple sequence $x = (x_{mnk})$ is called triple entire sequence if

$$\left|x_{mnk}\right|^{\frac{1}{m+n+k}} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty.$$

The space of all triple entire sequences are usually denoted by Γ^3.
A triple sequence $x = (x_{mnk})$ is called triple chi sequence if

$$\left((m + n + k)!\left|x_{mnk}\right|\right)^{\frac{1}{m+n+k}} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty.$$

The space of all triple chi sequences are usually denoted by χ^3.
The space Λ^3 and Γ^3 is a metric space with the metric

$$d(x, y) = sup_{m,n,k} \left\{\left|x_{mnk} - y_{mnk}\right|^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, ...\right\}, \quad (0.1)$$

for all $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ $in\, \Gamma^3$.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [10] as follows

$$Z\left(\Delta\right) = \left\{x = (x_k) \in w : (\Delta x_k) \in Z\right\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Let $w^3, \chi^3\left(\Delta_{mnk}\right), \Lambda^3\left(\Delta_{mnk}\right)$ be denote the spaces of all, triple gai difference sequence space and triple analytic difference sequence space respectively. The difference triple sequence space was introduced by Debnath et al. (see [2]) and is defined as
$\Delta x_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$ and $\Delta^0 x_{mnk} = \left\langle x_{mnk}\right\rangle$.

1. Definitions and Preliminaries

Throughout the article $w^3, \chi^3\left(\Delta\right), \Lambda^3\left(\Delta\right)$ denote the spaces of all, triple gai difference sequence spaces and triple analytic difference sequence spaces respectively.

For a triple sequence $x \in w^3$, Murugesan et al. introduced by ([13]), the spaces $\Gamma^3\left(\Delta\right), \Lambda^3\left(\Delta\right)$ as follows:

$$\Gamma^3(\Delta) = \left\{x \in w^3 : |\Delta x_{mnk}|^{1/m+n+k} \to 0\, as\, m,n,k \to \infty\right\}$$

$$\Lambda^3(\Delta) = \left\{x \in w^3 : sup_{m,n,k} |\Delta x_{mnk}|^{1/m+n+k} < \infty\right\}.$$

The spaces $\Gamma^3(\Delta), \Lambda^3(\Delta)$ are metric spaces with the metric

$$d(x,y) = sup_{m,n,k} \left\{|\Delta x_{mnk} - \Delta y_{mnk}|^{1/m+n+k} : m,n,k = 1,2,\cdots\right\}$$

for all $x = (x_{mnk})$ and $y = (y_{mnk})$ in $\Gamma^3(\Delta), \Lambda^3(\Delta)$.

1.1. **Definition.** An Orlicz function ([see [9]) is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$, then this function is called modulus function.

"Linden-strauss and Tzafriri ([11])" used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mnk})$ defined by

$$g_{mnk}(v) = sup\{|v|\, u - (f_{mnk})(u) : u \ge 0\}, m,n,k = 1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, (see [12]) the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0\, as\, m,n,k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an exteneded real number.

1.2. **Definition.** Let X, Y be a real vector space of dimension m, where $n \le m$. A real valued function $d_p(x_1,\ldots,x_n) = \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = 0$ if and only if $d_1(x_1,0),\ldots,d_n(x_n,0)$ are linearly dependent,

(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = |\alpha|\, \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n)) = (d_X(x_1,x_2,\cdots x_n)^p + d_Y(y_1,y_2,\cdots y_n)^p)^{1/p}$ $for 1 \le p < \infty$; (or)

(v) $d((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)) := \{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\}$, for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces (see [13]) .

1.3. **Definition.** Let X be a linear metric space. A function $\rho : X \to \mathbb{R}$ is called paranorm, if
(1) $\rho\left(x\right) \geq 0$, for all $x \in X$;
(2) $\rho\left(-x\right) = \rho\left(x\right)$, for all $x \in X$;
(3) $\rho\left(x+y\right) \leq \rho\left(x\right) + \rho\left(y\right)$, for all $x, y \in X$;
(4) If $\left(\sigma_{mnk}\right)$ is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n, k \to \infty$ and $\left(x_{mnk}\right)$ is a sequence of vectors with $\rho\left(x_{mnk} - x\right) \to 0$ as $m, n \to \infty$, then $\rho\left(\sigma_{mnk}x_{mnk} - \sigma x\right) \to 0$ as $m, n, k \to \infty$.
A paranorm w for which $\rho\left(x\right) = 0$ implies $x = 0$ is called total paranorm and the pair $\left(X, w\right)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm.

