Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **4**(2)(2015), 197-204

Pointwise almost periodicity in in a generalized shift dynamical system

Fatemah Ayatollah Zadeh Shirazi [1] and Meysam Miralaei [2]
[1] Faculty of Math., Stat. and Computer Science, College of Science, University of Tehran, Tehran, Iran (fatemah@khayam.ut.ac.ir)
[2] Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran (m.miralaei@math.iut.ac.ir)

Abstract. In the following text we prove that in a generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$ for discrete X with at least two elements, arbitrary nonempty Γ and bijection $\varphi : \Gamma \to \Gamma$, the following statements are equivalent:

- $(X^\Gamma, \sigma_\varphi)$ is pointwise recurrent;
- $(X^\Gamma, \sigma_\varphi)$ is pointwise almost periodic;
- $(X^\Gamma, \sigma_\varphi)$ is pointwise regularly almost periodic;
- $(X^\Gamma, \sigma_\varphi)$ is compactly almost periodic;
- $\mathrm{Per}(\varphi) = \Gamma$ ($\varphi : \Gamma \to \Gamma$ is pointwise periodic).

Keywords: Almost periodic, Compactly almost periodic, Compactly recurrent, Dynamical system, Generalized shift, Periodic, Pointwise almost periodic, Pointwise periodic, Pointwise regularly almost periodic, Pointwise recurrent, Recurrent.

2000 Mathematics subject classification: 37B05, 54H20.

1. Preliminaries

Let Y be an arbitrary set. We call the collection $\mathcal{F}$ of subsets of $Y \times Y$ a uniformity on Y if:

- for all $\alpha \in \mathcal{F}$ we have $\Delta_Y \subseteq \alpha$;
- for all $\alpha, \beta \in \mathcal{F}$ we have $\alpha \cap \beta \in \mathcal{F}$;

[1] Corresponding author: fatemah@khayam.ut.ac.ir

Received: 11 July 2014
Revised: 25 April 2015
Accepted: 24 May 2015

- for all $\alpha \in \mathcal{F}$ and $\beta \subseteq Y \times Y$ with $\alpha \subseteq \beta$ we have $\beta \in \mathcal{F}$;
- for all $\alpha \in \mathcal{F}$ there exists $\beta \in \mathcal{F}$ with $\beta \circ \beta^{-1} \subseteq \alpha$;

where $\Delta_Y = \{(y,y) : y \in Y\}$, $\alpha \circ \beta = \{(x,y) :$ there exists z such that $(x,z) \in \alpha$ and $(z,y) \in \beta\}$, and $\alpha^{-1} = \{(y,x) : (x,y) \in \alpha\}$ for $\alpha, \beta \subseteq Y \times Y$. We call an element of uniformity $\mathcal{F}$ an index. Moreover, for $\alpha \in \mathcal{F}$ and $x \in Y$ let $\alpha[x] = \{y : (x,y) \in \alpha\}$.
If $\mathcal{F}$ is a uniformity on Y, we call $(Y, \mathcal{F})$ a uniform space and equip it with topology $T = \{U \subseteq Y :$ for all $x \in U$ there exists $\alpha \in \mathcal{F}$ with $\alpha[x] \subseteq U\}$ (topology generated by $\mathcal{F}$). For nonempty set Λ if $\{(Y_\theta, \mathcal{F}_\theta) : \theta \in \Lambda\}$ is a collection of uniform spaces, then $\prod_{\theta \in \Lambda} Y_\theta$ under product topology is a uniform space too, and we may consider the following uniformity over it:

$$\{\gamma \subseteq \prod_{\theta \in \Lambda} Y_\theta \times \prod_{\theta \in \Lambda} Y_\theta : \text{there exist } \theta_1, \ldots, \theta_n \in \Lambda \text{ and } \alpha_1 \in \mathcal{F}_{\theta_1}, \ldots, \alpha_n \in \mathcal{F}_{\theta_n} \text{ with } \gamma \subseteq \kappa_{(\alpha_1, \cdots, \alpha_n)}\},$$

where $\kappa_{(\alpha_1, \cdots, \alpha_n)}$ is the following set:

$$\{((x_\theta)_{\theta \in \Lambda}, (y_\theta)_{\theta \in \Lambda}) \in \prod_{\theta \in \Lambda} Y_\theta \times \prod_{\theta \in \Lambda} Y_\theta : \forall i \in \{1, \ldots, n\} \left((x_{\theta_i}, y_{\theta_i}) \in \alpha_i\right)\}.$$

The topological space W is uniformazable if there exists a uniformity $\mathcal{G}$ on W such that the topology generated by $\mathcal{G}$ coincides with original topology of W and in this case we call $\mathcal{G}$ an admissible uniformity on W. If Y is compact Hausdorff, then it admits a unique admissible uniformity $\{\alpha \subseteq Y \times Y : \Delta_Y$ is a subset of the interior of $\alpha\}$. See [7] for more details.

