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Existence of infinitely many solutions for coupled system of Schrödinger-Maxwell's equations

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Abstract. In this paper we study the existence of infinitely many large energy solutions for the coupled system of Schrödinger-Maxwell's equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = H_v(x,u,v) & \text{in } \mathbb{R}^3 \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3 \\ \\ -\Delta v + V(x)v + \psi v = H_u(x,u,v) & \text{in } \mathbb{R}^3 \\ -\Delta \psi = v^2 & \text{in } \mathbb{R}^3, \end{cases}$$

via the Fountain theorem under Cerami condition. More precisely, we consider the More general case and weaken V_1^* with respect to the condition V_1 in [6].

Keywords: SchrödingerMaxwell system ,Cerami condition,Variational methods, Strongly indefinite functionals.

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1. Introduction

In this paper, we study the nonlinear coupled system of Schrödinger-Maxwell's equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = H_v(x,u,v) & \text{in } \mathbb{R}^3 \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3 \\ \\ -\Delta v + V(x)v + \psi v = H_u(x,u,v) & \text{in } \mathbb{R}^3 \\ -\Delta \psi = v^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $V \in C(\mathbb{R}^3, \mathbb{R})$ and $H \in C^1(\mathbb{R}^3, \mathbb{R})$ which are satisfied in some suitable conditions.

In the classical model, the interaction of a charge particle with an electromagnetic field can be described by the nonlinear Schrödinger-Maxwell's equations. In this article, we want to study the interaction of two charge particles Simultaneously with same potential function $V(x)$ and different scalar potential ϕ and ψ which are satisfied in suitable conditions. For more details on the physical aspects see [1] and the references therein. More precisely, we have to solve the system 1.1 if we want to find electrostatic-type solutions.

In [2] Zhang et al. considered the system 1.1 without the scalar potential ϕ and ψ, so-called the Hamiltonian elliptic system

$$\begin{cases} -\Delta u + V(x)u = H_v(x,u,v) & \text{in } \mathbb{R}^3 \\ \\ -\Delta v + V(x)v = H_u(x,u,v) & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

They assumed the potential $V(x)$ is non-periodic and sing changing, and $H(x,z)$ is non-periodic and asymptotically quadratic in $z = (u,v) : \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}$. For the case of a bounded domain the system 1.2 were studied by many authors [3, 4, 5] and the references therein.

In [6] Li and Chen, studied the existence of infinitely many large energy solutions for the superlinear Schrödinger-Maxwell'equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3 \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

They assumed on this paper that the potential function $V(x)$ is bounded from below with a positive constant and they are used to solve the problem 1.3, the Fountain theorem in critical point theory. In [7] Schaftingen et al. studied positive bound states for the equation

$$\left\{ -\epsilon^2 \Delta u + V(x)u = K(x)f(u) \quad \text{in } \mathbb{R}^N, \right. \tag{1.4}$$

where $\epsilon > 0$ is real parameter and V and K are radial positive potentials. The problem 1.4 is Nonlinear Schrödinger equation with unbounded or vanishing potentials. The equations kind of 1.4 can be considered as nonlinear parametric Schrödinger equation which were studied by many authors see [8, 9, 10]. The infinitely many large solutions for the equation 1.3 are obtained in [11] with the following variant Ambrosetti-Rabinowitz type condition [6], there exist $\mu > 4$ such that for any $s \in \mathbb{R}$ and $x \in \mathbb{R}^3$

$$\mu F(x,u) := \mu \int_0^s f(x,t)dt \leq sf(x,s).$$

Existence of solutions are obtained via Fountain theorem in critical point theory. More precisely, in this paper we consider the more general case and weaken the condition of V_1 in [6] and we assume that the potential V is non-periodic and sing changing. We assume the following conditions :

V_1^*) $V \in C(\mathbb{R}^3, \mathbb{R})$ and there exists some $M > 0$ such that the set $\Omega_M = \{x \in \mathbb{R}^3 \ ; \ V(x) \leq M\}$ is not nonempty and has finite Lebesgue measure.

H_1) $H \in C^1(\mathbb{R}^3 \times \mathbb{R}^2, \mathbb{R})$ and for some $2 < p < 2^* = 6$, and $M_1, M_2 > 0$,

$$|H_u(x,u,v)| \leq M_1|u| + M_1|u|^{p-1} \ and \ |H_v(x,u,v)| \leq M_2|v| + M_2|v|^{p-1},$$

for a.e $x \in \mathbb{R}^3$ and $u, v : \mathbb{R}^3 \to \mathbb{R}$, and also

$$\lim_{u\to 0} \frac{H_u(x,u,v)}{u} = 0 \ and \ \lim_{u\to 0} \frac{H_v(x,u,v)}{v} = 0,$$

uniformly for $x \in \mathbb{R}^3$ and $u, v \in \mathbb{R}$.

H_2) $\lim\limits_{|(u,v)|\to\infty} \frac{H(x,u,v)}{|(u,v)|^4} = +\infty$, uniformly in $x \in \mathbb{R}^3$ and $(u,v) \in \mathbb{R}^2$ and

$$H(x,0,0) = 0, H(x,u,v) \geq 0$$

for all $(x,u,v) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$.

H_3) There exists a constant $\theta \geq 1$ such that

$$\theta \hat{H}(x,u,v) \geq \hat{H}(x,su,sv)$$

for all $x \in \mathbb{R}^3, (u,v) \in \mathbb{R}^2$ and $t, s \in [0,1]$, where

$$\hat{H}(x,u,v) = H_u(x,tu,v)tu + H_v(x,u,sv)sv - 4H(x,tu,sv).$$

H_4) $H(x,-u,v) = H(x,u,v)$ and $H(x,u,-v) = H(x,u,v)$ for all $x \in \mathbb{R}^3$ and $(u,v) \in \mathbb{R}^2$.

