Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **8**(1)(2019), 18-34

Fuzzy filters in ordered $\Gamma-$semirings

MARAPUREDDY MURALI KRISHNA RAO [1]
Department of Mathematics, GIT, GITAM University,
Visakhapatnam- 530 045, Andhra Pradesh, INDIA.

Abstract. We introduce the notion of ideal, prime ideal, filter, fuzzy ideal, fuzzy prime ideal, fuzzy filter of ordered $\Gamma-$semiring and study their properties and relations between them. We characterize the prime ideals and filters of ordered $\Gamma-$semiring with respect to fuzzy ideals and fuzzy filters respectively.

Keywords: Fuzzy prime ideal, fuzzy filter, ordered $\Gamma-$semiring.

2000 Mathematics subject classification: 08A72, 20N25..

1. Introduction

In 1995, M. Murali Krishna Rao [9,10.11,12] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring. The notion of a semiring is an algebraic structure with two associative binary operations where one distributes over the other, was first introduced by H. S. Vandiver [23] in 1934 but semirings had appeared in earlier studies on the theory of ideals of rings. In structure, semirings lie between semigroups and rings. The results which hold in rings but not in semigroups hold in semirings since semiring is a generalization of ring. The study of rings shows that multiplicative structure of ring is an independent of additive structure whereas in semiring multiplicative structure of semiring is not an independent of additive structure of semiring. The additive and the multiplicative structures of a semiring play an important role in determining the structure of a semiring. The

[1]Corresponding author: mmarapureddy@gmail.com
Received: 3 April 2017
Accepted: 29 July 2018

theory of rings and theory of semigroups have considerable impact on the development of theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solution of graph theory, optimization theory, in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches.

As a generalization of ring, the notion of a $\Gamma-$ring was introduced by N. Nobusawa [20] in 1964. In 1981, M. K. Sen [21] introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [4] in 1932, Lister [6] introduced the notion of a ternary ring . The set of all negative integers Z is not a semiring with respect to usual addition and multiplication but Z forms a $\Gamma-$semiring where $\Gamma = Z$. The important reason for the development of $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, ternary semirings and semigroups The fuzzy set theory was developed by L. A. Zadeh [24] in 1965. The fuzzification of algebraic structure was introduced by A. Rosenfeld [19] and he introduced the notion of fuzzy subgroups in 1971. K.L. N. Swamy and U. M. Swamy [22] studied fuzzy prime ideals in rings in 1988. In 1982, W. J. Liu [5] defined and studied fuzzy subrings as well as fuzzy ideals in rings. Applying the concept of fuzzy sets to the theory of $\Gamma-$ring, Y. B. Jun and C. Y. Lee [1] introduced the notion of fuzzy ideals in $\Gamma-$ring and studied the properties of fuzzy ideals of $\Gamma-$ring. D. Mandal[7] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. studied fuzzy soft $\Gamma-$semiring and fuzzy soft $k-$ideal over $\Gamma-$semiring. N. Kuroki [3] studied fuzzy interior ideals in semigroups. M. Murali Krishna Rao and B .Venkateswarlu [17] studied $\Gamma-$incline and field $\Gamma-$semiring. In 1988, Zhang [24] studied prime $L-$fuzzy ideals in rings where L is completely distributive lattice. The concept of $L-$fuzzy ideal and normal $L-$fuzzy ideal in semirings were studied by Jun, Neggers and Kim [2].

In this paper, we introduce the notion of ideal, prime ideal, filter, fuzzy ideal, fuzzy prime ideal, fuzzy filter in an ordered $\Gamma-$semiring and study their properties and relations between them. We prove that if $\phi : M \to N$ be an onto homomorphism. and f is a ϕ homomorphism invariant fuzzy filter of ordered $\Gamma-$semiring M then $\phi(f)$ is a fuzzy filter of ordered $\Gamma-$semiring N,characterize the prime ideals and filters of ordered $\Gamma-$semiring with respect to fuzzy ideals and fuzzy filters respectively.

2. Preliminaries

In this section we recall some of the fundamental concepts and definitions which are necessary for this paper.and

Definition 2.1. A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i). Addition is a commutative operation.
(ii). Multiplication distributes over addition both from the left and from the right.
(iii). There exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Definition 2.2. Let M and Γ be two non-empty sets. Then M is called a $\Gamma-$semigroup if it satisfies

(i) $x\alpha y \in M$
(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.3. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. A $\Gamma-$semigroup M is said to be $\Gamma-$semiring M if it satisfies the following axioms, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$.

Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation as the usual semiring multiplication

Definition 2.4. A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M$.

Example 2.5. Let M be the additive semi group of all $m \times n$ matrices over the set of non negative rational numbers and Γ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers, then with respect to usual matrix multiplication M is a $\Gamma-$semiring.

Definition 2.6. Let M be a $\Gamma-$semiring and A be a non-empty subset of M. A is called a $\Gamma-$subsemiring of $\Gamma-$semiring M if A is a sub-semigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

Definition 2.7. Let M be a $\Gamma-$semiring. A subset A of M is called a left (right) ideal of $\Gamma-$semiring M if A is closed under addition and $M\Gamma A \subseteq A$ $(A\Gamma M \subseteq A)$. A is called an ideal of M if it is both a left ideal and a right ideal of M .

Definition 2.8. A $\Gamma-$semiring M is called an ordered $\Gamma-$semiring if it admits a compatible relation $\leq$. i.e. $\leq$ is a partial ordering on M satisfies the following conditions. If $a \leq b$ and $c \leq d$ then

(i) $a + c \leq b + d$ (ii) $a\alpha c \leq b\alpha d$ (iii) $c\alpha a \leq d\alpha b$, for all $a, b, c, d \in M, \alpha \in \Gamma$.