By a (topological) dynamical system $((Z, \mu), h)$ or briefly (Z, h) we mean a Hausdorff uniform topological space Z (phase space) equipped with uniformity μ and a homeomorphism $h : Z \to Z$. In dynamical system (Z, h), we call nonempty subset W invariant if $h(W) = W$. We call the dynamical system (Z, h) [8], [9]:
• *periodic*, if there exists $n \geq 1$ with $h^n = \mathrm{id}_Z$, where $\mathrm{id}_Z : Z \to Z$ is the identity map, $\mathrm{id}_Z(x) = x$, $x \in Z$;
• *pointwise periodic*, if for all $x \in Z$ there exists $n \geq 1$ with $h^n(x) = x$;
• *pointwise recurrent*, if for all $z \in Z$ and all open neighborhood U of z there exists $n \geq 1$ such that $h^n(z) \in U$;
• *pointwise almost periodic*, if for all $z \in Z$ and all open neighborhood U of z, there exists $N \geq 1$ such that for all $p \in \mathbb{Z}$ there exists $n \in \{p, p+1, \ldots, p+N-1\}$ with $h^n(z) \in U$;
• *pointwise regularly almost periodic*, if for all $z \in Z$ and all open neighborhood U of z, there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $h^{nm}(z) \in U$ for all $m \in \mathbb{N}$.

• *recurrent* (or *uniformly recurrent*) if for all $\alpha \in \mu$, there exists $n \geq 1$ with $\{(h^n(z), z) : z \in X\} \subseteq \alpha$;
• *almost periodic* (or *uniformly almost periodic*), if for all $\alpha \in \mu$, there exists $N \geq 1$ such that for all $p \in \mathbb{Z}$ there exists $n \in \{p, p+1, \ldots, p+N-1\}$ with $\{(h^n(z), z) : z \in Z\} \subseteq \alpha$;
• *regularly almost periodic* (or *uniformly regularly almost periodic*), if for all $\alpha \in \mu$, there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $\{(h^{nm}(z), z) : z \in Z, m \in \mathbb{N}\} \subseteq \alpha$;
• *compactly almost periodic* (or *uniformly compactly almost periodic*), if for all compact subset B of Z, $\overline{\bigcup\{h^n(B) : n \in \mathbb{Z}\}}$ is compact and for all compact invariant subset W of Z, $(W, h\restriction_W)$ is almost periodic;
• *compactly recurrent* (or *uniformly compactly recurrent*), if for all compact subset B of Z, $\overline{\bigcup\{h^n(B) : n \in \mathbb{Z}\}}$ is compact and for all compact invariant subset W of Z, $(W, h\restriction_W)$ is recurrent (the concept of compactly recurrence is introduced here, imitating the concept of compactly almost periodicity in [9]). For nonempty arbitrary sets Γ, X and map $\varphi : \Gamma \to \Gamma$, we call $\sigma_\varphi : X^\Gamma \to X^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ (for $(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$), a generalized shift [3]. Whenever $\Gamma = \mathbb{N}$ and $\varphi(n) = n + 1$ ($n \in \mathbb{N}$), $\sigma_\varphi : X^\mathbb{N} \to X^\mathbb{N}$ is the familiar one sided shift, also whenever $\Gamma = \mathbb{Z}$ and $\varphi(n) = n + 1$ ($n \in \mathbb{Z}$), $\sigma_\varphi : X^\mathbb{Z} \to X^\mathbb{Z}$ is the well-known two sided shift. On the other hand, if X is a topological space and X^Γ is equipped with product topology, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is continuous.

Remark 1.1. For nonempty arbitrary sets Γ, X and map $\varphi : \Gamma \to \Gamma$, with $|X| \geq 2$, the map $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is bijective if and only if $\varphi : \Gamma \to \Gamma$ is bijective. Hence if X is a topological space and X^Γ is equipped with product topology, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is a homeomorphism if and only if $\varphi : \Gamma \to \Gamma$ is bijective.