Here, we express the Cerami condition which was established by G. Cerami in [12]

Definition 1.1. Suppose that functional I is C^1 and $c \in \mathbb{R}$, if any sequence $\{u_n\}$ satisfying $I(u_n) \to c$ and $(1 + \|u_n\|)I^{'}(u_n) \to 0$ has a convergence subsequence, we say the I satisfies Cerami condition at the level c.

To approach the main result, we need the following critical point theorem.

Theorem 1.2. *(Fountain theorem under Cerami condition) Let X be a Banach space with the norm $\|.\|$ and let X_j be a sequence of subspace of X with $dimX_j < \infty$ for any $j \in \mathbb{N}$. Further, $X = \overline{\oplus_{j\in\mathbb{N}} Xj}$, the closure of the direct sum of all X_j. Set $W_k = \oplus_{j=0}^{k} X_j$, $Z_k \overline{\oplus_{j=k}^{\infty} X_j}$. Consider an even functional $I \in C^1(X, \mathbb{R})$, that is , $I(-u) = I(u)$ for any $u \in X$. Also suppose that for any $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

I_1) $a_k := \max_{u\in W_k,\|u\|=\rho_k} I(u) \leq 0$,

I_2) $b_k := \inf_{u\in Z_k,\|u\|=r_k} I(u) \to +\infty$ as $k \to \infty$,

I_3) the Cerami condition holds at any level $c > 0$. Then the functional I has an unbounded sequence of critical values.

Now, our main result is the following :

Theorem 1.3. *Let V_1^*, $H_1 - H_4$ be satisfied. Then the system 1.1 has infinitely many solutions $\{((u_k, \phi_k), (v_k, \psi_k))\}$ in product space $Y_{HD} \times Y_{HD}$ (see section 2)which satisfies in*

$$\frac{1}{2}\int\limits_{\mathbb{R}^3} [\ |\nabla u_k|^2 + |\nabla v_k|^2 + V(x)(u_k^2 + v_k^2)\ ]dx - \frac{1}{4}\int\limits_{\mathbb{R}^3} [\ |\nabla \phi_k|^2 \ + \ |\nabla \psi_k|^2\ ]dx$$

$$+ \ \frac{1}{2}\int\limits_{\mathbb{R}^3} [\ \phi_k u_k^2 + \ \psi_k v_k^2\ ]dx - \int\limits_{\mathbb{R}^3} H(x, u, v)dx \to +\infty.$$

Remark 1.4. The assumption V_1^* implies that the potential V is not periodic and changes sing. This is different from condition V_1 in [6]. Also conditions H_1 and H_3 modified for the system 1.1 with respect to the system 1.1 which considered by Li and Chen in [6].

2. SOME AUXILIARY RESULTS AND NOTATIONS

In this section we give some notations and definitions on the function product space. We set

$$H^1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) \mid |\nabla u| \in L^2(\mathbb{R}^3)\}, \tag{2.1}$$

with the norm

$$\|u\|_{H^1} := (\int\limits_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx)^{\frac{1}{2}}, \tag{2.2}$$

and we consider the function space

$$D^{1,2}(\mathbb{R}^3) := \{u \in L^{2^*}(\mathbb{R}^3) \mid |\nabla u| \in L^2(\mathbb{R}^3)\}, \tag{2.3}$$

with the norm

$$\|u\|_{D^{1,2}} := (\int_{\mathbb{R}^3} |\nabla u|^2 dx)^{\frac{1}{2}}. \tag{2.4}$$

Now, we consider the function space

$$E := \{u \in H^1(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx < \infty\}.$$

Then E is Hilbert space [13] with the inner product

$$(u, v)_E := \int_{\mathbb{R}^3} (\nabla u.\nabla v + V(x)uv)dx \tag{2.5}$$

and the norm $\|u\|_E := (u, u)_E^{\frac{1}{2}}$. We set

$$X_E := E \times E, Y_{HD} := H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \textit{ and } Z_{ED} := E \times D^{1,2}(\mathbb{R}^3).$$

Hence , we can define an inner product on X_E as

$$((u, v)(w, z))_{X_E} := (u, w)_E + (v, z)_E \tag{2.6}$$

and the corresponding norm on X_E by this inner product as following

$$\|(u, v)\|_{X_E} := (\|u\|_E^2 + \|v\|_E^2)^{\frac{1}{2}} = ((u, u)_E + (v, v)_E)^{\frac{1}{2}}. \tag{2.7}$$

Let H be a Hilbert space and $\mathcal{H} = H \times H$ with the inner product

$$((u, v), (w, z))_{\mathcal{H}} = (u, w)_H + (v, z)_H$$

and the corresponding generated norm by this inner product. Suppose that T is operator on Hilbert space H. We assume that there is an orthogonal decomposition

$$H = H^- \oplus H^+ \oplus H^0$$

such that T is negative definite (resp. positive definite) in $H^-(resp.H^+)$ and $H^0 = kerT$. Let $\mathcal{H}^+ = H^+ \times H^-$, $\mathcal{H}^- = H^- \times H^+$ and $\mathcal{H}^0 = H^0 \times H^0$. Then for any $z = (u, v) \in \mathcal{H}$ we have $z = z^- + z^+ + z^0$, where $z^+ = (u^+, v^-)$, $z^- = (u^-, v^+)$ and $z^0 = (u^0, v^0)$. Therefore, $\mathcal{H}^+, \mathcal{H}^-$ and $\mathcal{H}^0$ are orthogonal. Hence, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \oplus \mathcal{H}^0$, for details see [2].

