Common Fixed Point Theorems for Generalized Weakly Contractive Mappings under The Weaker Meir-Keeler Type Function

Sirous Moradi and Ebrahim Analouei Audeganı

1 Department of Mathematics, Faculty of Science, Arak University, Arak, 38156-8-8349, Iran
2 Department of Mathematics, Faculty of Science, Arak University, Arak, 38156-8-8349, Iran

Abstract. In this paper, we prove some common fixed point theorems for multivalued mappings and we present some new generalization contractive conditions under the condition of weak compatibility. Our results extends Chang-Chen’s results as well as Ćirić results. An example is given to support the usability of our results.

Keywords: Metric space; Common fixed point; Contractive mapping; Weakly compatible.


1. Introduction

It is common that the contractive-type conditions are very important in the study a fixed point theory. The first important result of fixed points for contractive-type mapping was the well-known Banach-Caccioppoli theorem published for the first time in 1922 in and also found in.

In recent years, many authors had proved fixed point theorems for mappings in metric spaces satisfying general contractive integral type

---

1 Corresponding author: sirousmoradi@gmail.com, s-moradi@araku.ac.ir
Received: 22 Oct 2012
Revised: 20 Aug 2013
Accepted: 19 Sep 2013
inequalities for example see \cite{[12, 18, 19, 15, 17, 16]} and weakly compatible mappings, see for example \cite{[1, 2, 4, 7, 9, 10, 13, 20, 21]}.

The purpose of this paper is to establish the existence of fixed point theorem for generalized contractive multivalued mappings.

2. Preliminaries

At first we recall the notion of the Meir-Keeler and weaker Meir-Keeler type functions as follows.

**Definition 2.1** \cite{[14]}. A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler type function, if for each $\eta \in \mathbb{R}^+$, there exists $\delta = \delta(\eta) > 0$ such that for $t \in \mathbb{R}^+$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$.

**Definition 2.2** \cite{[6]}. The function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a weaker Meir-Keeler type function, if for each $\eta > 0$, there exists $\delta > \eta$ such that for $t \in \mathbb{R}^+$ with $\eta \leq t < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

Throughout this paper, let $\mathcal{B}(X)$ stand for the set of all nonempty bounded subsets of $X$ and two functions $\delta, D : \mathcal{B}(X) \times \mathcal{B}(X) \to [0, +\infty)$ are defined be:

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\},$$

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

The following definition is given by Fisher in \cite{[8]}.

**Definition 2.3.** Let $\{A_n : n = 1, 2, \ldots\}$ be a sequence of $2^X - \emptyset$. We say that the sequence $\{A_n\}$ converges to a subset $A$ of $X$ if

(i) each point $a$ in $A$ is the limit of a convergent sequence $\{a_n\}$ with $\{a_n \in A_n : n = 1, 2, \ldots\}$;

(ii) For arbitrary $\epsilon > 0$, there exists an integer $N$ such that $A_n \subset A_\epsilon$ for $n > N$, where $A_\epsilon$ is a the union of all open spheres with center in $A$ and radius $\epsilon$.

The set $A$ is then said to be the limit of the sequence $\{A_n\}$.

The following lemmas that appear in \cite{[8]} and \cite{[10]}, are useful for the main results of this paper.

**Lemma 2.4** \cite{[8]}. If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded sets of $(X, d)$ which converge to the bounded sets $A$ and $B$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.

**Lemma 2.5** \cite{[10]}. If $\{A_n\}$ is a sequences of bounded sets in the complete metric space $(X, d)$ and if $\lim_{n \to \infty} \delta(A_n, \{y\}) = 0$, for some $y \in X$, then $\{A_n\} \to \{y\}$. 
Let $T : X \to \mathcal{B}(X)$ be a multivalued mapping. If $U$ is any nonempty subset of $X$ then we define

$$\text{ } T(U) = \bigcup_{x \in U} Tx.$$ 

Also, if $f$ is a self mapping of $X$, then by $T(X) \subseteq f(X)$, we mean

$$\text{ } T(U) = \bigcup_{x \in U} Tx \subseteq f(X),$$

that is, for all $x \in U$, we have $Tx \subseteq f(X)$.

