

Collocation Method using Compactly Supported Radial Basis Function for Solving volterra's Population Model

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ABSTRACT. In this paper, indirect collocation approach based on compactly supported radial basis function (CSRBF) is applied for solving Volterras population model. The method reduces the solution of this problem to the solution of a system of algebraic equations. Volterras model is a non-linear integro-differential equation where the integral term represents the effect of toxin. To solve the problem, we use the well-known CSRBF: *Wendland*_{3,5}. Numerical results and residual norm ($\|R(t)\|^2$) show good accuracy and rate of convergence.

Keywords: Volterras population model, Compact support radial basis functions, Collocation method.

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1. INTRODUCTION

The Volterras model for population growth of a species within a closed system is given in [1, 2] as

$$\frac{dp}{dt} = ap - bp^2 - cp \int_0^t p(x)dx, \quad p(0) = p_0, \quad (1.1)$$

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where $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient and $c > 0$ is the toxicity coefficient. the coefficient c indicates the essential behaviour of the population evolution before its level falls to zero in the long term. p_0 is the initial population and $p = p(t)$ denotes the population at time t . This model includes the well-known terms of a logistic equation, and in addition it, includes an integral term $cp \int_0^t p(x)dx$ that characterizes the accumulated toxicity produced since time zero[2, 3]. We apply scale time and population by introducing the non-dimensional variables

$$t = \frac{tc}{b}, \quad u = \frac{pb}{a}, \quad (1.2)$$

to obtain the non-dimensional problem

$$\kappa \frac{du}{dt} = u - u^2 - u \int_0^t u(x)dx, \quad u(0) = u_0, \quad (1.3)$$

where $u(t)$ is the scaled population of identical individuals at time t and $\kappa = \frac{c}{ab}$ is a prescribed non-dimensional parameter. The only equilibrium solution of Eq. (1.3) is the trivial solution $u(t) = 0$ and the analytical solution [2]

$$u(t) = u_0 \exp\left(\frac{1}{\kappa} \int_0^t [1 - u(\tau) - \int_0^\tau u(x)dx]d\tau\right) \quad (1.4)$$

shows that $u(t) > 0$ for all t if $u_0 > 0$.

The solution of Eq. (1.1) has been of considerable concern. Although a closed form solution has been achieved in [1, 4], it was formally shown that the closed form solution cannot lead to any insight into the behaviour of the population evolution [1]. Some approximate and numerical solutions for Volterras population model have been reported. the successive approximations method was suggested for the solution of Eq. (1.3), but was not implemented. In this case, the solution $u(t)$ has a smaller amplitude compared to the amplitude of $u(t)$ for the case $\kappa \ll 1$.

TeBeest [2] obtained several numerical algorithms namely the Euler method, the modified Euler method, the classical fourth-order Runge-Kutta method and Runge-Kutta-Fehlberg method for the solution of Eq. (1.3). Moreover, a phase-plane analysis is implemented. In [2], the numerical results are correlated to give insight on the problem and its solution without using perturbation techniques. However, the performance of the traditional numerical techniques is well-known in that it using provides grid points only, and in addition, it requires a large amounts of calculations.

The series solution method and the decomposition method are implemented independently to Eq. (1.3) and to a related non-linear ordinary differential equation used in [3]. Furthermore, the Padé approximations

are used in the analysis to capture the essential behaviour of the populations $u(t)$ of identical individuals and approximation of u_{max} and exact value of u_{max} for different κ were compared. Small [4] solved the Volterra population Model by the singular perturbation method. This author scaled out the parameters of Eq. (1.1) as much as possible and considered four different ways to do this. He considered two cases $\kappa = \frac{c}{ab}$ small and $\kappa = \frac{c}{ab}$ large.

It is shown in [4] that for the case $\kappa \ll 1$, where populations are weakly sensitive to toxins, a rapid rise occurs along the logistic curve that will reach a peak and then is followed by a slow exponential decay. And, for large κ , where populations are strongly sensitive to toxins, the solutions are proportional to $sech^2(t)$.

Adomian decomposition method and Sinc-Galerkin method were compared for the solution of the same integral equation in [5]. This showed that Adomian decomposition method is more efficient and easier to use for the solution of Volterra Population Model.

Ramezani [7] applied an approach based upon composite spectral functions approximations. The properties of composite spectral functions consisting of few terms of orthogonal functions utilized to reduce the solution of the Volterra model to the solution of a system of algebraic equations.

Rational Chebyshev and Hermite functions collocation approach were compared for the solution of Volterra Population Model growth model of a species within a closed system by Parand et al. [6]. They reduced the solution of this problem to the solution. of a system of algebraic equations.