Lemma 2.1. [2] *If V_1^* holds. Then $\mathcal{H} \hookrightarrow L^p(\mathbb{R}^N, \mathbb{R}^2)$ is continuous for $p \in [2, 2^*]$ and $\mathcal{H} \hookrightarrow L^p_{loc}(\mathbb{R}^N, \mathbb{R}^2)$ is compact for $p \in [2, 2^*)$.*

Therefore, the system 1.1 is the Euler-Lagrange equations of the functional

$$J : Z_{ED} \times Z_{ED} \to \mathbb{R}$$

define by

$$J((u,\phi),(v,\psi)) := \frac{1}{2}\|(u,v)\|^2_{X_E} - \frac{1}{4}\int\limits_{\mathbb{R}^3}[\ |\nabla\phi|^2 + |\nabla\psi|^2\ ]dx$$

$$+\frac{1}{2}\int\limits_{\mathbb{R}^3}[\ \phi u^2 + \psi v^2\ ]dx - \int\limits_{\mathbb{R}^3} H(x,u,v)dx. \quad (2.8)$$

The functional $J \in C^1(Z_{ED} \times Z_{ED}, \mathbb{R})$ and its critical points are the solutions of system 1.1.

Remark 2.2. [6] the functional J exhibits a strong indefiniteness, that is, it is unbounded both from below and from above on infinitely dimensional subspaces. This indefiniteness can be removed using the reduction method described in [14], by which we are led to study a one variable functional that does not present such a strongly indefiniteness nature. For any $u \in E$ the Lax-Milgram theorem [15] implies there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u = u^2$$

in a distributional (or weak) sense. Then we can obtain an integral expression for ϕ_u :

$$\phi_u = \frac{1}{4\pi}\int\limits_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}dy, \quad (2.9)$$

for any $u \in E$.

Lemma 2.3. [11] *For any* $u \in E$

i) $\|\phi_u\|_{D^{1,2}} \leq M_3\|u\|^2_{L^{\frac{12}{5}}}$, *where* M_3 *is positive constant which does not depend on* u. *In particular, there exists positive constant* M_4 *such that*

$$\int\limits_{\mathbb{R}^3}\phi_u u^2 dx \leq M_4\|u\|^4_E; \quad (2.10)$$

ii) $\phi_u \geq 0$.

Now, by the lemma 2.3 we define the functional $I : X_E \to \mathbb{R}$ by

$$I(u,v) := J((u,\phi_u),(v,\psi_v)).$$

Remark 2.4. Using the relation $-\Delta\phi_u = u^2$ and integration by parts, we can obtain

$$\int\limits_{\mathbb{R}^3}|\nabla\phi_u|^2 dx = \int\limits_{\mathbb{R}^3}\phi_u u^2 dx.$$

Then, we can consider the functional 2.8 as following

$$I(u,v)=\frac{1}{2}\|(u,v)\|^2_{X_E}+\frac{1}{4}\int\limits_{\mathbb{R}^3}[\ \phi_u u^2+\psi_v v^2\ ]dx-\int\limits_{\mathbb{R}^3}H(x,u,v)dx. \quad (2.11)$$

It is well-known that I is C^1-functional with derivative given by

$$\langle I^{'}(u,v),(u,v)\rangle=\int\limits_{\mathbb{R}^3}[\ \nabla u.\nabla u+V(x)u^2+\phi_u u^2-H_u(x,u,v)\ ]dx$$

$$+\int\limits_{\mathbb{R}^3}[\ \nabla v.\nabla v+V(x)v^2+\psi_v v^2-H_v(x,u,v)\ ]dx. \quad (2.12)$$

Now, using the proposition 2.3 in [6] we can consider the following proposition for our functional J :

Proposition 2.5. *The following statements are equivalent :*
i) $((u,\phi_u),(v,\psi_v))\in Z_{ED}\times Z_{ED}$ *is a critical point of* J;
ii) (u,v) *is a critical point of functional* I *and* $(\phi,\psi)=(\phi_u,\psi_v)$.

Proof. It follows using the remark 2.2 and theorem 2.3 in [14]. □

3. PROOF OF MAIN THEOREM

We take an orthogonal basis $\{(e_i,e_j)\}$ of product space $X:=X_E=E\times E$ and we define $W_k:=span\{(e_i,e_j)\}_{i,j=1,\dots,k}$, $Z_k:=W_k^{\perp}$.

Lemma 3.1. [11] *for any* $p\in[2,2^*)$,

$$\beta_k:=\sup_{u\in Z_k,\|u\|=1}\|u\|_{L^p}\to 0$$

as $k\to\infty$.

Now, we prove that the functional $I:X_E\to\mathbb{R}$ satisfies the Cerami condition .