The following definitions were given by Jungck and Rhoades [11].

**Definition 2.6.** Let $f : X \to X$ and $S : X \to \mathcal{B}(X)$ be two mappings. The pair $(f, S)$ is said to be weakly compatible if $f$ and $S$ commute at coincidence; i.e., for each point $u$ in $X$ such that $Su = \{fu\}$, we have $Sfu = fSu$.

**Definition 2.7.** Let $\Phi$ denotes all function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions:

(i) $\phi(0) = 0$, and $\phi(t) > 0$ for all $t \in \mathbb{R}^+$,

(ii) $\phi$ is continuous from the right and

(iii) $\phi$ is nondecreasing on $\mathbb{R}^+$.

**Definition 2.8.** Let $\Psi$ denotes all function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy weaker Meir-Keeler type function such that for $t > 0$ with $\psi(t) < t$ and $\{\psi^n(t)\}_{n \in \mathbb{N}}$ is non-increasing.

The following Lemma is useful for the main results of this paper, that appear in [22].

**Lemma 2.9.** Let $\phi \in \Phi$. If $\lim_{n \to \infty} \phi(\epsilon_n) = 0$, for $\{\epsilon_n\} \subset \mathbb{R}^+$, then $\lim_{n \to \infty} \epsilon_n = 0$.

**Definition 2.10 ([6]).** Let $(X, d)$ be a metric space, and let $T, S : E \to \mathcal{B}(E)$. If the following inequality holds:

$$\phi(\delta(Sx, Ty)) \leq \psi(\phi(M(x, y))) \quad (2.1)$$

for all $x, y \in X$, where

$$M(x, y) := \max \{d(x, y), \delta(Sx, x), \delta(y, Ty), \frac{1}{2}[D(x, Ty) + D(Sx, y)]\}, \quad (2.2)$$

then we call that the pair $(T, S)$ having the $(\phi, \psi)$- contraction property.

Using this definition Chang and Chen proved the following theorem and extended the Ćirić results.
Theorem 2.11 (see [6, Theorem 1]). Let \((X, d)\) be a complete metric
space and let \(T, S : X \to \mathcal{B}(X)\). If \((T, S)\) have the \((\phi, \psi)\)-contraction
property, where \(\phi \in \Phi\) and \(\psi \in \Psi\), then \(S\) and \(T\) have a unique common
fixed point \(a\) in \(X\). Moreover, \(S a = T a = \{a\}\).

Now, we define a generalized \((\phi, \psi)\)-contractive for the pair \((T, S)\) that
\(T, S : E \to \mathcal{B}(E)\) as follows:

Definition 2.12. Two mappings \(T, S : X \to \mathcal{B}(X)\) are called gener-
alized \((\phi, \psi)\)-contractive if there exist two maps \(f, g : X \to X\) such that
\[
\phi(\delta(Sx, Ty)) \leq \psi(\phi(M(x, y)))
\]
for all \(x, y \in X\), where
\[
M(x, y) := \max \{d(fx, gy), \delta(Sx, fx), \delta(gy, Ty), \frac{1}{2}[D(fx, Ty) + D(Sx, gy)]\}.
\]

In next section, we give a new fixed point theorem for \((\phi, \psi)\)-contractive
mappings and extend Chang-Chen’s Theorem. After that an example
shows that our results extend Chang-Chen’s Theorem.