The notion of ideal convergence was introduced first by Kostyrko et al. [14]. as a generalization of statistical convergence which was further studied in topological spaces by Gunawan et al. [8] and also more applications of ideals can be deals with various authors by B.Hazarika.

1.4. **Definition.** A family $I \subset 2^{Y \times Y \times Y}$ of subsets of a non empty set Y is said to be an ideal in Y if
(1) $\phi \in I$
(2) $A, B \in I$ imply $A \bigcup B \in I$
(3) $A \in I, B \subset A$ imply $B \in I$.

while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. Given $I \subset 2^{\mathbb{N}^3}$ be a non trivial ideal in $\mathbb{N}^3$. A sequence $\left(x_{mnk}\right)_{m,n,k \in \mathbb{N}^3}$ in X is said to be $I-$ convergent to $0 \in X$, if for each $\epsilon > 0$ the set $A\left(\epsilon\right) = \left\{m, n, k \in \mathbb{N}^3 : \|(d_1(x_1), \ldots, d_n(x_n)) - 0\|_p \geq \epsilon\right\}$ belongs to I.

1.5. **Definition.** A non-empty family of sets $F \subset 2^{X \times X \times X}$ is a filter on X if and only if
(1) $\phi \in F$
(2) for each $A, B \in F$, we have imply $A \bigcap B \in F$
(3) each $A \in F$ and each $A \subset B$, we have $B \in F$.

1.6. **Definition.** An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^{X \times X \times X}$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X.

1.7. **Definition.** A non-trivial ideal $I \subset 2^{X \times X \times X}$ is called (i) admissible if and only if $\{\{x\} : x \in X\} \subset I$. (ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

If we take $I = I_f = \left\{A \subseteq \mathbb{N}^3 : A\, is\, a\, finite\, subset\,\right\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \left\{A \subseteq \mathbb{N} \times \mathbb{N}^3 : \delta(A) = 0\right\}$ where $\delta(A)$ denote the asyptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}^3$ and the corresponding convergence coincides with the statistical convergence.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R} \times \mathbb{N}$. For $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$, $d(X,Y) = max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$.

Then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space. Also the relation $\leq$ is a partial order on D. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R} \times \mathbb{R}$ i.e. a mapping $X : R \to J(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

1.8. **Definition.** A fuzzy number X is said to be (i) convex if $X(t) \geq X(s) \wedge X(r) = min\{X(s), X(r)\}$, where $s < t < r$. (ii) normal if there exists $t_0 \in \mathbb{R} \times \mathbb{R}$ such that $X(t_0) = 1$. (iii) upper semi-continuous if for each $\epsilon > 0, X^{-1}([0, a + \epsilon])$ for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R}^3$.

Let $R(J)$ denote the set of all fuzzy numbers which are upper semi-continuous and have compact support, i.e. if $X \in \mathbb{R}(J)^3$ the for any $\alpha \in [0,1], [X]^\alpha$ is compact, where $[X]^\alpha = \left\{t \in \mathbb{R}^3 : X(t) \geq \alpha,\ if\ \alpha \in [0,1]\right\}$, $[X]^0$ =closure of $\left(\left\{t \in \mathbb{R}^3 : X(t) > \alpha, if \alpha = 0\right\}\right)$.

The set $\mathbb{R}$ of real numbers can be embedded $\mathbb{R}(J)^3$ if we define $\bar{r} \in \mathbb{R}^3(J)$ by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r : \\ 0, & \text{if } t \neq r \end{cases}$$

The absolute value, $|X|$ of $X \in \mathbb{R}(J)$ is defined by

$$|X|(t) = \begin{cases} max\{X(t), X(-t)\}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Define a mapping $\bar{d} : \mathbb{R}^3(J) \to \mathbb{R}^+ \cup \{0\}$by

$$\bar{d}(X,Y) = sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha).$$

It is known that $\left(\mathbb{R}(J), \bar{d}\right)$ is a complete metric space.