Proposition 3.2. *under the conditions* H_1-H_3, *the functional* $I(u,v)$ *satisfies the Cerami condition at any positive level.*

Proof. Let $\{(u_n,v_n)\}$ be a sequence in X_E such that for some $c\in\mathbb{R}$,

$$I(u_n,v_n)=\frac{1}{2}\|(u_n,v_n)\|^2_{X_E}+\frac{1}{4}\int\limits_{\mathbb{R}^3}[\ \phi_{u_n}u_n^2+\psi_{v_n}v_n^2\ ]dx-\int\limits_{\mathbb{R}^3}H(x,u_n,v_n)dx\to c \quad (3.1)$$

and

$$(1+\|(u_n,v_n)\|_{X_E})I^{'}(u_n,v_n)\to 0,\ as\ n\to\infty. \quad (3.2)$$

From 3.1 and 3.2 for n large enough

$$1+c\geq I(u_n,v_n)-\frac{1}{4}\langle I^{'}(u_n,v_n),(u_n,v_n)\rangle=$$

$$\frac{1}{2}\|(u_n,v_n)\|^2_{X_E}+\frac{1}{4}\int\limits_{\mathbb{R}^3}[\ \phi_{u_n}u_n^2+\psi_{v_n}v_n^2\ ]dx-\int\limits_{\mathbb{R}^3}H(x,u_n,v_n)dx-$$
$$\frac{1}{4}[\int\limits_{\mathbb{R}^3}[\ \nabla u_n.\nabla u_n+V(x)u_n^2+\phi_{u_n}u_n^2-H_{u_n}(x,u_n,v_n)u_n\ ]dx+$$
$$\int\limits_{\mathbb{R}^3}[\ \nabla v_n.\nabla v_n+V(x)v_n^2+\psi_{v_n}v_n^2-H_{v_n}(x,u_n,v_n)v_n\ ]dx\ ]$$
$$\frac{1}{2}\|(u_n,v_n)\|^2_{X_E}-\int\limits_{\mathbb{R}^3}H(x,u_n,v_n)dx-\frac{1}{4}[\int\limits_{\mathbb{R}^3}[\ \nabla u_n.\nabla u_n+V(x)u_n^2\ ]dx+\int\limits_{\mathbb{R}^3}[\ \nabla v_n.\nabla v_n+V(x)v_n^2\ ]dx\ ]$$
$$+\frac{1}{4}[\int\limits_{\mathbb{R}^3}H_{u_n}(x,u_n,v_n)u_ndx+\int\limits_{\mathbb{R}^3}H_{v_n}(x,u_n,v_n)v_ndx\ ]=$$
$$\frac{1}{4}\|(u_n,v_n)\|^2_{X_E}+\frac{1}{4}[\int\limits_{\mathbb{R}^3}H_{u_n}(x,u_n,v_n)u_ndx+\int\limits_{\mathbb{R}^3}H_{v_n}(x,u_n,v_n)v_ndx\ ]-\int\limits_{\mathbb{R}^3}H(x,u_n,v_n)dx. \tag{3.3}$$

Now, we shall show that the sequence $\{(u_n,v_n)\}$ is bounded. Suppose that $\|(u_n,v_n)\|_{X_E}\to\infty$ as $n\to\infty$. Then we consider

$$(w_n,z_n):=(\frac{u_n}{\|u_n\|},\frac{v_n}{\|v_n\|})\in X_E,$$

so the sequence $\{(w_n,z_n)\}$ is bounded. For some $(w,z)\in X_E$, and subsequence of (w_n,z_n), we can imply that

$$(w_n,z_n)\rightharpoonup(w,z)\ as\ n\to\infty\ in\ X_E,$$

and

$$(w_n,z_n)\to(w,z)\ in\ L^t(\mathbb{R}^3)\times L^s(\mathbb{R}^3)\ for\ t,s\in[2,2^*)$$

and

$$w_n(x)\to w(x)\ and\ z_n(x)\to z(x) \tag{3.4}$$

for a.e. $x\in\mathbb{R}^3$. Now, we consider two cases. In first case we suppose that $(w,z)\neq(0,0)$ in X_E. By dividing relation 3.1 with $\|(u_n,v_n)\|^4_{X_E}$ and lemma 2.3 we get that

$$\int\limits_{\mathbb{R}^3}\frac{H(x,u_n,v_n)}{\|(u_n,v_n)\|^4_{X_E}}dx=\frac{1}{2\|(u_n,v_n)\|^2_{X_E}}+\int\limits_{\mathbb{R}^3}\frac{(\phi_{u_n}u_n^2+\psi_{v_n}v_n^2)dx-c}{4\|(u_n,v_n)\|^4_{X_E}}$$
$$+O(\|(u_n,v_n)\|^{-4}_{X_E})\le M_5<\infty, \tag{3.5}$$

where M_5 is a positive constant. we consider $\Omega:=\{x\in\mathbb{R}^3\mid w(x),z(x)\neq 0\}$. By condition H_2 for all $x\in\Omega$,

$$\frac{H(x,u_n,v_n)}{\|(u_n,v_n)\|^4_{X_E}}=\frac{H(x,u_n,v_n)}{|(u_n,v_n)|^4}(w_n^4(x),z_n^4(x))\to+\infty,\ as\ n\to\infty.$$

Since $|\Omega| > 0$, by Fatou's lemma,

$$\int_{\mathbb{R}^3} \frac{H(x, u_n, v_n)}{\|(u_n, v_n)\|^4_{X_E}} dx \to +\infty \ as \ n \to \infty.$$

This contradicts with 3.5. In second case, we assume that $(w_n z_n) = (0, 0)$ then we define sequence $(t_n, s_n) \in \mathbb{R}^2$ by

$$I(t_n u_n, s_n v_n) \max_{(t,s)\in[0,1]\times[0,1]} I(tu, sv).$$

Let $m > 0$ be fixed and let

$$(\bar{w}, \bar{z}) := (\sqrt{4m}\frac{u_n}{\|u_n\|_E}, \sqrt{4m}\frac{u_n}{\|u_n\|_E}) = 2\sqrt{m}(w_n, z_n) \in X_E.$$