3. Main result

The following theorem extends Chang-Chen’s Theorem

Theorem 3.1. Let \((X, d)\) be a complete metric space, and let \(E\) be a
nonempty closed subset of \(X\). Let \(T, S : E \to \mathcal{B}(E)\) be two generalized
\((\phi, \psi)\)-contractive, where \(\phi \in \Phi\) and \(\psi \in \Psi\), and \(f, g : E \to X\) verifying
the following:

(A) \((f, S)\) and \((g, T)\) are weakly compatible;
(B) \(T(E) \subseteq f(E)\) and \(S(E) \subseteq g(E)\).

Assume that \(f(E)\) or \(g(E)\) is a closed subset of \(X\). Then \(f, T, g\) and \(S\) have a unique common fixed point, that is, there exist \(x \in E\) such that

\[
\{fx\} = \{gx\} = Tx = Sx = \{x\}.
\]

Proof. Lex \(x_0 \in E\) be arbitrary. Using \((B)\), we choose \(x_1 \in E\) such that
\(gx_1 \in Sx_0 = A_0\). There exists \(x_2 \in E\) such that \(fx_2 \in Tx_1 = A_1\), and
so on. Using induction, we can define a sequence \(\{x_n\}\) in \(E\) as follows:

\[
gx_{2n+1} \in Sx_{2n} = A_{2n}, \quad fx_{2n+2} \in Tx_{2n+1} = A_{2n+1},
\]

for \(n = 0, 1, \ldots\). We break the argument into three steps.

Step 1. \(\lim_{n \to \infty} \delta(A_n, A_{n+1}) = 0\).
Proof. Using (2.3), have
\[ \phi(\delta(Sx_{2n}, Tx_{2n+1})) = \phi(\delta(A_{2n}, A_{2n+1})) \leq \psi(\phi(M(x_{2n}, x_{2n+1}))), \] (3.2)
where
\[ M(x_{2n}, x_{2n+1}) = \max \{ d(fx_{2n}, gx_{2n+1}), \delta(Sx_{2n}, fx_{2n}), \delta(gx_{2n+1}, Tx_{2n+1}), \]
\[ \frac{1}{2}[D(fx_{2n}, Tx_{2n+1}) + D(Sx_{2n}, gx_{2n+1})] \]
\[ = \max \left\{ d(fx_{2n}, gx_{2n+1}), \delta(A_{2n}, fx_{2n}), \delta(gx_{2n+1}, A_{2n+1}), \right\} \]
\[ \leq \max \{ \delta(A_{2n-1}, A_{2n}), \delta(A_{2n}, A_{2n+1}), \frac{1}{2} \delta(A_{2n-1}, A_{2n+1}) \} \]
\[ \leq \max \{ \delta(A_{2n-1}, A_{2n}), \delta(A_{2n}, A_{2n+1}), \]
\[ \frac{1}{2}[\delta(A_{2n-1}, A_{2n}) + \delta(A_{2n}, A_{2n+1})] \}
\[ = \max \{ \delta(A_{2n-1}, A_{2n}), \delta(A_{2n}, A_{2n+1}) \} = \delta(A_{2n-1}, A_{2n}). \]
If \( \{ \delta(A_{2n-1}, A_{2n}), \delta(A_{2n}, A_{2n+1}) \} = \delta(A_{2n-1}, A_{2n}) \), then By (3.2), we have
\[ \phi(\delta(A_{2n}, A_{2n+1})) = \phi(\delta(Sx_{2n}, Tx_{2n+1})) \leq \psi(\phi(M(x_{2n}, x_{2n+1}))) = \psi(\phi(\delta(A_{2n}, A_{2n+1}))) \leq \phi(\delta(A_{2n}, A_{2n+1})), \] (3.4)
where that is a contradiction. Hence
\[ \delta(A_{2n}, A_{2n+1}) \leq \delta(A_{2n-1}, A_{2n}). \] (3.5)
Similarly,
\[ \delta(A_{2n+1}, A_{2n+2}) \leq \delta(A_{2n+1}, A_{2n}). \] (3.6)
Consequently for each \( n \in \mathbb{N} \), we have
\[ \delta(A_n, A_{n+1}) \leq \delta(A_n, A_{n-1}), \] (3.7)
and hence,
\[ \psi(\delta(A_n, A_{n+1})) \leq \psi(\phi(\delta(A_{n-1}, A_n))) \leq \psi^n(\phi(\delta(A_0, A_1))). \] (3.8)
Since \( \{ \psi^n(\phi(\delta(A_0, A_1))) \}_{n \in \mathbb{N}} \) is nonincreasing and so it must converges to some \( \eta \geq 0 \). We claim that \( \eta = 0 \). On the contrary, assume that \( \eta > 0 \). From \( \psi(t) < t \) we get \( \phi(\delta(A_0, A_1)) \geq \eta \), then by the definition of
the weaker Meir-Keeler type function, there exists $\delta > \eta$ such that for $\phi(\delta(A_0, A_1)) > 0$ with $\eta \leq \phi(\delta(A_0, A_1)) < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\phi(\delta(A_0, A_1))) < \delta$. Since