1.9. **Definition.** A metric on $\mathbb{R}^3(J)$ is said to be translation invariant if $\bar{d}(X + Z, Y + Z) = \bar{d}(X,Y)$, for $X, Y, Z \in \mathbb{R}^3(J)$.

1.10. **Definition.** A sequence $X = (X_{mnk})$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for every $\epsilon > 0$, there exists a positive integer n_0 such that $\bar{d}(X_{mnk}, X_0) < \epsilon$ for all $m, n, k \geq n_0$.

1.11. **Definition.** A sequence $X = (X_{mnk})$ of fuzzy numbers is said to be (i) I-convergent to a fuzzy number X_0 if for each $\epsilon > 0$ such that

$$A = \left\{m, n, k \in \mathbb{N}^3 : \bar{d}\left(X_{mnk}, X_0\right) \geq \epsilon\right\} \in I.$$

The fuzzy number X_0 is called I-limit of the sequence (X_{mnk}) of fuzzy numbers and we write $I - lim X_{mnk} = X_0$. (ii) I-bounded if there exists $M > 0$ such that

$$\left\{m, n, k \in \mathbb{N}^3 : d\left(X_{mn}, \bar{0}\right) > M\right\} \in I.$$

1.12. **Definition.** A sequence space E_F of fuzzy numbers is said to be (i) solid (or normal) if $(Y_{mnk}) \in E_F$ whenever $(X_{mnk}) \in E_F$ and $\bar{d}\left(Y_{mnk}, \bar{0}\right) \leq \bar{d}\left(X_{mnk}, \bar{0}\right)$ for all $m, n, k \in \mathbb{N}^3$. (ii) symmetric if $(X_{mnk}) \in E_F$ implies $\left(X_{\pi(mnk)}\right) \in E_F$ where π is a permutation of $\mathbb{N}^3$.

Let $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$ and E be a sequence space. A K-step space of E is a sequence space

$$\lambda^E_{mnk} = \left\{\left(X_{m_p n_p k_p}\right) \in w^3 : \left(m_p n_p k_p\right) \in E\right\}.$$

A canonical preimage of a sequence $\left\{\left(x_{m_p n_p k_p}\right)\right\} \in \lambda^E_K$ is a sequence $\{y_{mnk}\} \in w^3$ defined as

$$y_{mnk} = \begin{cases} x_{mnk}, & \text{if } m, n, k \in E \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ^E_K is a set of canonical preimages of all elements in λ^E_K, i.e. y is in canonical preimage of λ^E_K if and only if y is canonical preimage of some $x \in \lambda^E_K$.

1.13. **Definition.** A sequence space E_F is said to be monotone if E_F contains the canonical pre-images of all its step spaces.

1.14. **Lemma.** A sequence space E_F is normal implies E_F is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [9], page 53).

1.15. **Lemma.** If $I \subset 2^{\mathbb{N}^3}$ is a maximal ideal, then for each $A \subset \mathbb{N}^3$ we have either $A \in I$ or $\mathbb{N}^3 - A \in I$.

2. Some new integrated sequence spaces of fuzzy numbers

The main aim of this article to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let $p = (p_{mnk})$ be a sequence of positive real numbers for all $m, n, k \in \mathbb{N}^3$. $f = (f_{mnk})$ be a Musielak-Orlicz function, $\left(X, \left\|\left(d\left(x_1\right), d\left(x_2\right), \cdots, d\left(x_{n-1}\right)\right)\right\|_p\right)$ be a $p-$metric space, and (λ_{mnk}) be a sequence of non-zero scalars,

$\mu_{mnk}(X) = \bar{d}\left(\lambda_{mnk}\left((m+n+k)!\Delta^m X_{mnk}\right)^{1/m+n+k}, \bar{0}\right)$ and