By condition H_1 we have

$|H_u(x, u, v)| \leq M_1|u|^2 + M_1|u|^{p-1}$ *and* $|H_v(x, u, v)| \leq M_2|v|^2 + M_2|u|^{p-1}$

for any $x \in \mathbb{R}^3$ and $(u, v) \in \mathbb{R}^2$, and

$$H(x, u, v) = \int_0^1 H_u(x, tu, v)tudt + \int_0^1 H_v(x, u, sv)svds.$$

Therefore,

$$H(x, u, v) \leq \frac{M_1}{2}|u|^2 + \frac{M_1}{p}|u|^p + \frac{M_2}{2}|v|^2 + \frac{M_2}{p}|v|^p. \tag{3.6}$$

Using the relation 3.5 we get that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} H(x, \bar{w}_n, \bar{z}_n)dx \leq \lim_{n\to\infty}[\int_{\mathbb{R}^3} \frac{M_1}{2}|\bar{w}_n|^2 + \frac{M_1}{p}|\bar{w}_n|^p + \frac{M_2}{2}|\bar{z}_n|^2 + \frac{M_2}{p}|\bar{z}_n|^p \ ]dx$$

$$\leq \lim_{n\to\infty} M_6 \int_{\mathbb{R}^3} [\ |\bar{w}_n|^2 + |\bar{z}_n|^2 \ ]dx + M_7 \int_{\mathbb{R}^3} [\ |\bar{w}_n|^p + |\bar{z}_n|^p \ ]dx,$$

where $M_6 := \max\{\frac{M_2}{2}, \frac{M_3}{2}\}$ and $M_7 := \max\{\frac{M_2}{p}, \frac{M_3}{p}\}$. Hence,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} H(x, \bar{w}_n, \bar{z}_n)dx \leq \lim_{n\to\infty} M_6 \int_{\mathbb{R}^3} [\ |\bar{w}_n|^2 + |\bar{z}_n|^2 \ ]dx + \lim_{n\to\infty} M_7 \int_{\mathbb{R}^3} [\ (|\bar{w}_n|^2 + |\bar{z}_n|^2)^p \ ]dx$$

$$\lim_{n\to\infty} M_6 \int_{\mathbb{R}^3} |(\bar{w}_n, \bar{z}_n)|^2 dx + M_7 \int_{\mathbb{R}^3} (|(\bar{w}_n, \bar{z}_n)|^2)^p dx = 0. \tag{3.7}$$

Therefore, for n large enough,

$$I((t_n u_n, s_n v_n)) \geq I((\bar{w}_n, \bar{z}_n)) = 2m + \frac{1}{4}\int_{\mathbb{R}^3} [\ \phi_{\bar{w}_n}\bar{w}_n^2 + \psi_{\bar{z}_n}\bar{z}_n^2 \ ]dx$$

$$- \int_{\mathbb{R}^3} H(x, \bar{w}_n, \bar{z}_n)dx \geq m. \tag{3.8}$$

By relation 3.8, $\lim_{n\to\infty} I(t_nu_n, s_nv_n) = +\infty$. Since $I(0,0) = 0$ and $I(u_n, v_n) \to c$ then $t_ns_n \in (0,1)$. Hence, for $t_n, s_n \in (0,1)$ and n large enough we obtain

$$\int_{\mathbb{R}^3} \nabla t_nu_n.\nabla t_nu_n + V(x)t_nu_nt_nu_n + \phi_{t_nu_n}t_nu_nt_nu_n - H_{t_nu_n}(x, t_nu_n, v_n)t_nu_ndx$$

$$+\int_{\mathbb{R}^3} \nabla s_nv_n.\nabla s_nv_n + V(x)s_nv_ns_nv_n + \phi_{s_nv_n}s_nv_ns_nv_n - H_{s_nv_n}(x, u_n, s_nv_n)s_nv_ndx$$

$$= \langle I^{'}((t_nu_n, s_nv_n)), (t_nu_n, s_nv_n)\rangle = 0.$$

Then by H_3 we can get that

$$I((u_n, v_n)) - \frac{1}{4}\langle I^{'}(u_n, v_n), (u_n, v_n)\rangle =$$

$$\frac{1}{4}\|(u_n, v_n)\|^2_{X_E} + \frac{1}{4}\int_{\mathbb{R}^3} [H_{u_n}(x, u_n, v_n)u_n + H_{v_n}(x, u_n, v_n)]dx - \int_{\mathbb{R}^3} H(x, u_n, v_n)dx =$$

$$\frac{1}{4}\|(u_n, v_n)\|^2_{X_E} + \frac{1}{4}\int_{\mathbb{R}^3} \hat{H}(x, u_n, v_n)dx \geq \frac{1}{4\theta}\|(u_n, v_n)\|^2_{X_E} +$$

$$\frac{1}{4\theta}\int_{\mathbb{R}^3} [H_{t_nu_n}(x, t_nu_n, v_n)t_nu_n + H_{s_nv_n}(x, u_n, s_nv_n)s_nv_n]dx - \int_{\mathbb{R}^3} H(x, t_nu_n, s_nv_n)dx$$

$$\frac{1}{\theta}I((t_nu_n, s_nv_n)) - \frac{1}{4\theta}\langle I^{'}((t_nu_n, s_nv_n)), (t_nu_n, s_nv_n)\rangle \to +\infty.$$