$$\lim_{n \to \infty} \psi^n(\phi(\delta(A_0, A_1))) = 0,$$

and this is a contradiction. Consequently, $\eta = 0$. Thus By (3.8), we have

$$\lim_{n \to \infty} \psi(\delta(A_n, A_{n+1})) = 0,$$

so $\lim_{n \to \infty} \delta(A_n, A_{n+1}) = 0$. □

**Step 2.** $\{A_n\}$ is Cauchy.

*Proof.* For each $m \in \mathbb{N}$, we suppose $C_m = \delta(A_m, A_{m+1})$ and we claim that following result holds:

$$\forall \gamma > 0 \exists n_0(\gamma) \in \mathbb{N} \ s.t \ \forall m, n > n_0(\gamma), \ \delta(A_m, A_n) < \gamma. \tag{3.11}$$

Suppose (3.11), is not held. Then using **Step 1** there exists some $\gamma > 0$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ that $m_k - n_k > 3$ and $m_k$ is even, $n_k$ is odd and

$$\delta(A_{m_k}, A_{n_k}) \geq \gamma. \tag{3.12}$$

For every $k \in \mathbb{N}$, let $m_k$ be the smallest even number satisfying (3.12).

Since $\lim_{m \to \infty} C_m = 0$, there exist $k_0 \in \mathbb{N}$ such that for $m \geq k_0$, $\delta(A_m, A_{m+1}) < \gamma$. Thus By (3.12), we have

$$\gamma \leq \delta(A_{m_k}, A_{n_k}) \leq \delta(A_{m_k}, A_{m_k-1}) + \delta(A_{m_k-1}, A_{m_k-2}) + \delta(A_{m_k-2}, A_{n_k}) \leq \delta(A_{m_k}, A_{m_k-1}) + \delta(A_{m_k-1}, A_{m_k-2}) + \gamma \tag{3.13}$$