$\eta_{mnk}(X) = \bar{d}\left(\lambda_{mnk}(\Delta^m X_{mnk})^{1/m+n+k}, \bar{0}\right)$ are sequence spaces of fuzzy numbers, we define the following sequence spaces as follows:

$$\left[\chi^{3q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)} = \left\{(r,s,t) \in \mathbb{N}^3 : \left[f_{mnk}\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}} \geq \epsilon\right\} \in I,$$

$$\left[\Lambda^{3q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)} = \left\{(r,s,t) \in \mathbb{N}^3 : \left[f_{mnk}\left(\|\eta_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}} \geq K\right\} \in I.$$

2.1. **Theorem.** Let $f = (f_{mnk})$ be a Musielak-Orlicz function, $q = (q_{mnk})$ be a triple analytic sequence of strictly positive real numbers, the sequence spaces $\left[\chi^{3q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$ and $\left[\Lambda^{3q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$ are linear spaces.

Proof: We prove the result only for the space $\left[\chi^{3q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$. The other spaces can be treated, similarly. Let $X = (X_{mnk})$ and $Y = (Y_{mnk})$ be three elements $\left[\chi^{3q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$. We have

$$A_{\frac{\epsilon}{2}} = \left\{(r,s,t) \in \mathbb{N}^3 : \left[f_{mnk}\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}} \geq \frac{\epsilon}{2}\right\} \in I$$

and

$$B_{\frac{\epsilon}{2}} = \left\{(r,s,t) \in \mathbb{N}^3 : \left[f_{mnk}\left(\|\mu_{mnk}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq \frac{\epsilon}{2}\right\} \in I.$$

Let α and β be two scalars. By the Musielak continuity of the function $f = (f_{mnk})$ the following inequality holds:

$$\left[f_{mnk}\left(\left\|\frac{\mu_{mnk}(\alpha x+\beta y)}{|\alpha|+|\beta|}, (d(x_1), d(x_2), \cdots, d(x_{n-1}))\right\|_p\right)\right]^{q_{mnk}} \leq D\left[\frac{|\alpha|}{|\alpha|+|\beta|} f_{mnk}\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}} + D\left[\frac{|\beta|}{|\alpha|+|\beta|} f_{mnk}\left(\|\mu_{mnk}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq D\left[f_{mnk}\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}} + D\left[f_{mnk}\left(\|\mu_{mnk}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}}$$

. From the above

relation we obtain the following:
$\left\{(r,s,t)\in\mathbb{N}^3:\left[f_{mn}\left(\left\|\frac{\mu_{mnk}(\alpha x+\beta y)}{|\alpha|+|\beta|},\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mnk}}\geq\epsilon\right\}\subseteq$
$\left\{(r,s,t)\in\mathbb{N}^3:DK\left[f_{mnk}\left(\left\|\mu_{mnk}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mnk}}\geq\frac{\epsilon}{2}\right\}\cup$
$\left\{(r,s,t)\in\mathbb{N}^3:DK\left[f_{mnk}\left(\left\|\mu_{mnk}\left(y\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mnk}}\geq\frac{\epsilon}{2}\right\}\in$
I. This completes the proof.

2.2. **Remark.** It is easy to verify $\left[\Lambda^{3q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$ is a linear space

2.3. **Theorem.** The classes of sequences $\left[\chi^{3q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{F}$ and $\left[\Lambda^{3q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{F}$ are paranormed spaces paranormed by g, defined by

$$g\left(X\right)=$$
$$inf\left\{\frac{q_{mnk}}{H}:sup_{mnk}f_{mnk}\left(\left\|\mu_{mnk}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\leq1\right\}$$

where $H=max\left\{1,sup_{mnk}q_{mnk}\right\}.$

Proof: Clearly $g\left(X\right)\geq0,g\left(-X\right)=g\left(X\right)$ and $g\left(X+Y\right)\leq g\left(X\right)+g\left(Y\right)$. Next we show the continuity of the product. Let α be fixed and $g\left(X\right)\to0$. Then it is obvious that $g\left(\alpha X\right)\to0$. Next let $\alpha\to0$ and X be fixed. Since f_{mnk} are continuous, we have $f_{mnk}\left(\alpha\left\|\mu_{mnk}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\to$ 0, as $\alpha\to0$. Thus we have

$$inf\left\{\frac{q_{mnk}}{H}:sup_{mnk}f_{mnk}\left(\left\|\mu_{mnk}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\leq1\right\}\to 0,\text{ as }\alpha\to0.$$

Hence $g\left(\alpha X\right)\to0$ as $\alpha\to0$. Therefore g is a paranorm.