Definition 2.9. An ordered $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M, \alpha \in \Gamma$.

Definition 2.10. Let M be an ordered $\Gamma-$semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.11. An ordered $\Gamma-$semiring M is said to be commutative $\Gamma-$semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.12. Let M be an ordered $\Gamma-$semiring. An element $a \in M$ is said to be an idempotent of M if $a = a\alpha a$ for all $\alpha \in \Gamma$.

Example 2.13. Let $M = [0, 1], \Gamma = N,$ $+$ and ternary operation be defined as $x + y = \max\{x, y\}, x\gamma y = \min\{x, \gamma, y\}$ for all $x, y \in M, \gamma \in \Gamma$. Then M is an ordered $\Gamma-$semiring with respect to usual ordering.

Definition 2.14. An ordered $\Gamma-$semiring M is said to be totally ordered $\Gamma-$semiring M if any two elements of M are comparable.

Definition 2.15. In an ordered $\Gamma-$semiring M

(i) semigroup $(M, +)$ is said to be positively ordered if $a \leq a + b$ and $b \leq a + b$ for all $a, b \in M$.
(ii) semigroup $(M, +)$ is said to be negatively ordered if $a + b \leq a$ and $a + b \leq b$ for all $a, b \in M$.
(iii) $\Gamma-$semigroup M is said to be positively ordered if $a \leq a\alpha b$ and $b \leq a\alpha b$ for all $\alpha \in \Gamma, a, b \in M$.
(iv) $\Gamma-$semigroup M is said to be negatively ordered if $a\alpha b \leq a$ and $a\alpha b \leq b$ for all $\alpha \in \Gamma, a, b \in M$.

Theorem 2.16. *If M is an ordered $\Gamma-$semiring in which $(M, +)$ is positively ordered,then the zero element of a ordered $\Gamma-$semiring $M$is the least element of M.*

Theorem 2.17. *If M is an ordered $\Gamma-$semiring in which $\Gamma-$semigroup M is negatively ordered with unity 1 then 1 is the greatest element of M.*

Theorem 2.18. *Let M be an ordered $\Gamma-$semiring in which $(M, +)$ is positively ordered and $\Gamma-$semigroup M is a negatively ordered with unity 1 and zero element 0. If $a \in M$ then $0 \leq a \leq 1$.*

Definition 2.19. A non-empty subset A of an ordered $\Gamma-$semiring M is called a $\Gamma-$subsemiring M if $(A,+)$ is a subsemigroup of $(M,+)$ and $a\alpha b \in A$ for all $a,b \in A$ and $\alpha \in \Gamma$.

Definition 2.20. Let M be an ordered $\Gamma-$semiring. A non-empty subset A of M is called a left (right) ideal of an ordered $\Gamma-$semiring M if A is closed under addition, $M\Gamma A \subseteq A$ $(A\Gamma M \subseteq A)$ and if for any $a \in M,\ b \in A,\ a \leq b \Rightarrow a \in A$. A is called an ideal of M if it is both a left ideal and a right ideal of M.

Definition 2.21. Let M be a non-empty set. A mapping $f: M \to [0,1]$ is called a fuzzy subset of $\Gamma-$semiring M. If f is not a constant function then f is called a non-empty fuzzy subset

Definition 2.22. The complement of a fuzzy subset μ of a $\Gamma-$semiring M is denoted by μ^c and is defined as $\mu^c(x) = 1 - \mu(x), \ \text{ for all } x \in M.$

Definition 2.23. Let S and T be two sets and $\phi : S \to T$ be any function. A fuzzy subset μ of S is called a $\phi-$invariant if $\phi(x) = \phi(y) \Rightarrow \mu(x) = \mu(y)$.

Definition 2.24. Let M be an ordered $\Gamma-$semiring. A fuzzy subset μ of M is called a fuzzy $\Gamma-$subsemiring of M if
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq min\{\mu(x), \mu(y)\}$

Definition 2.25. Let μ be a non-empty fuzzy subset of an ordered $\Gamma-$semiring M. Then μ is called a fuzzy left (right) ideal of M if
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq \mu(y)\big(\mu(x)\big)$
(iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y),$ for all $x, y \in M, \alpha \in \Gamma$.

By fuzzy ideal we mean, it is both a fuzzy left ideal as well as a fuzzy right ideal of an ordered $\Gamma-$semiring M.

3. Fuzzy filters in an ordered $\Gamma-$semiring

In this section, we introduce the notion of ideal, prime ideal, filter, fuzzy ideal, fuzzy prime ideal and fuzzy filter in an ordered $\Gamma-$semiring and we study some of their properties.

Definition 3.1. Let M be an ordered $\Gamma-$semiring. A $\Gamma-$subsemiring P of M is called a prime ideal of M if
(i) $a \leq b, a \in M, b \in P \Rightarrow a \in P$
(ii) $a\gamma b \in P, a, b \in M, \gamma \in \Gamma \Rightarrow a \in P$ or $b \in P$

Definition 3.2. Let μ be a non-empty fuzzy subset of an ordered $\Gamma-$semiring M. Then μ is called a fuzzy prime ideal of M if

(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) = max\{\mu(x), \mu(y)\}$
(iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y),$ for all $x, y \in M, \alpha \in \Gamma.$

Definition 3.3. Let M be an ordered $\Gamma-$semiring. A $\Gamma-$subsemiring F of M is called a filter of M if

(i) $a \leq b,\ a \in F, b \in M \Rightarrow b \in F$
(ii) $a\gamma b \in F \Rightarrow a \in F$ and $b \in F,$ for any $a, b \in M, \gamma \in \Gamma.$