letting $k \to \infty$, we get

$$\lim_{k \to \infty} \delta(A_{m_k}, A_{n_k}) = \gamma. \tag{3.14}$$

So with using (2.3),

$$\phi(\delta(A_{m_k}, A_{n_k})) = \phi(\delta(Sx_{m_k}, Tx_{n_k})) \leq \psi(\phi(M(x_{m_k}, x_{n_k}))), \tag{3.15}$$
where
\[
M(x_{mk}, x_{nk}) = \max \left\{ d(fx_{mk}, gx_{nk}), \delta(Sx_{mk}, fx_{mk}), \delta(gx_{nk}, Tx_{nk}), \frac{1}{2}[D(fx_{mk}, Tx_{nk}) + D(Sx_{mk}, gx_{nk})] \right\}
\]
\[
\leq \max \left\{ \delta(A_{mk-1}, A_{nk-1}), \delta(A_{mk}, A_{mk-1}) \right\}, \delta(A_{nk-1}, A_{nk})
\]
\[
\frac{1}{2}[\delta(A_{mk-1}, A_{nk}) + \delta(A_{mk}, A_{nk-1})]
\]
\[
\leq \max \left\{ \delta(A_{mk-1}, A_{nk}) + \delta(A_{mk}, A_{nk}) + \delta(A_{nk-1}, A_{nk}) \right\}, \delta(A_{nk}, A_{nk-1})
\]
\[
\frac{1}{2}[\delta(A_{mk-1}, A_{nk}) + \delta(A_{mk}, A_{nk}) + \delta(A_{nk-1}, A_{nk}) + \delta(A_{nk}, A_{nk-1})]
\]
\[
= \max \left\{ C_{mk-1} + \delta(A_{mk}, A_{nk}) + C_{nk-1}, C_{nk-1}, C_{nk} \right\}
\]
\[
\leq C_{mk-1} + C_{nk-1} + \delta(A_{mk}, A_{nk}).
\]
Now with combining (3.15), (3.16), and letting \( k \to \infty \), we have
\[
\phi(\gamma) \leq \psi(\phi(\gamma)),
\]
and this is a contraction. So \( \{A_n\} \) is Cauchy.

\[
\square
\]

**Step 3.** \( T, S, g \) and \( f \) have a common fixed point.

Proof. If \( a_n \) be an arbitrary point in \( A_n \) for \( n = 0, 1, \ldots \), it follows that
\[
\lim_{n,m \to \infty} d(a_n, a_m) \leq \lim_{n,m \to \infty} \delta(A_n, A_m) = 0.
\]
Therefore, the sequence \( \{a_n\} \) and hence any subsequence thereof is a Cauchy sequence in \( X \). Since \( gx_{2n+1} \in Sx_{2n} = A_{2n} \) for \( n = 0, 1, \ldots \), we have
\[
\lim_{n,m \to \infty} d(gx_{2n+1}, gx_{2m+1}) \leq \lim_{n,m \to \infty} \delta(A_{2n}, A_{2m}) = 0.
\]
Therefore, the sequence \( \{gx_{2n+1}\} \) is Cauchy. So there exists \( z \in X \) such that
\[
\lim_{n \to \infty} gx_{2n+1} = z. \quad \text{Since } E \text{ is closed and } \{gx_{2n+1}\} \subseteq X, \text{ we have}
\]
\[
z \in E. \quad \text{Since } g(E) \text{ is closed, then there exists } u \in E \text{ such that } z = gu.
\]
But, \( fx_{2n} \in Tx_{2n-1} = A_{2n-1} \), so that we have
\[
\lim_{n \to \infty} d(fx_{2n}, gx_{2n+1}) \leq \lim_{n \to \infty} \delta(A_{2n-1}, A_{2n}) = 0.
\]
Consequently, \( \lim_{n \to \infty} fx_{2n} = z. \) Now we prove \( Tu = \{z\} \). By using (2.3), we have
\[
\phi(\delta(Sx_{2n}, Tu)) \leq \psi(\phi(M(x_{2n}, u))),
\]
where
\[
\delta(gu, Tu) \leq M(x_{2n}, u) = \max \left\{ d(fx_{2n}, gu), \delta(Sx_{2n}, fx_{2n}), \delta(gu, Tu), \frac{1}{2}[D(fx_{2n}, Tu) + D(Sx_{2n}, gu)] \right\}
\]
(3.22)
\[
\leq \max \left\{ \delta(fx_{2n}, gu), \delta(Sx_{2n}, fx_{2n}), \delta(gu, Tu), \frac{1}{2}[\delta(fx_{2n}, Tu) + \delta(Sx_{2n}, gu)] \right\}.
\]
(3.23)

letting \( n \to \infty \) in above inequality we noted
\[
\lim_{n \to \infty} \delta(Sx_{2n}, z) = 0.
\]
(3.24)

Consequently, by combining (3.22), (3.24) and letting \( n \to \infty \), we have got
\[
\lim_{n \to \infty} M(x_{2n}, u) = \delta(z, Tu).
\]
(3.25)

From (3.21) and (3.25) by letting \( n \to \infty \) we have
\[
\phi(\delta(z, Tu)) \leq \psi(\phi(\delta(z, Tu))).
\]
(3.26)

Hence \( \delta(z, Tu) = 0 \). So \( Tu = \{z\} \). Consequently, \( \{gu\} = Tu = \{z\} \).