For mapping $f : A \to A$, we call $a \in A$ a *periodic* point of $f : A \to A$ if there exists $n \geq 1$ with $f^n(a) = a$. Let $\mathrm{Per}(f) = \{a \in A : a$ is a periodic point of $f : A \to A\}$.
In the following text suppose X is a discrete topological space with at least two elements, Γ is an infinite set, $\varphi : \Gamma \to \Gamma$ is bijective, and consider X^Γ under product (pointwise convergence) topology.

2. Pointwise periodicity in generalized shift dynamical systems

In this section we prove that the generalized shift $(X^\Gamma, \sigma_\varphi)$ is pointwise periodic (resp. periodic) if and only if $\varphi : \Gamma \to \Gamma$ is periodic.

Remark 2.1. For maps $\lambda, \theta : \Gamma \to \Gamma$, we have $\sigma_\lambda = \sigma_\theta$ if and only if $\lambda = \theta$.

Proof. If $\theta \neq \lambda$, there exists $\beta \in \Gamma$ with $\theta(\beta) \neq \lambda(\beta)$. Choose distinct $p, q \in X$, and let $x_\alpha = p$ for $\alpha \neq \theta(\beta)$ and $x_{\theta(\beta)} = q$. Then for $(y_\alpha)_{\alpha\in\Gamma} := \sigma_\theta((x_\alpha)_{\alpha\in\Gamma})$ and $(z_\alpha)_{\alpha\in\Gamma} := \sigma_\lambda((x_\alpha)_{\alpha\in\Gamma})$ we have $z_\beta = x_{\theta(\beta)} = q$ and $y_\beta = x_{\lambda(\beta)} = p$ (since $\lambda(\beta) \neq \theta(\beta)$). So $z_\beta \neq y_\beta$ and $\sigma_\theta((x_\alpha)_{\alpha\in\Gamma}) \neq \sigma_\lambda((x_\alpha)_{\alpha\in\Gamma})$, which leads to $\sigma_\theta \neq \sigma_\lambda$ and completes the proof. □

Remark 2.2. For maps $\lambda, \theta : \Gamma \to \Gamma$, we have $\sigma_\lambda \circ \sigma_\theta = \sigma_{\theta\circ\lambda}$.

Theorem 2.3. *In the generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$, the following statements are equivalent:*

1. *$(X^\Gamma, \sigma_\varphi)$ is periodic (i.e., there exists $n \geq 1$ such that $\sigma_\varphi^n = \mathrm{id}_{X^\Gamma}$);*
2. *$(X^\Gamma, \sigma_\varphi)$ is pointwise periodic (i.e., $\mathrm{Per}(\sigma_\varphi) = X^\Gamma$);*
3. *$\varphi : \Gamma \to \Gamma$ is periodic (i.e., there exists $m \geq 1$ with $\varphi^m = \mathrm{id}_\Gamma$).*

Proof. It is clear that if $(X^\Gamma, \sigma_\varphi)$ is periodic, then it is pointwise periodic. Now suppose $(X^\Gamma, \sigma_\varphi)$ is pointwise periodic and choose distinct $p, q \in X$. Suppose $\beta \in \Gamma$. Let $x_\alpha = p$ for $\alpha \neq \beta$ and $x_\beta = q$. Since σ_φ is pointwise periodic, there exists $n \geq 1$ with $\sigma_\varphi^n((x_\alpha)_{\alpha\in\Gamma}) = (x_\alpha)_{\alpha\in\Gamma}$. So $(x_\alpha)_{\alpha\in\Gamma} = (x_{\varphi^n(\alpha)})_{\alpha\in\Gamma}$ and $q = x_\beta = x_{\varphi^n(\beta)}$ which leads to $\varphi^n(\beta) = \beta$. Hence $\varphi : \Gamma \to \Gamma$ is pointwise periodic. For $\alpha \in \Gamma$ let $n_\alpha = \min\{n \geq 1 : \varphi^n(\alpha) = \alpha\}$, it's evident that for all $\alpha \in \Gamma$ we have $n_\alpha = n_{\varphi(\alpha)}$ (note to the fact that $\varphi : \Gamma \to \Gamma$ is bijective). In the following claim we prove that $\sup\{n_\alpha : \alpha \in \Gamma\}$ is finite.
Claim. $\sup\{n_\alpha : \alpha \in \Gamma\} < \infty$.
Proof of Claim. If $\sup\{n_\alpha : \alpha \in \Gamma\} = +\infty$, then there exists a strinctly increasing sequence $\{n_{\alpha_k}\}_{k\in\mathbb{N}}$. Let:

$$x_\alpha = \begin{cases} q & \alpha \in \{\alpha_k : k \in \mathbb{N}\}\,, \\ p & \text{otherwise}\,. \end{cases}$$