Definition 3.4. Let M be an ordered $\Gamma-$semiring. A fuzzy $\Gamma-$subsemiring μ of M is called a fuzzy filter of M if

(i) $\mu(x+y) \leq \max\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) = \min\{\mu(x), \mu(y)\}$
(iii) $x \leq y \Rightarrow \mu(x) \leq \mu(y),$ for all $x, y \in M, \alpha \in \Gamma.$

Definition 3.5. Let R and M be ordered $\Gamma-$semirings. Then a mapping f from R to M is called a homomorphism of ordered $\Gamma-$semirings R and S if

(i) $f(a+b) = f(a) + (b)$
(ii) $f(a\alpha b) = f(a)\alpha f(b)$
(iii) $a \leq b \Rightarrow f(a) \leq f(b),$ for all $a, b \in R, \alpha \in \Gamma.$

Theorem 3.6. *Let μ and γ be fuzzy filters of an ordered $\Gamma-$semiring M. Then $\mu \cap \gamma$ is a fuzzy filter of anordered $\Gamma-$semiring M.*

Proof. Let μ and γ be fuzzy filters of the ordered $\Gamma-$semiring M, $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
\mu \cap \gamma(x+y) &= \min\{\mu(x+y), \gamma(x+y)\} \\
&\leq \min\Big\{ \max\{\mu(x), \mu(y)\}, \max\{\gamma(x), \gamma(y)\}\Big\} \\
&= \max\Big\{ \min\{\mu(x), \gamma(x)\}, \min\{\mu(y), \gamma(y)\}\Big\} \\
&= \max\Big\{\mu \cap \gamma(x), \mu \cap \gamma(y)\Big\} \\
\mu \cap \gamma(x\alpha y) &= \min\{\mu(x\alpha y), \gamma(x\alpha y), \} \\
&= \min\Big\{ \min\{\mu(x), \mu(y)\}, \min\{\gamma(x), \gamma(y)\}\Big\} \\
&= \min\Big\{ \min\{\mu(x), \gamma(x)\}, \min\{\mu(y), \gamma(y)\}\Big\} \\
&= \min\Big\{\mu \cap \gamma(x), \mu \cap \gamma(y)\Big\}
\end{aligned}$$

If $x \leq y$ then $\mu(x) \leq \mu(y)$ and $\gamma(x) \leq \gamma(y)$

$$\begin{aligned}
\Rightarrow \mu \cap \gamma(x) &= \min\Big\{\mu(x), \gamma(x)\Big\} \\
&\leq \min\Big\{\mu(y), \gamma(y)\Big\} \\
&= \mu \cap \gamma(y).
\end{aligned}$$

Hence $\mu \cap \gamma$ is a fuzzy filter of the ordered $\Gamma-$semiring M. □

Corollary 3.7. *Let μ and γ be fuzzy filters of an ordered $\Gamma-$semiring M. Then $\mu \cup \gamma$ is a fuzzy filter of an ordered $\Gamma-$semiring M.*

Proof of the following theorems are trivial, so we omit the proofs.

Theorem 3.8. *If μ and μ' be a fuzzy subset and complement of μ respectively of an ordered $\Gamma-$semiring M then the following are equivalent for all $x, y \in M, \alpha \in \Gamma$.*

(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
$\mu(x\alpha y) \geq max\{\mu(x), \mu(y)\}$
$x \leq y \Rightarrow \mu(x) \geq \mu(y$

(ii) $\mu'(x+y) \leq max\{\mu'(x), \mu'(y)\}$
$\mu'(x\alpha y) \leq min\{\mu'(x), \mu'(y)\}$
$x \leq y \Rightarrow \mu'(x) \leq \mu'(y$

Theorem 3.9. *μ is a fuzzy filter of an ordered $\Gamma-$semiring M if and only if for any $t \in [0, 1]$, $\phi \neq \mu_t$ is a filter of an ordered $\Gamma-$semiring M*

Theorem 3.10. *μ is a fuzzy filter of an ordered $\Gamma-$semiring M if and only if μ' is a fuzzy prime ideal of an ordered $\Gamma-$semiring M.*

Theorem 3.11. *Let M be an ordered $\Gamma-$semiring and $\phi \neq F \subseteq M$. F is a filter of M if and only if the characteristic set χ_F is a fuzzy filter of an ordered $\Gamma-$semiring M.*

Let M be an ordered $\Gamma-$semiring , $a \in M$ andμ be a fuzzy filter of M. The set $\{x \in M \mid \mu(a) \leq \mu(x)\}$ is denoted by $F_{\mu(a)}$.

Theorem 3.12. *Let M be an ordered $\Gamma-$semiring and μ be a fuzzy filter of M. Then $F_{\mu(a)}$. is a filter of an ordered $\Gamma-$semiring M.*

Proof. Let μ be a fuzzy filter of the ordered $\Gamma-$semiring$M$ and $b, c \in F_{\mu(a)}, \alpha \in \Gamma$.