This is contradiction with 3.3. Hence, the sequence $\{(u_n, v_n)\}$ is bounded in X_E. Since $(u_n, v_n) \hookrightarrow (u, v)$ in X_E, so by lemma 2.1, $(u_n, v_n) \to (u, v)$ in for any $s, t \in [2, 2^*)$. By 2.12 we can get that

$$\|(u_n, v_n) - (u, v)\|^2_{X_E} = \|(u_n - u, v_n - v)\|^2_{X_E} = ((u_n - u, v_n - v), (u_n - u, v_n - v))_{X_E} =$$

$$\int_{\mathbb{R}^3} \nabla(u_n - u).\nabla(u_n - u)V(x)(u_n - u)^2dx + \int_{\mathbb{R}^3} \nabla(v_n - v).\nabla(v_n - v)V(x)(v_n - v)^2dx =$$

$$\langle I^{'}(u_n, v_n) - I^{'}(u, v), (u_n, v_n) - (u, v)\rangle - \int_{\mathbb{R}^3} [\ (\phi_{u_n}u_n - \phi_uu)(u_n - u)\ ]dx +$$

$$\int_{\mathbb{R}^3} [\ (\psi_{v_n}v_n - \psi_vv)(v_n - v)\ ]dx + \int_{\mathbb{R}^3} [\ (H_{u_n}(x, u_n, v_n) - H_u(x, u, v))(u_n - u)\ ]dx$$

$$+ \int_{\mathbb{R}^3} [\ (H_{v_n}(x, u_n, v_n) - H_v(x, u, v)(v_n - v))\ ]dx.$$

Since $(u_n, v_n) \hookrightarrow (u, v)$ in X_E, and $(u_n, v_n) \to (u, v)$ in for any $s, t \in [2, 2^*)$, we can imply

$$\langle I^{'}(u_n, v_n) - I^{'}(u, v), (u_n, v_n) - (u, v)\rangle \to 0,$$

as $n \to \infty$. On the other hand,

$$|\int_{\mathbb{R}^3}[\ (\phi_{u_n}u_n - \phi_u u)(u_n - u)\ ]dx + \int_{\mathbb{R}^3}[\ (\psi_{v_n}v_n - \psi_v v)(v_n - v)\ ]dx\ | \leq$$

$$|\int_{\mathbb{R}^3}[\ (\phi_{u_n}u_n - \phi_u u)(u_n - u)\ ]dx\ | + |\ \int_{\mathbb{R}^3}[\ (\psi_{v_n}v_n - \psi_v v)(v_n - v)\ ]dx\ |.$$

By Höder inequality and 2.3,

$$|\phi_{u_n}u_n(u_n - u)dx| \leq \|\phi_{u_n}u_n\|_{L^2}\|u_n - u\|_{L^2} \leq \|\phi_{u_n}\|_{L^6}\|u_n\|_{L^3}\|u_n - u\|_{L^2}$$
$$M_8\|\phi_{u_n}\|_{D^{1,2}}\|u_n\|_{L^3}\|u_n - u\|_{L^2} \leq M_3M_8\|u_n\|^2_{L^{\frac{12}{5}}}\|u_n\|_{L^3}\|u_n - u\|_{L^2}, \tag{3.9}$$

where M_8 is positive constant. We can similarity conclude the following inequalities :

$$|\phi_u u(u_n - u)dx| \leq \|\phi_u u\|_{L^2}\|u_n - u\|_{L^2} \leq \|\phi_u\|_{L^6}\|u_n\|_{L^3}\|u_n - u\|_{L^2}$$
$$M_9\|\phi_u\|_{D^{1,2}}\|u_n\|_{L^3}\|u_n - u\|_{L^2} \leq M_3M_9\|u\|^2_{L^{\frac{12}{5}}}\|u\|_{L^3}\|u_n - u\|_{L^2}, \tag{3.10}$$

and

$$|\psi_{v_n}v_n(v_n - v)dx| \leq \|\psi_{v_n}v_n\|_{L^2}\|v_n - v\|_{L^2} \leq \|\psi_{v_n}\|_{L^6}\|v_n\|_{L^3}\|v_n - v\|_{L^2}$$
$$M_{10}\|\psi_{v_n}\|_{D^{1,2}}\|v_n\|_{L^3}\|v_n - v\|_{L^2} \leq M_3M_{10}\|v_n\|_{{}^2L^{\frac{12}{5}}}\|v_n\|_{L^3}\|v_n - v\|_{L^2}, \tag{3.11}$$

and

$$|\psi_v v(v_n - v)dx| \leq \|\psi_v v\|_{L^2}\|v_n - v\|_{L^2} \leq \|\psi_v\|_{L^6}\|v_n\|_{L^3}\|v_n - v\|_{L^2}$$
$$M_{11}\|\psi_v\|_{D^{1,2}}\|v_n\|_{L^3}\|v_n - v\|_{L^2} \leq M_3M_{11}\|v\|^2_{L^{\frac{12}{5}}}\|v\|_{L^3}\|v_n - v\|_{L^2}, \tag{3.12}$$

where M_9, M_{10} and M_{11} are positive constants.
Therefore, by relations 3.9, 3.10, 3.11 and 3.12 we obtain that

$$|\int_{\mathbb{R}^3}[\ (\phi_{u_n}u_n - \phi_u u)(u_n - u)\ ]dx + \int_{\mathbb{R}^3}[\ (\psi_{v_n}v_n - \psi_v v)(v_n - v)\ ]dx\ | \leq$$
$$[\ M_3M_8\ \|u_n\|^2_{L^{\frac{12}{5}}}\ \|u_n\|_{L^3}\ +\ M_3M_9\|u\|^2_{L^{\frac{12}{5}}}\ \|u\|_{L^3}\ ]\ \|u_n - u\|_{L^2}$$
$$+[\ M_3M_{10}\ \|v_n\|_{{}^2L^{\frac{12}{5}}}\ \|v_n\|_{L^3}\ +\ M_3M_{11}\|v\|^2_{L^{\frac{12}{5}}}\ \|v\|_{L^3}\ ]\ \|v_n - v\|_{L^2}.$$