2.4. **Proposition.** $\left[\chi^{3q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}\subset$ $\left[\Lambda^{2q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$ and the inclusion is proper

Proof: Let $I\left(F\right)=I,f_{mnk}\left(\left\|\mu_{mnk}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)=$ $\left(-1\right)^{m+n+k},\lambda_{mnk}=q_{mnk}=m,n,k=1$ then $\mu\left(x\right)=\left[\Lambda^{3q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$ but $\left(x_{mnk}\right)\notin\left[\chi^{3q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$

2.5. **Theorem.** The spaces $\left[\chi^{3q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$ and
$\left[\Lambda^{3q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$ are neither solid nor monotone in general

Proof: Let (x_{mnk}) be a given sequence and (α_{mnk}) be a sequence of scalars such that $|\alpha_{mnk}| \leq 1$, for all $m, n, k \in \mathbb{N}^3$. Then we have
$\left[f_{mnk}\left(\|\mu_{mnk}(\alpha x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}} \leq$
$\left[f_{mnk}\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}}$, for all $m, n, k \in \mathbb{N}^3$.
If $\Delta_{mnk} = 1$ then solidness follows above inequality. The monotonicity follows by lemma 2.12.
The first part of the proof follows from the following example:
Example: Let $I(F) = I$, $\left[f_{mnk}\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}} =$
$\left[f\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}} =$
$\left[\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}}$, $m, n, k = 1, \lambda_{mnk} = 1$ for all $m, n, k \in \mathbb{N}$, $q_{mnk} = 1$ for m, n, k odd, $q_{mnk} = 3$ for m, n, k even, $(x_{mnk}) = (mnk)^{m+n+k}$ for all $m, n, k \in \mathbb{N}^3$ belongs to $\left[\Lambda^{3q}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$.
For E, a sequence space, consider its step space E_J defined by $(y_{mnk}) \in E_J$ implies $y_{mnk} = 0$ for all m, n, k odd and $y_{mnk} = x_{mnk}$ for m, n, k even. Then
$(y_{mnk}) \in \left[\Lambda^{3q}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}_{J}$. Hence the spaces are not monotone. Hence are not solid.

2.6. **Theroem.** The spaces $\left[\chi^{3q}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$ and
$\left[\Lambda^{3q}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$ are not convergence free

Example: Let $I(F) = I$, $\left[f_{mnk}\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}} =$
$\left[f\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}} =$
$\left[\left(\|\mu_{mnk}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mnk}}$, $m, n, k = 1, \lambda_{mnk} = 1$ for all $m, n, k \in \mathbb{N}$, $q_{mnk} = 1$ for m, n, k odd, $q_{mnk} = 2$ for m, n, k even, consider the sequence $(x_{mnk}) = (mnk)^{-(m+n+k)}$ for all $m, n, k \in \mathbb{N}^3$ belongs to each of $\left[\chi^{3q}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$. and $\left[\Lambda^{3q}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$.
Consider the sequence (y_{mnk}) defined by $(y_{mnk})^{1/m+n+k} = m^3n^3k^3$, for all $m, n, k \in \mathbb{N}^3$. Then (y_{mnk}) neither belongs to

$\left[\chi_{\mu}^{3q}, \left\|\left(d\left(x_1\right), d\left(x_2\right), \cdots, d\left(x_{n-1}\right)\right)\right\|_p\right]^I$ nor $\left[\Lambda_{\mu}^{3q}, \left\|\left(d\left(x_1\right), d\left(x_2\right), \cdots, d\left(x_{n-1}\right)\right)\right\|_p\right]^I$. Hence the spaces are not convergence free.

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