$$\begin{aligned}
&\text{Then } \mu(a) \leq \mu(b) \text{ and } \mu(a) \leq \mu(c)\\
&\Rightarrow \mu(a) \leq \max\{\mu(b), \mu(c)\} \leq \mu(b+c)\\
&\Rightarrow \mu(a) \leq \mu(b+c)\\
&\text{Therefore } b+c \in F_{\mu(a)}.\\
\text{Now } &\mu(a) \leq \max\{\mu(b), \mu(c)\} = \mu(b\alpha c)\\
&\Rightarrow \mu(a) \leq \mu(b\alpha c),\\
&\text{Therefore } b\alpha c \in F_{\mu(a)}.\\
&\text{Suppose } b\alpha c \in F_a \text{ and } b, c \in F_{\mu(a)}, \alpha \in \Gamma.\\
&\Rightarrow \mu(a) \leq \ \mu(b\alpha c) = \min\{\mu(b), \mu(c)\}\\
&\Rightarrow \ \mu(a) \leq \mu(b) \text{ and } \mu(a) \leq \mu(c)\\
&\text{Therefore } b, c \in F_{\mu(a)}.\\
\text{Let } \ &x \in F_a, x \leq y \text{ and } y \in M\\
&\Rightarrow \mu(a) \leq \mu(x) \leq \mu(y)\\
&\Rightarrow \mu(a) \leq \mu(y).\\
&\text{Therefore } y \in F_{\mu(a)}.
\end{aligned}$$

Hence $F_{\mu(a)}$ is a filter of the ordered $\Gamma-$semiring M. □

Definition 3.13. Let μ be a fuzzy subset of $S \times S$ and γ be a fuzzy subset of S. Then μ is said to be a fuzzy relation on γ if

$$\mu(x, y) \leq min\{\gamma(x), \gamma(y)\}, \text{ for all } x, y \in S.$$

Definition 3.14. Let γ be a fuzzy subset on a set S. Then μ_γ is said to be strongest fuzzy relation on γ if

$$\mu_\gamma(x, y) = min\{\gamma(x), \gamma(y)\}, \text{ for all } x, y \in S.$$

Theorem 3.15. *Let μ_γ be the strongest fuzzy relation on an ordered $\Gamma-$semiring M. Then γ is a fuzzy filter of M if and only if μ_γ is a fuzzy filter of an ordered $\Gamma-$semiring $M \times M$.*

Proof. Let γ be a fuzzy filter of the ordered $\Gamma-$semiring $M,\ (x_1, x_2), (y_1, y_2) \in M \times M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
\mu_\gamma\big\{(x_1, x_2) + (y_1, y_2)\big\} &= \mu_\gamma\big\{(x_1 + y_1), (x_2 + y_2)\big\} \\
&= \min\big\{\gamma(x_1 + y_1), \gamma(x_2 + y_2)\big\} \\
&\leq min\Big\{max\{\gamma(x_1), \gamma(y_1)\}, maxn\{\gamma(x_2), \gamma(y_2)\}\Big\} \\
&= \min\Big\{max\{\gamma(x_1), \gamma(x_2)\}, max\{\gamma(y_1), \gamma(y_2)\}\Big\} \\
&= \max\Big\{min\{\gamma(x_1), \gamma(x_2)\}, min\{\gamma(y_1), \gamma(y_2)\}\Big\} \\
&= \max\Big\{\mu_\gamma(x_1, x_2), \mu_\gamma(y_1, y_2)\Big\} \\
\mu_\gamma\big\{(x_1, x_2)\alpha(y_1, y_2)\big\} &= \mu_\gamma\big\{(x_1\alpha y_1), (x_2\alpha y_2)\big\} \\
&= \min\big\{\gamma(x_1\alpha y_1), \gamma(x_2\alpha y_2)\big\} \\
&= \min\Big\{min\{\gamma(x_1), \gamma(y_1)\}, min\{\gamma(x_2), \gamma(y_2)\}\Big\} \\
&= \min\Big\{min\{\gamma(x_1), \gamma(x_2)\}, min\{\gamma(y_1), \gamma(y_2)\}\Big\} \\
&= \min\Big\{\mu_\gamma(x_1, x_2), \mu_\gamma(y_1, y_2)\Big\}. \\
\text{Suppose } (x_1, x_2) \leq (y_1, y_2) &\Rightarrow x_1 \leq y_1,\ x_2 \leq y_2 \\
&\Rightarrow \gamma(x_1) \leq \gamma(y_1), \gamma(x_2) \leq \gamma(y_2) \\
\mu_\gamma(x_1, x_2) &= min\{\gamma(x_1), \gamma(x_2)\} \\
&\leq min\{\gamma(y_1), \gamma(y_2)\} \\
&= \mu_\gamma(y_1, y_2).
\end{aligned}$$

Hence μ_γ is a fuzzy filter of the ordered $\Gamma-$semiring $M \times M$.

Conversely suppose that μ_γ is a fuzzy filter of the ordered $\Gamma-$semiring M. $(x_1, x_2), (y_1, y_2) \in M \times M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
min\big\{\gamma(x_1 + y_1), \gamma(x_2 + y_2)\}\big\} &= \mu_\gamma\big(x_1 + y_1), (x_2 + y_2)\big\} \\
&= \mu_\gamma\big\{(x_1, x_2) + (y_1, y_2)\big\} \\
&\leq \max\Big\{\mu_\gamma(x_1, x_2), \mu_\gamma(y_1, y_2)\Big\} \\
&= \max\Big\{\ \min\{\gamma(x_1), \gamma(x_2)\}, min\{\gamma(y_1), \gamma(y_2)\}\Big\}.
\end{aligned}$$

Now put $x_1 = x, x_2 = 1, y_1 = y, y_2 = 1$ then we get

$$\min\{\gamma(x+y), \gamma(1)\} \leq \max\Big\{ \min\{\gamma(x), \gamma(1)\}, \min\{\gamma(y), \gamma(1)\}\Big\}$$
$$\Rightarrow \ \gamma(x+y) \leq \max\{\gamma(x), \gamma(y)\}, \text{ since } \gamma(x) \leq \gamma(1), \text{ for all } \ x \in M.$$

$$\begin{aligned}
\min\big\{\gamma\{(x_1\alpha y_1), \gamma(x_2\alpha y_2)\}\big\} &= \mu_\gamma\big\{(x_1\alpha y_1), (x_2\alpha y_2)\big\} \\
&= \mu_\gamma\big\{(x_1, x_2)\alpha(y_1, y_2)\big\} \\
&= max\Big\{\mu_\gamma(x_1, x_2), \mu_\gamma(y_1, y_2)\Big\} \\
&= max\Big\{ \min\{\gamma(x_1), \gamma(x_2)\}, \min\{\gamma(y_1), \gamma(y_2)\}\Big\}.
\end{aligned}$$