Since \( T(E) \subseteq f(E) \) and \( Tu \in T(E) \), so there exists \( w \in E \) exists such that \( Tu = \{fw\} = \{gu\} \). Now we prove \( Sw = \{z\} \). By using (2.3), we have
\[
\phi(\delta(Sw, Tu)) \leq \psi(\phi(\delta(Sw, Tu))),
\]
(3.27)

where
\[
M(w, u) = \max \left\{ d(fw, gu), \delta(Sw, fw), \delta(gu, Tu), \frac{1}{2}[D(fw, Tu) + D(Sw, gu)] \right\}
\]
(3.28)
\[
= \max \left\{ d(fw, gu), \delta(Sw, fw), \delta(fw, Tu), \frac{1}{2}[0 + D(Sw, fw)] \right\}
\]
\[
= \delta(Sw, fw) = \delta(Sw, z).
\]

Now by combining (3.27), (3.28) and \( Tu = \{z\} \) we get
\[
\phi(\delta(Sw, z)) \leq \psi(\phi(\delta(Sw, z))).
\]
(3.29)

From \( \psi(t) < t \) for all \( t > 0 \). We conclude that \( \delta(Sw, z) = 0 \), so \( Sw = \{z\} \).

It follows that \( \{gu\} = \{fw\} = Tu = Sw = \{z\} \).
Since the pair \((T, g)\) is weakly compatible, then \(gz = gTu = Tgu = Tz\). Now we prove that \(Tz = \{z\}\). Using \(2.3\), we have
\[
\phi(\delta(Sw, Tz)) \leq \psi(\phi(M(w, z))),
\]
where
\[
M(w, z) = \max \{d(fw, gz), \delta(Sw, fw), \delta(gz, Tz), \frac{1}{2}[D(fw, Tz) + D(Sw, gz)]\}
\]
(3.31)
Now by combining (3.30), (3.31), we get
\[
\phi(\delta(z, Tz)) \leq \psi(\phi(Tz, z)).
\]
(3.32)
Consequently, \(\delta(Tz, z) = 0\), so \(Tz = \{z\}\). Hence \(\{gz\} = Tz = \{z\}\).
Similarly, \(\{fz\} = Sz = \{z\}\). Therefore, we obtain \(\{fz\} = Sz = \{z\} = \{gz\} = Tz\).

Uniqueness of the common fixed point follows from \(2.3\). Similarly, if \(f(E)\) is closed, we can conclude by a similar argument as noted above that theorem is holds. This completes the proof.

**Remark 3.2.** By taking \(f = g = I_X\) and \(E = X\) in Theorem 3.1, we conclude Theorem 2.11.

The following corollaries are direct results of Theorem 3.1 that extends Ćirić’s result [7].

**Corollary 3.3.** Let \((X, d)\) be a complete metric space, and let \(E\) be a nonempty closed subset of \(X\). Let \(T, S, f, g : E \to E\) be four mappings verifying the following conditions:
(A) \((f, S)\) and \((g, T)\) are weakly compatible,
(B) \(T(E) \subseteq f(E)\) and \(S(E) \subseteq g(E)\);
(C) \(\phi(\delta(Sx, Ty)) \leq \psi(\phi(M(x, y)))\), for all \(x, y \in E\), where \(\phi \in \Phi\), \(\psi \in \Psi\) and where
\[
M(x, y) := \max \{d(fx, gy), d(Sx, fx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(Sx, gy)]\}.
\]
Assume that \(f(E)\) or \(g(E)\) is a closed subset of \(X\). Then \(f, T, g\) and \(S\) have a unique common fixed point, that is, there exists \(x \in X\) such that \(fx = gx = Tx = Sx = x\).