Then

$$\int_{\mathbb{R}^3}[\ (\phi_{u_n}u_n - \phi_u u)(u_n - u)\ ]dx + \int_{\mathbb{R}^3}[\ (\psi_{v_n}v_n - \psi_v v)(v_n - v)\ ]dx \to 0.$$

Now, by condition H_1 and Höder inequality we obtain that

$$|\int_{\mathbb{R}^3} H_{u_n}(x,u_n,v_n)(u_n-u)dx| \le \int_{\mathbb{R}^3} |H_{u_n}(x,u_n,v_n)|\ |u_n-u|dx \le$$

$$\int_{\mathbb{R}^3} [\ \frac{M_1}{2}|u_n| \ + \ \frac{M_1}{p}|u_n|^{p-1}\ ]\ |u_n-u|dx \le$$

$$[\ \frac{M_1}{2}\ \|u_n\|_{L^2}^2 \ + \ \frac{M_1}{p}\ \|u_n\|_{L^P}^{p-1}\ ]\ \|u_n-u\|_{L^P}. \tag{3.13}$$

Similarly, we can get the following inequalities by condition H_1 and Höder inequality:

$$|\int_{\mathbb{R}^3} H_{u}(x,u,v)(u_n-u)dx| \le \int_{\mathbb{R}^3} |H_{u}(x,u,v)|\ |u_n-u|dx \le$$

$$\int_{\mathbb{R}^3} [\ \frac{M_1}{2}|u| \ + \ \frac{M_1}{p}|u|^{p-1}\ ]\ |u_n-u|dx \le$$

$$[\ \frac{M_1}{2}\ \|u\|_{L^2}^2 \ + \ \frac{M_1}{p}\ \|u\|_{L^P}^{p-1}\ ]\ \|u_n-u\|_{L^P} \tag{3.14}$$

and

$$|\int_{\mathbb{R}^3} H_{v_n}(x,u_n,v_n)(v_n-v)dx| \le \int_{\mathbb{R}^3} |H_{v_n}(x,u_n,v_n)|\ |v_n-v|dx \le$$

$$\int_{\mathbb{R}^3} [\ \frac{M_2}{2}|v_n| \ + \ \frac{M_2}{p}|v_n|^{p-1}\ ]\ |v_n-v|dx \le$$

$$[\ \frac{M_2}{2}\ \|v_n\|_{L^2}^2 \ + \ \frac{M_2}{p}\ \|v_n\|_{L^P}^{p-1}\ ]\ \|v_n-v\|_{L^P} \tag{3.15}$$

and

$$|\int_{\mathbb{R}^3} H_{v}(x,u,v)(v_n-v)dx| \le \int_{\mathbb{R}^3} |H_{v}(x,u,v)|\ |v_n-v|dx \le$$

$$\int_{\mathbb{R}^3} [\ \frac{M_2}{2}|v| \ + \ \frac{M_2}{p}|v|^{p-1}\ ]\ |v_n-v|dx \le$$

$$[\ \frac{M_2}{2}\ \|v\|_{L^2}^2 \ + \ \frac{M_2}{p}\ \|v\|_{L^P}^{p-1}\ ]\ \|v_n-v\|_{L^P}. \tag{3.16}$$

Hence, by relations 3.13, 3.14, 3.15 and 3.16 we obtain

$$\int_{\mathbb{R}^3} [\ (H_{u_n}(x,u_n,v_n) - H_u(x,u,v))(u_n-u)\ ]dx + \int_{\mathbb{R}^3} [\ (H_{v_n}(x,u_n,v_n) - H_v(x,u,v)(v_n-v))\ ]dx$$

$$\leq 2[\ \frac{M_1}{2}\ \|u_n\|_{L^2}^2+\frac{M_1}{p}\ \|u_n\|_{L^P}^{p-1}\ ]\ \|u_n-u\|_{L^P}+2[\ \frac{M_2}{2}\ \|v_n\|_{L^2}^2+\frac{M_2}{p}\ \|v_n\|_{L^P}^{p-1}\ ]\ \|v_n-v\|_{L^P}.$$

Since, $(u_n,v_n)\to(u,v)$ in $L^t(\mathbb{R}^3)\times L^s(\mathbb{R}^3)$ for any $t,s\in[2,2^*)$ then

$$\int_{\mathbb{R}^3}[\ (H_{u_n}(x,u_n,v_n)-H_u(x,u,v))(u_n-u)\ ]dx+\int_{\mathbb{R}^3}[\ (H_{v_n}(x,u_n,v_n)-H_v(x,u,v)(v_n-v))\ ]dx\to 0,$$

as $n\to\infty$.