Now put $x_1 = x, x_2 = 1$,$y_1 = y, y_2 = 1$, then we get

$$\begin{aligned}
\min\{\gamma(x\alpha y), \gamma(1)\} &= \max\Big\{ \min\{\gamma(x), \gamma(1)\}, min\{\gamma(y), \gamma(1)\}\Big\} \\
&= \max\Big\{\gamma(x), \gamma(y)\Big\}.
\end{aligned}$$

Therefore $\gamma(x\alpha y) = \max\{\gamma(x), \gamma(y)\}$.

Suppose $x \leq y, x, y \in M$. Then

$$\begin{aligned}
&(x, 1) \leq (y, 1) \\
\Rightarrow &\mu_\gamma(x, 1) \leq \mu_\gamma(y, 1) \\
\Rightarrow &\min\{\gamma(x), \gamma(1)\} \leq min\{\gamma(y), \gamma(1)\} \\
\Rightarrow &\gamma(x) \leq \gamma(y).
\end{aligned}$$

Hence γ is a fuzzy filter of the ordered Γ−semiring M. □

Definition 3.16. Let μ and γ be fuzzy subsets of X. The cartesian product of μ and γ is defined by

$$(\mu \times \gamma)(x, y) = \min\{\mu(x), \gamma(y)\}, \text{ for all } x, y \in X.$$

Theorem 3.17. *Let μ and γ be fuzzy filters of an ordered* Γ−*semiring* M*. Then $\mu \times \gamma$ is a fuzzy filter of an ordered* Γ−*semiring* $M \times M$*.*

Proof. Let μ and γ be fuzzy filters of the ordered $\Gamma-$semiring M and $(x_1,x_2),(y_1,y_2)\in M\times M,\alpha\in\Gamma$. Then

$$\begin{aligned}(\mu\times\gamma)\Big((x_1,x_2)+(y_1,y_2)\Big)&=\mu\times\gamma\Big(x_1+y_1,x_2+y_2\Big)\\&=min\{\mu(x_1+y_1),\gamma(x_2+y_2)\}\\&\leq min\Big\{max\{\mu(x_1),\mu(y_1)\},\max\{\gamma(x_2),\gamma(y_2)\}\Big\}\\&=min\Big\{max\{\mu(x_1),\gamma(x_2)\},\max\{\mu(y_1),\gamma(y_2)\}\Big\}\\&=max\Big\{min\{\mu(x_1),\gamma(x_2)\},\min\{\mu(y_1),\gamma(y_2)\}\Big\}\\&=max\Big\{(\mu\times\gamma)(x_1,x_2),(\mu\times\gamma)(y_1,y_2)\Big\}\end{aligned}$$

$$\begin{aligned}(\mu\times\gamma)\Big((x_1,x_2)\alpha(y_1,y_2)\Big)&=\mu\times\gamma\Big(x_1\alpha y_1,x_2\alpha y_2\Big)\\&=\min\{\mu(x_1\alpha y_1),\gamma(x_2\alpha y_2)\}\\&=\min\Big\{min\{\mu(x_1),\mu(y_1)\},\min\{\gamma(x_2),\gamma(y_2)\}\Big\}\\&=\min\Big\{min\{\mu(x_1),\gamma(x_2)\},\min\{\mu(y_1),\gamma(y_2)\}\Big\}\\&=\min\Big\{(\mu\times\gamma)(x_1\alpha x_2),(\mu\times\gamma)(y_1\alpha y_2)\Big\}\end{aligned}$$

If $(x_1,x_2)\leq(y_1,y_2)$ then $x_1\leq y_1$ and $x_2\leq y_2$

$$\begin{aligned}(\mu\times\gamma)(x_1,x_2)&=\min\{\mu(x_1),\gamma(x_2)\}\\&\leq\min\{\mu(y_1),\gamma(y_2)\}\\&=(\mu\times\gamma)(y_1,y_2).\end{aligned}$$

Therefore $\mu\times\gamma$ is a fuzzy filter of the ordered $\Gamma-$semiring $M\times M$. □

.

Definition 3.18. Let μ be a fuzzy subset of X and $\alpha\in[0,1-\sup\{\mu(x)\mid x\in X\}]$ The mapping $\mu^T_\alpha:X\to[0,1]$ is called a fuzzy translation of μ if $\mu^T_\alpha(x)=\mu(x)+\alpha$

Definition 3.19. Let μ be a fuzzy subset of X and $\beta\in[0,1]$. Then mapping
$\mu^M_\beta:X\to[0,1]$ is called a fuzzy multiplication of μ if $\mu^M_\beta(x)=\beta\mu(x)$.

Definition 3.20. Let μ be a fuzzy subset of X and $\alpha\in[0,1-\sup\{\mu(x)\mid x\in X\}]$, $\beta\in[0,1]$. Then mapping $\mu^{MT}_{\beta,\alpha}:X\to[0,1]$ is called a magnified translation of μ if $\mu^{MT}_{\beta,\alpha}(x)=\beta\mu(x)+\alpha$, for all $x\in X$.