**Corollary 3.4.** Let \((X, d)\) be a complete metric space, and let two mappings verifying the following conditions:
(A) \((g, T)\) are weakly compatible;
(B) \(T(E) \subseteq g(E)\);

(C) \( \phi(d(Tx, Ty)) \leq \psi(\phi(M(x, y))) \), for all \( x, y \in E \),
where \( \phi \in \Phi, \psi \in \Psi \) and where

\[
M(x, y) := \max \{d(gx, gy), d(gx, Tx), d(gy, Ty), \frac{1}{2}[d(gx, Ty) + d(gy, Tx)]\}.
\]

Assume that \( g(E) \) is a closed subset of \( X \). Then \( T \) and \( g \) have a unique common fixed point, that is, there exists \( x \in X \) such that \( Tx = gx = x \).

**Proof.** Let \( T = S \) and \( f = g \) and apply Corollary 3.3 \( \square \)

The following example shows that Theorem 3.1 is a real extension of Theorem 2.11.

**Example 3.5.** Let \( X = [0, +\infty) \) endowed with the Euclidean metric and let \( E = [0, 1] \). Let \( f, g : E \to X \) and \( T, S : E \to B(E) \) defined by \( fx = 2x, gx = 4x^2, Tx = [0, \frac{6}{7}] \) and \( Sx = [0, \frac{6}{7}] \) for all \( x \in X \).

Also, we define \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) and \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) as follows:

\[
\phi(t) = t \quad \forall t \in \mathbb{R}^+, \text{ and } \psi(t) = \frac{1}{7}t \quad \forall t \in \mathbb{R}^+.
\]

Obviously \( E \) and \( f(E) \) and \( g(E) \) are nonempty closed subsets of \( X \). Also \( \phi \in \Phi \) and \( \psi \in \Psi \) and \( (f, S) \) and \( (g, T) \) are weakly compatible in \( x = 0 \) and \( T(E) \subseteq f(E) \) and \( S(E) \subseteq g(E) \).

We next verify inequality (2.3) of Theorem 3.1. For all \( x, y \in E \) where \( x \neq y \)

\[
\delta(Sx, Ty) = \max\{\frac{x^3}{6}, \frac{y^6}{6}\} = \max\{\frac{1}{7} \frac{7x^3}{6}, \frac{1}{7} \frac{7y^6}{6}\}
\]

\[
\leq \max\{\frac{1}{7} 2x, \frac{1}{7} 4y^2\}
\]

\[
\leq \frac{1}{7} \max\{2x, 4y^2, |4y^2 - 2x|\}
\]

\[
\leq \frac{1}{7} \max\{2x, 4y^2, |4y^2 - 2x|, \frac{1}{2}[D(x, Ty) + D(Sx, y)]\}
\]

\[
\leq \frac{1}{7} M(x, y) = \psi(M(x, y))
\]

Hence all conditions of Theorem 3.1 are satisfied. So \( f, T, g \) and \( S \) have a unique common fixed point in \( x = 0 \).

The condition (2.1) of Theorem 2.11 is not satisfied for \( x = 0 \) and \( y = 1 \). Because

\[
\frac{1}{6} = \delta(S0, T1) \geq \psi(M(0, 1))
\]

\[
= \psi(\max\{d(0, 1), \delta(S0, 0), \delta(1, T1), \frac{1}{2}[D(0, T1) + D(1, S0)]\})
\]

\[
= \psi(1) = \frac{1}{7}
\]
Therefore, our example does not satisfy the condition of Theorem 2.11. Hence Theorem 3.1 is a real extension of Theorem 2.11.

REFERENCES