Parand et al. [8] applied two common collocation approaches based on radial basis functions to solve Volterra Population Model.

Marzban et al. [9] proposed a numerical method based on Hybrid function approximations to solve Volterra Population Model. These Hybrid functions consist of block-pulse and Lagrange-interpolating polynomials.

Also, in [10] Volterra population growth model of a species within a closed system is approximated by collocation method based on two orthogonal functions, Sinc and Rational Legendre functions. Momani et al. [11] and Xu [12] used a numerical and analytical algorithm for approximate solutions of a fractional population growth model, respectively. The first algorithm is based on Adomian decomposition method (ADM) with Pad approximants and the second algorithm is based on homotopy analysis method (HAM).

2. ICSRBF METHOD

2.1. **CSRBF.** The use of the RBF [13, 14] is an one of the popular meshfree method for solving the differential equations [15, 16]. For many years the global radial basis functions such as Gaussian, Multi quadric, Thin plate spline, Inverse multiquadric and etc was used. These functions are globally supported and generate a system of equations with ill-condition full matrix. To convert the ill-condition matrix to a well-condition matrix, CSRBFs can be used instead of global RBFs. CSRBFs[17] can convert the global scheme into a local one with banded matrices, Which makes the RBF method more feasible for solving large-scale problem [18].

Wendland's functions. The most popular family of CSRBF are Wendland functions. This function introduced by Holger Wendland in 1995 [19]. Wendland starts with the truncated power function $\phi_l(r) = (1-r)_+^l$ which be strictly positive definite and radial on \mathbb{R}^s for $l \geq \lfloor \frac{s}{2} \rfloor + 1$, and then he walks through dimension by repeatedly applying the operator I . **Definition** [20] with $\phi_l(r) = (1-r)_+^l$ we define

$$\phi_{s,k} = I^k \phi_{\lfloor \frac{s}{2} \rfloor + k + 1}, \quad (2.1)$$

it turns out that the functions $\phi_{s,k}$ are all supported on $[0,1]$.

Theorem 1 [20] The function $\phi_{s,k}$ are strictly positive definite (SPD) and radial on \mathbb{R}^s and are of the form

$$\phi_{s,k}(r) = \begin{cases} p_{s,k}(r) & r \in [0, 1], \\ 0 & r > 1, \end{cases}$$

with a univariate polynomial $p_{s,k}$ of degree $\lfloor \frac{s}{2} \rfloor + 3k + 1$. Moreover, $\phi_{s,k} \in C^{2k}(R)$ are unique up to a constant factor, and the polynomial degree is minimal for given space dimension s and smoothness $2k$ [20]. Wendland gave recursive formulas for the functions $\phi_{s,k}$ for all s, k . We instead list the explicit formulas of [21].

Theorem 2 [20] The function $\phi_{s,k}$, $k = 0, 1, 2, 3$, have form

$$\begin{aligned} \phi_{s,0} &= (1-r)_+^l, \\ \phi_{s,1} &\doteq (1-r)_+^{l+1}[(l+1)r+1], \\ \phi_{s,2} &\doteq (1-r)_+^{l+2}[(l^2+4l+3)r^2+(3l+6)r+3], \\ \phi_{s,3} &\doteq (1-r)_+^{l+3}[(l^3+9l^2+23l+15)r^3+(6l^2+36l+45)r^2 \\ &\quad +(15l+45)r+15], \end{aligned}$$

where $l = \lfloor \frac{s}{2} \rfloor + k + 1$, and the symbol \doteq denotes equality up to a multiplicative positive constant.

The case $k = 0$ follows directly from the definition. application of the

definition for the case $k = 1$ yields

$$\begin{aligned}\phi_{s,1}(r) &= (I\phi_l)(r) = \int_r^\infty t\phi_l(t)dt \\ &= \int_r^\infty t(1-t)_+^l dt = \int_r^1 t(1-t)^l dt \\ &= \frac{1}{(l+1)(l+2)}(1-r)^{l+1}[(l+1)r+1],\end{aligned}$$

where the compact support of ϕ_l reduces the improper integral to a definite integral which can be evaluated using integration by parts. The other two cases are obtained similarly by repeated application of I . [20] We showed the most of wendland functions in table 1 . Wu's and Buh-

TABLE 1. Wendland's compactly supported radial function for various choices of k and $s=3$.