Theorem 3.21. *A fuzzy subset μ is a fuzzy filter of an ordered $\Gamma-$semiring M if and only if μ^T_α is a fuzzy filter of an ordered $\Gamma-$semiring M*

Proof. Suppose μ is a fuzzy filter of the ordered $\Gamma-$semiring M and $x, y \in M, \gamma \in \Gamma$.

$$\begin{aligned}
\mu_\alpha^T(x+y) &= \mu(x+y)+\alpha \\
&\leq \max\{\mu(x), \mu(y)\}+\alpha \\
&= \max\{\mu(x)+\alpha, \mu(y)+\alpha\} \\
&= \max\{\mu_\alpha^T(x), \mu_\alpha^T(y)\} \\
\mu_\alpha^T(x\gamma y) &= \mu(x\gamma y)+\alpha \\
&= \min\{\mu(x), \mu(y)\}+\alpha \\
&= \min\{\mu(x)+\alpha, \mu(y)+\alpha\} \\
&= \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}
\end{aligned}$$

$$\begin{aligned}
&\text{Let } x \leq y. \text{Then } \mu(x) \leq \mu(y) \\
&\Rightarrow \mu(x)+\alpha \leq \mu(y)+\alpha \\
&\Rightarrow \mu_\alpha^T(x) \leq \mu_\alpha^T(y).
\end{aligned}$$

Hence μ_α^T is a fuzzy filter of the ordered $\Gamma-$semiring M.

Conversely suppose that μ_α^T is a fuzzy filter of the ordered $\Gamma-$semiring $M,\ x, y \in M$ and $\gamma \in \Gamma$.

$$\begin{aligned}
\mu(x+y)+\alpha &= \mu_\alpha^T(x+y) \\
&\leq \max\{\mu_\alpha^T(x), \mu_\alpha^T(y)\} \\
&= \max\{\mu(x)+\alpha, \mu(y)+\alpha\} \\
&= \max\{\mu(x), \mu(y)\}+\alpha \\
\text{Therefore } \mu(x+y) &\leq \max\{\mu(x), \mu(y)\}. \\
\mu(x\gamma y)+\alpha &= \mu_\alpha^T(x\gamma y) \\
&= \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\} \\
&= \min\{\mu(x)+\alpha, \mu(y)+\alpha\} \\
&= \min\{\mu(x), \mu(y)\}+\alpha \\
\text{Therefore } \mu(x\gamma y) &= \min\{\mu(x), \mu(y)\}.
\end{aligned}$$

$$\begin{aligned}
&\text{Let } x \leq y. \text{Then } \mu_\alpha^T(x) \leq \mu_\alpha^T(y). \\
&\Rightarrow \mu(x)+\alpha \leq \mu(y)+\alpha \\
&\Rightarrow \mu(x) \leq \mu(y).
\end{aligned}$$

Hence μ is a fuzzy filter of the ordered $\Gamma-$semiring M. □

Theorem 3.22. *A fuzzy subset μ is a fuzzy filter of an ordered $\Gamma-$semiring M if and only if μ_β^M is a fuzzy filter of an ordered $\Gamma-$semiring M.*

Proof. Suppose μ is a fuzzy filter of the ordered $\Gamma-$semiring M and $x, y \in M, \gamma \in \Gamma$. Then

$$\begin{aligned}
\mu_\beta^M(x+y) &= \beta\mu(x+y) \\
&= \beta \max\{\mu(x), \mu(y)\} \\
&= \max\{\beta\mu(x), \beta\mu(y)\} \\
&= \max\{\mu_\beta^M(x), \mu_\beta^M(y)\}. \\
\mu_\beta^M(x\gamma y) &= \beta\mu(x\gamma y) \\
&\leq \beta \min\{\mu(x), \mu(y)\} \\
&= \min\{\beta\mu(x), \beta\mu(y)\} \\
&= \min\{\mu_\beta^M(x), \mu_\beta^M(y)\}.
\end{aligned}$$

$$\begin{aligned}
&\text{Let } x \leq y. \text{Then } \mu(x) \leq \mu(y) \\
&\Rightarrow \beta\mu(x) \leq \beta\mu(y) \\
&\Rightarrow \mu_\beta^M(x) \leq \beta_\beta^M\mu(y).
\end{aligned}$$

Hence μ_β^M is a fuzzy filter of the ordered $\Gamma-$semiring M.

Conversely, suppose that μ_β^M is a fuzzy filter of the ordered $\Gamma-$semiring M and $x, y \in M, \gamma \in \Gamma$. Then

$$\begin{aligned}
\mu_\beta^M(x+y) &\leq \max\{\mu_\beta^M(x), \mu_\beta^M(y)\} \\
\Rightarrow \ \beta\mu(x+y) &\leq \max\{\beta\mu(x), \beta\mu(y)\} \\
&= \beta \max\{\mu(x), \mu(y)\} \\
\text{Therefore } \mu(x+y) &\leq \max\{\mu(x), \mu(y)\}. \\
\mu_\beta^M(x\gamma y) &\leq \min\{\mu_\beta^M(x), \mu_\beta^M(y)\} \\
&= \beta \min\{\mu(x), \mu(y)\} \\
\beta\mu(x\gamma y) &= \beta \min\{\mu(x), \mu(y)\} \\
\text{Therefore } \mu(x\gamma y) &= \min\{\mu(x), \mu(y)\}.
\end{aligned}$$

$$\begin{aligned}
&\text{Let } x \leq y. \text{Then } \mu_\beta^M(x) \leq \beta_\beta^M\mu(y) \\
&\Rightarrow \beta\mu(x) \leq \beta\mu(y) \\
&\Rightarrow . \mu(x) \leq \mu(y).
\end{aligned}$$