proof of theorem 1.3 By proposition 3.2 the functional $I(u,v)$ satisfies in Cerami condition. Now, we show that $I(u,v)$ satisfies in condition I_1 and I_2 in theorem 1.2. From H_2,

$$\lim_{|(u,v)|\to\infty}\frac{H(x,u,v)}{|(u,v)|^4}=+\infty,$$

for any $M>0$ there exists $N>0$ such that $\forall x\in\mathbb{R}^3$, $|(u,v)|\geq N$,

$$H(x,u,v)\geq\frac{1}{4}M\ |(u,v)|^4,$$

for any $x\in\mathbb{R}^3$ and $(u,v)\in\mathbb{R}^2$. Therefore,

$$I(u,v)=\frac{1}{2}\|(u,v)\|_{X_E}^2+\frac{1}{4}\int_{\mathbb{R}^3}[\phi_u u^2+\psi_v v^2]dx-\int_{\mathbb{R}^3}H(x,u,v)dx$$

$$\leq\frac{1}{2}\|(u,v)\|_{X_E}^2+[\ \frac{M_4}{4}(\|u\|_E^4+\|v\|_E^4)\ ]-\frac{M}{4}\|(u,v)\|_{X_E}^4+\tilde{M}\|(u,v)\|_{X_E}^2,$$

where $\tilde{M}:=\sup\limits_{|(u,v)|<N}(\frac{M}{4}|(u,v)|^4-\frac{H(x,u,v)}{|(u,v)|^2})$. Then

$$H(x,u,v)\geq\frac{M}{4}\|(u,v)\|_{X_E}^4-\tilde{M}\|(u,v)\|_{X_E}^2.$$

Hence,

$$I(u,v)\leq\frac{1}{2}\|(u,v)\|_{X_E}^2+\frac{M_4}{4}(\ \|u\|_E^4\ +\ \|v\|_E^4\ )-\frac{M_{12M}}{4}\ [\ \|u\|_E^4\ +\ \|v\|_E^4\ ]$$
$$+\ M_{12}\tilde{M}\ (\ \|u\|_E^2\ +\ \|v\|_E^2\ ),$$

where M_{12} is positive constant. Since, $\frac{M_4}{4}-\frac{M_{12}M}{4}<0$, so for M large enough, it follows that

$$a_k:=\max_{(u,v)\in W_k,\|(u,v)\|_{X_E}=\rho_k}I(u,v)\leq 0,$$

for some positive constant large enough. Now, using the lemma 2.3 and 2.1 we show that $I(u,v)$ satisfies in condition I_2.

$$I(u,v)=\frac{1}{2}\|(u,v)\|_{X_E}^2+\frac{1}{4}\int_{\mathbb{R}^3}[\phi_u u^2+\psi_v v^2]dx-\int_{\mathbb{R}^3}H(x,u,v)dx$$

$$\geq \frac{1}{2}\|(u,v)\|^2_{X_E} - \Big(\frac{M_1}{2}\|u\|^2_{L^2} + \frac{M_1}{p}\|u\|^p_{L^p} + \frac{M_2}{2}\|v\|^2_{L^2} + \frac{M_2}{p}\|v\|^p_{L^p} \Big)$$

$$\geq \frac{1}{2}[\ \|u\|^2_E + \|v\|^2_E\ ] - \frac{M_1C_{emb}}{2}\|u\|^2_E - \frac{\beta^p_k M_1 C_{emb}}{p}\|u\|^p_E - \frac{M_2C_{emb}}{2}\|v\|^2_E - \frac{M_2C_{emb}\alpha^p_k}{p}\|v\|^p_E$$

$$= (\frac{1}{2} - \frac{M_1C_{emb}}{2})\|u\|^2_E + (\frac{1}{2} - \frac{M_2C_{emb}}{2})\|v\|^2_E - [\ \frac{M_1C_{emb}\beta^p_k}{p}\|u\|^p_E + \frac{M_2C_{emb}\alpha^p_k}{p}\|v\|^p_E\ ].$$

We choose $r_k := (\frac{M_1C_{emb}}{\beta^p_k} + \frac{M_2C_{emb}}{\alpha^p_k})^{\frac{1}{p-2}}$. Hence,

$$b_k := \inf_{(u,v)\in Z_k, \|(u,v)\|_{X_E}=r_k} I(u,v) \geq \inf_{(u,v)\in Z_k, \|(u,v)\|_{X_E}=r_k} \Big\{ [\ (\frac{1}{2} - \frac{M_1C_{emb}}{2})\|u\|^2_E - \frac{M_1C_{emb}\beta^p_k}{p}\|u\|^p_E\ ]$$

$$+ (\frac{1}{2} - \frac{M_2C_{emb}}{2})\|v\|^2_E - \frac{M_2C_{emb}\alpha^p_k}{p}\|v\|^p_E \Big\}$$

$$\geq \inf_{(u,v)\in Z_k, \|(u,v)\|_{X_E}=r_k} M_{13}(\ \|u\|^2_E\ +\ \|v\|^2_E\ ) - M_{14}(\ \|u\|^p_E\ +\ \|v\|^p_E\ ),$$

where $M_{13} := min\{\frac{1-M_1C_{emb}}{2}\ ,\ \frac{1-M_2C_{emb}}{2}\}$ and $M_{14} := min\{\frac{1-M_1\beta^p_kC_{emb}}{p}\ ,\ \frac{1-M_2\alpha^p_kC_{emb}}{p}\}$ are positive constants. Therefore,

$$I(u,v) \geq \inf_{(u,v)\in Z_k, \|(u,v)\|_{X_E}=r_k} M_{13}\|(u,v)\|^2_{X_E} - M_{14}\|(u,v)\|^p_{X_E} \geq M_{15}r^2_k(1 - r^{p-2}_k),$$

where $M_{15} := min\{M_{13}\ ,\ M_{14}\}$ is positive constant. Now, if $k \to \infty$, then by lemma 3.1 we have $r_k \to +\infty$. Therefore, by theorem 1.2 the system 1.1 has infinitely many solutions. □

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