$\phi_{s,k}$	smoothness	SPD
$\phi_{3,0}(r) = (1-r)_+^2$	C^0	\mathbb{R}^3
$\phi_{3,1}(r) \doteq (1-r)_+^4(4r+1)$	C^2	\mathbb{R}^3
$\phi_{3,2}(r) \doteq (1-r)_+^6(35r^2+18r+3)$	C^4	\mathbb{R}^3
$\phi_{3,3}(r) \doteq (1-r)_+^8(32r^3+25r^2+8r+1)$	C^6	\mathbb{R}^3
$\phi_{3,4}(r) \doteq (1-r)_+^{10}(429r^4+450r^3+210r^2+50r+5)$	C^8	\mathbb{R}^3
$\phi_{3,5}(r) \doteq (1-r)_+^{12}(2048r^5+2697r^4+1644r^3+566r^2+108r+9)$	C^{10}	\mathbb{R}^3

mann's functions are the other kind of CSRBFs [22, 23]. For obtaining Wu's functions operator \mathbf{D} is used on convolution function

$$\phi(r) = (1-r^2)_+^l, l \in \mathbb{N}$$

and Buhmann's functions contain a logarithmic term in addition to a polynomial. Buhmann's functions have the general form

$$\phi(r) = \int_0^\infty \left(1 - \frac{r^2}{t}\right)_+^\lambda t^\alpha (1-t^\delta)_+^\rho dt.$$

where $0 < \delta \leq 0.5$ and $\rho \geq 1$. α and λ values change on construting SPD functions on \mathbb{R}^s for different s [20].

2.2. Interpolation by CSRBFs. The one-dimensional function $y(x)$ to be interpolated or approximated can be represented by an CSRBF as

$$y(x) \approx y_n(x) = \sum_{i=1}^N \xi_i \phi_i(x) = \Phi^T(x) \Xi,$$

where

$$\begin{aligned}\phi_i(x) &= \phi\left(\frac{\|x - x_i\|}{r_\omega}\right), \\ \Phi^T(x) &= [\phi_1(x), \phi_2(x), \dots, \phi_N(x)], \\ \Xi &= [\xi_1, \xi_2, \dots, \xi_N]^T,\end{aligned}\tag{2.2}$$

x is the input, r_ω is the local support domain and ξ_i s are the set of coefficients to be determined. By using the local support domain, we mapped the domain of problem to CSRBF local domain. By choosing N interpolate nodes ($x_j, j = 1, 2, \dots, N$) in domain:

$$y_j = \sum_{i=1}^N \xi_i \phi_i(x_j) (j = 1, 2, \dots, N).$$

To summarize the discussion on the coefficients matrix, we define

$$A\Xi = Y,\tag{2.3}$$

where :

$$\begin{aligned}Y &= [y_1, y_2, \dots, y_N]^T, \\ A &= [\Phi^T(x_1), \Phi^T(x_2), \dots, \Phi^T(x_N)]^T \\ &= \begin{pmatrix} \phi_1(x_1) & \phi_2(x_1) & \dots & \phi_N(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \dots & \phi_N(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_N) & \phi_2(x_N) & \dots & \phi_N(x_N) \end{pmatrix}.\end{aligned}$$

Note that $\phi_i(x_j) = \phi\left(\frac{\|x_i - x_j\|}{r_\omega}\right)$, by solving the system (2.3), the unknown coefficients ξ_i will be achieved.

2.3. ICSRBF method. In the indirect method, the formulation of the problem starts with the decomposition of the highest order derivative under consideration into CRBF. The obtained derivative expression is then integrated to yield expressions for lower order derivatives and finally for the original function itself.

We approximate $\frac{du}{dt}$ for solving the model by ICSRBF:

$$\frac{du}{dt} \simeq \hat{u}_N(t) = \sum_{i=1}^N \xi_i \phi_i(t) = \Phi^T(t)\Xi,\tag{2.4}$$

by using integral operator $I_{\vartheta}f(t) = \int_0^t f(x)dx$ we have

$$\int_0^t du \simeq \int_0^t \hat{u}_N(v)dv = I_{\vartheta}\Phi^T(t)\Xi,$$

$$u(t) = I_{\vartheta}\Phi^T(t)\Xi + u_0, \tag{2.5}$$

$$I_{\vartheta}u = I_{\vartheta}^2\Phi^T(t)\Xi + u_0t. \tag{2.6}$$

Now, to obtain $\{\xi_i\}_{i=1}^N$ we define the residual functions by substituting Eqs. (1.4)-(2.6) in Eq. (1.3)

$$\hat{R}(t) = \kappa\Phi^T(t)\Xi - (I_{\vartheta}\Phi^T(t)\Xi + u_0)(1 - I_{\vartheta}^T(t)\Xi - u_0 - I_{\vartheta}^2\Phi^T(t)\Xi - u_0t). \tag{2.7}$$