Hence μ is a fuzzy filter of the ordered $\Gamma-$semiring M. □

Theorem 3.23. *A fuzzy subset μ is a fuzzy filter of an ordered $\Gamma-$semiring M if and only if $\mu_{\beta,\alpha}^{MT} : X \to [0,1].\mu_{\beta,\alpha}^{MT}$ is a fuzzy filter of the ordered $\Gamma-$semiring M.*

Proof. Suppose μ is a fuzzy filter of the ordered $\Gamma-$semiring M.
$\Leftrightarrow$ μ_β^M is a fuzzy filter of ordered $\Gamma-$semiring M, by Theorem 3.22
$\Leftrightarrow$ $\mu_{\beta,\alpha}^{MT}$ is a fuzzy filter of the ordered $\Gamma-$semiring M. $\square$

Definition 3.24. Let a function $\phi : M \to N$ be a homomorphism of ordered $\Gamma-$semirings M, N and μ be a fuzzy subset of M. Then μ is said to be ϕ homomorphism invariant if $\phi(a) \leq \phi(b)$ then $\mu(a) \leq \mu(b)$, for $a, b \in M$.

Theorem 3.25. *Let M and N be ordered $\Gamma-$semirings and $\phi : M \to N$ be an onto homomorphism. If f is a ϕ homomorphism invariant fuzzy filter of M then $\phi(f)$ is a fuzzy filter of N.*

Proof. Let M and N be ordered $\Gamma-$semirings, $\phi : M \to N$ be an onto homomorphism, f be a homomorphism ϕ invariant fuzzy filter of M and $a \in M$. Suppose $x \in N, t \in \phi^{-1}(x)$ and $x = \phi(a)$. Then $a \in \phi^{-1}(x) \Rightarrow \phi(t) = x = \phi(a)$,

$$\begin{aligned}
\phi(a\alpha b) =&\phi(a)\alpha\phi(b)\\
=&x\alpha y\\
\phi f(x\alpha y) =&f(a\alpha b)\\
=&\min\{f(a), f(b)\}\\
=&\min\{\phi(f(x)), \phi(f(y))\}
\end{aligned}$$

since f is a ϕ invariant, $f(t) = f(a) \Rightarrow \phi(f)(x) = \inf\limits_{t\in\phi^{-1}(x)} f(t) = f(a)$. Hence $\phi(f)(x) = f(a)$. Let $x, y \in N$. Then there exist $a, b \in M$ such that $\phi(a) = x, \phi(b) = y \Rightarrow \phi(a + b) = x + y \Rightarrow \phi(f)(x + y) = f(a + b) \leq \max\{f(a), f(b)\} = \min\{\phi(f)(x), \phi(f)(y)\}$.

Let $x, y \in N$ and $x \leq y$.
Then there exist $a, b \in M$ such that $\phi(a) = x, \phi(b) = y$
and $\phi(f)(x) = f(a), \phi(f)(y) = f(b)$.

$$\begin{aligned}
\text{Therefore } x \leq y \Rightarrow&\phi(a) \leq \phi(b)\\
\Rightarrow&f(a) \leq f(b)\\
\Rightarrow&\phi(f)(x) \leq \phi(f)(y).
\end{aligned}$$

Hence the theorem. $\square$

Theorem 3.26. *Let $f : M \to N$ be a homomorphism of ordered $\Gamma-$semirings and η be a fuzzy filter of N. If $\eta \circ f = \mu$ then μ is a fuzzy filter of M.*

Proof. Let $f: M \to N$ be a homomorphism of ordered $\Gamma-$semirings, η be a fuzzy filter of $N, \eta \circ f = \mu$ and $x, y \in M$.

$$\begin{aligned}\mu(x+y) &= \eta(f(x+y)) = \eta(f(x)+f(y)) \\ &\leq \max\{\eta(f(x)), \eta(f(y))\} \\ &= \max\{\mu(x), \mu(y)\} \\ \mu(x\alpha y) &= \eta(f(x\alpha y)) = \eta(f(x)\alpha f(y)) \\ &= \min\{\eta(f(x)), \eta(f(y))\} \\ &= \min\{\mu(x), \mu(y)\}.\end{aligned}$$

Suppose $x, y \in M$ and $x \leq y$. Since $f: M \to N$ be a homomorphism, we have

$$\begin{aligned}&f(x) \leq f(y) \\ \Rightarrow &\eta(f(x)) \leq \eta(f(y)) \\ \Rightarrow &\mu(x) \leq \mu(y)\end{aligned}$$

Hence μ is a fuzzy filter of the ordered $\Gamma-$semiring M. □

Definition 3.27. Let M and N be two ordered $\Gamma-$semirings and f be a function from M into N. If μ is a fuzzy ideal of N then the pre-image of μ under f is the fuzzy subset of M, defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in M$.

Theorem 3.28. *Let $f: M \to N$ be an onto homomorphism of ordered $\Gamma-$semirings. If μ is a fuzzy filter of N then $f^{-1}(\mu)$ is a fuzzy filter of M.*

Proof. Suppose $f: M \to N$ is an onto homomorphism of ordered $\Gamma-$semirings and μ is a fuzzy filter of N and $x_1, x_2 \in M, \alpha \in \Gamma$.

$$\begin{aligned}f^{-1}(\mu)(x_1+x_2) &= \mu(f(x_1+x_2) = \mu\big(f(x_1)+f(x_2)\big) \\ &\leq \max\{\mu(f(x_1)), \mu(f(x_2))\} = \max\{f^{-1}(\mu)(x_1)), f^{-1}(\mu)(x_2))\} \\ f^{-1}(\mu)(x_1\alpha x_2) &= \mu(f(x_1\alpha x_2)) = \min\{\mu(f(x_1)), \mu(f(x_2))\} \\ &= \min\{f^{-1}(\mu)(x_1)), f^{-1}(\mu)(x_2))\}\end{aligned}$$

$$\begin{aligned}&\text{Let } x, y \in M, \text{ and } x \leq y. \\ &\Rightarrow f(x) \leq f(y) \\ &\Rightarrow \mu(f(x)) \leq \mu(f(y)) \\ &\Rightarrow f^{-1}(\mu)(x) \leq f^{-1}(\mu)(y)\end{aligned}$$