The set of equations for obtaining the coefficients $\{\xi_i\}_{i=1}^N$ come from equalizing Eq. (2.6) to zero at N interpolate nodes $\{t_i\}_{i=1}^N$ from $t_j = L(\frac{j}{N})^{\rho}$, $j = 1, 2, \dots, N$ where L is a last interpolate node and ρ is a arbitrary parameter.

$$\hat{R}(t_j) = 0, \quad j = 1, 2, \dots, N. \tag{2.8}$$

3. APPLICATION

We applied the method presented in this paper to examine the mathematical structure of $u(t)$. Table (2) shows the maximum of $u(x)$ for some κ and $u_0 = 0.1$ by using in comparison with exact solution and ADM solution by Wazwaz [3]. The resulting graph of Eq. (1.3) is shown in Fig. (1).

TABLE 2. A comparison of ADM[3] and the present method with exact values for u_{max} .

κ	u_{max}	r_{ω}	ρ	N	ICSRBF	ADM
0.02	0.9234271	1	1.766000	15	0.92342716	0.9234270
0.04	0.8737199	1	1.780000	18	0.8737193	0.8612401
0.1	0.7697414	1	1.811000	18	0.7697414	0.7651130
0.2	0.6590503	2	1.032770	18	0.6590493	0.6579123
0.5	0.4851902	2	1.114035	27	0.4851903	0.4852823

The local support domain r_{ω} and arbitrary parameter ρ must be specified by the user. An important unsolved problem is to find a approach to determine the optimal size of r_{ω} [18]. The accuracy of these CSRBF depends on the choice of r_{ω} and ρ . By the meaning of residual function in case of Eq. (2.6), we try to minimize $\|R(t)\|^2$ by choosing good r_{ω} and ρ parameters. We define $\|R(t)\|^2$ as

$$\|R(t)\|^2 = \int_0^L R^2(t)dt \simeq \frac{L}{2} \sum_{j=0}^m \omega_j R^2(\frac{L}{2}s_j + \frac{L}{2}), \tag{3.1}$$

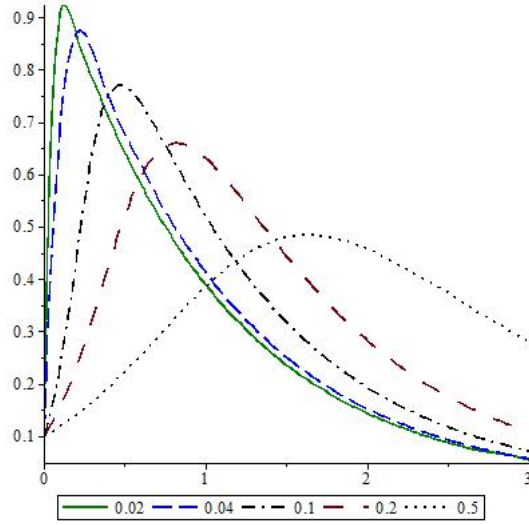


FIGURE 1. Plot of approximate solutions of Eq. (1.3) for $u_0 = 0.1$ and $\kappa = 0.02, 0.04, 0.1, 0.2, 0.5$.

TABLE 3. Minimum value of $\|Res\|^2$ which is obtained with r_ω and ρ for ICSRBF.

κ	$\ Res\ ^2$ -ICSRBF
0.5	$2.11e - 08$
0.2	$2.87e - 07$
0.1	$1.34e - 07$
0.04	$7.86e - 05$
0.02	$4.41e - 05$

were

$$\omega_j = \frac{L}{(1 - s_j^2)(\frac{d}{dx}P_{m+1}(s)|_{s=s_j})^2}, \quad j = 0, 1, \dots, m, \quad P_{m+1}(s_j) = 0, \quad j = 0, 1, \dots, m,$$

$P_{m+1}(x)$ is $(m + 1)$ th-order Legendre polynomial. Table (3) show the minimum of $\|R(t)\|^2$ which is obtained with local support domain r_ω and arbitrary parameter ρ .

4. CONCLUSION

A method has been presented for solving Volterra's population model which is an integro-ordinary differential equation, based on the compactly supported radial basis functions approximation. In this work, we applied two common ICSRBF methods on the Volterra's population

model without converting it to an ordinary differential equation. We used $Wendland_{3,5}$ function. This function are proposed to provide an effective but simple way to improve the convergence of the solution by collocation method. As appeared from the Figures, we have shown that, when the constant $\kappa = \frac{c}{ab}$ is small, this type of population is relatively insensitive to toxins, and when $c = ab$ is large, population of this type are extremely sensitive to toxins. Additionally, through the comparison with ADM, we have showed that the ICSRBF approach have good reliability and efficiency.

5. References

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