Hence $f^{-1}(\mu)$ is a fuzzy filter of the ordered $\Gamma-$semiring M. □

4. Conclusion:

We introduced the notion of fuzzy prime ideal, fuzzy filter in an ordered $\Gamma-$semiring and studied their properties and relations between them. We characterized filters in an ordered $\Gamma-$semiring using fuzzy filters. In continuous of this paper we wish to study fuzzy soft prime ideals and filters over ordered $\Gamma-$semirings.

5. Acknowledement

The author is thankful to the Editor and the Reviewers for their valuable advices for improving the paper.

References

[1] Y. B. Jun and C. Y. Lee, Fuzzy $\Gamma-$rings, Pusan Kyongnan Math. J., **8** (1991), 163-170.

[2] Y. B. Jun, J. Naggers and H. S. Kim, Normal $L-$fuzzy ideal in semirings, Fuzzy Sets and Systems, 82 (3) (1996), 383–386.

[3] N. Kuroki, On fuzzy semigroups, Information Sciences, **53** (3)(1991), 203 –236.

[4] H. Lehmer, A ternary analogue of abelian groups, American J. of Math., **59** (1932), 329–338.

[5] W. J. Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems, **8** (2) (1982), 133–139.

[6] W. G. Lister, Ternary rings, Tran. of American Math. Society, **154** (1971), 37–55.

[7] D. Mandal, Fuzzy ideals and fuzzy interior ideals in ordered semirings, Fuzzy Inf. Eng., **6** (2014), 101 –114.

[8] L.N. Mishra, S.K. Tiwari, V.N. Mishra, I.A. Khan, Unique Fixed Point Theorems for Generalized Contractive Mappings in Partial Metric Spaces, Journal of Function Spaces, Volume 2015 (2015), Article ID 960827, 8 pages.

[9] M. Murali Krishna Rao, $\Gamma-$semirings-I, Southeast Asian Bull. Math. **19** (1)(1995), 49–54.

[10] M. Murali Krishna Rao,$\Gamma-$semirings-II, Southeast Asian Bulletin of Mathematics, **21** (5), 281-287, 1997.

[11] M. Murali Krishna Rao,The Jacobson Radical of a $\Gamma-$semiring, Southeast Asian Bulletin of Mathematics **23**(1999),127-134

[12] Marapureddy Murali Krishna Rao,On$\Gamma-$Semiring with identity, Discussiones Mathematicae General Algebra and Applications, **37** (2017) 189–207, doi:10.7151/dmgaa.1276

[13] M. Murali Krishna Rao and B. Venkateswarlu, Fuzzy filters in $\Gamma-$semirings, Malaya J. Mat., **3** (1) (2015), 93-98.

[14] M. Murali Krishna Rao, B. Venkateswarlu, $L-$ fuzzy ideals in $\Gamma-$semirings, Annals of Fuzzy Mathematics and Informatics, **10** (1), (2015), 1-16.

[15] M. Murali Krishna Rao, Fuzzy soft $\Gamma-$semiring and fuzzy soft k ideal over $\Gamma-$semiring, Annals of Fuzzy Mathematics and Informatics, **9** (2)(2015), 12–25.

[16] . Murali Krishna Rao,T-fuzzy ideals in ordered semirings, Annals of Fuzzy Mathematics and Informatics , **13** (2) (2017), 253-276.

[17] . Murali Krishna Rao and B. Venkateswarlu, Regular $\Gamma-$semiring and field $\Gamma-$semiring, Novi Sad J. of Math., **45** (2) (2015), 155-171.

[18] . Murali Krishna Rao and B. Venkateswarlu, Fuzzy soft k ideals over semiring and fuzzy soft semiring homomorphism, Journal of Hyperstructures, **4** (2) (2015) 93–116.

[19] . Murali Krishna Rao and B. Venkateswarlu, An intuitionistic normal fuzzy soft k -ideal over a Γ -semiring, Annals of Fuzzy Mathematics and Informatics, **11** (3), (2016), 361 -376.

[20] . Murali Krishna Rao, Fuzzy prime ideals in ordered Γ -semirings, Joul. Int. Math. Virtual Inst. 7 (2017) 85–99.

[21] . Nobusawa, On a generalization of the ring theory, Osaka. J.Math., **1** (1964), 81 – 89.

[22] . Rosenfeld, Fuzzy groups, J. Math.Anal.Appl., **35** (1971), 512 – 517.

[23] . K. Sen, On $\Gamma-$semigroup, Proc. of International Conference of algebra and its application, (1981), Decker Publicaiton, New York, 301–308.

[24] . M. Swamy and K. L. N. Swamy, Fuzzy prime ideals of rings, J. Math. Anal. Appl., **134** (1988), 94– 103.

[25] . S. Vandiver, Note on a simple type of algebra in which cancellation law of addition does not hold, Bull. Amer. Math. Soc.(N.S.), **40** (1934), 914–920.

[26] . A. Zadeh, Fuzzy sets, Information and control, **8** (1965), 338–353.

[27] . Zhang, Prime $L-$fuzzy ideals and primary $L-$fuzzy ideals, Fuzzy Sets and Systems, **27** (3) (1988), 345–350.