Since $(X^\Gamma, \sigma_\varphi)$ is pointwise periodic, there exists $m \geq 1$ with $\sigma_\varphi^m((x_\alpha)_{\alpha\in\Gamma}) = (x_\alpha)_{\alpha\in\Gamma}$. So for all $k \geq 1$ we have $q = x_{\alpha_k} = x_{\varphi^m(\alpha_k)}$, which leads to $\varphi^m(\alpha_k) \in \{\alpha_l : l \in \mathbb{N}\}$, using $n_{\varphi^m(\alpha)} = n_\alpha$ and the fact that $\{n_{\alpha_l}\}_{l\in\mathbb{N}}$ is one to one, we conclude $\varphi^m(\alpha) = \alpha$. By $\varphi^m(\alpha) = \alpha$ we have $m \geq n_\alpha$, so $\sup\{n_\alpha : \alpha \in \Gamma\} \leq m$, which is a contradiction, hence $\sup\{n_\alpha : \alpha \in \Gamma\} < \infty$.
If $N = \sup\{n_\alpha : \alpha \in \Gamma\}$, then for all $\alpha \in \Gamma$ we have $\varphi^{N!}(\alpha) = \alpha$. Therefore, $\varphi^{N!} = \mathrm{id}_\Gamma$, and $\varphi : \Gamma \to \Gamma$ is periodic.
In order to complete the proof of theorem, note to the fact that if $\varphi^n = \mathrm{id}_\Gamma$, using Remarks 2.1 and 2.2 we have $\sigma_\varphi^n = \sigma_{\varphi^n} = \sigma_{\mathrm{id}_\Gamma} = \mathrm{id}_{X^\Gamma}$, and $(X^\Gamma, \sigma_\varphi)$ is periodic. □

3. Pointwise almost periodicity in generalized shift dynamical systems

In this section we prove that $(X^\Gamma, \sigma_\varphi)$ is any of pointwise recurrent, pointwise almost periodic, pointwise regularly almost periodic, compactly recurrent, compactly almost periodic if and only if $\mathrm{Per}(\varphi) = \Gamma$.

Lemma 3.1. *If* $(X^\Gamma, \sigma_\varphi)$ *is pointwise recurrent, then* $\mathrm{Per}(\varphi) = \Gamma$.

Proof. Suppose $\theta \in \Gamma$ and $(X^\Gamma, \sigma_\varphi)$ is pointwise recurrent. Choose distinct $p, q \in X$ and let:

$$U_\alpha = \begin{cases} \{p\} & \alpha = \theta\,, \\ X & \alpha \neq \theta\,, \end{cases} \quad \text{and} \quad x_\alpha = \begin{cases} p & \alpha = \theta\,, \\ q & \alpha \neq \theta\,. \end{cases} \qquad (*)$$

Since $(X^\Gamma, \sigma_\varphi)$ is pointwise recurrent and $\prod\limits_{\alpha\in\Gamma} U_\alpha$ is an open neighborhood of $(x_\alpha)_{\alpha\in\Gamma}$ there exists $n \geq 1$ with $(x_{\varphi^n(\alpha)})_{\alpha\in\Gamma} = \sigma_\varphi^n (x_\alpha)_{\alpha\in\Gamma} \in \prod\limits_{\alpha\in\Gamma} U_\alpha$. In particular $x_{\varphi^n(\theta)} \in U_\theta = \{p\}$ and $x_{\varphi^n(\theta)} = p$ which leads to $\varphi^n(\theta) = \theta$ by (*), and θ is periodic under φ. □

Lemma 3.2. *If* $\mathrm{Per}(\varphi) = \Gamma$*, then* $(X^\Gamma, \sigma_\varphi)$ *is pointwise regularly almost periodic.*

Proof. Suppose $\mathrm{Per}(\varphi) = \Gamma$. For $w = (w_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$ if U is an open neighborhood of w, then there exist $\theta_1, \ldots, \theta_k \in \Gamma$ such that $\prod\limits_{\alpha\in\Gamma} U_\alpha \subseteq U$, where:

$$U_\alpha = \begin{cases} \{w_\alpha\} & \alpha = \theta_1, \ldots, \theta_k\,, \\ X & \text{otherwise}\,. \end{cases}$$

For all $i \in \{1, \ldots, k\}$ there exists $r_i \geq 1$ such that $\varphi^{r_i}(\theta_i) = \theta_i$. For all $i \in \{1, \ldots, k\}$ and $t \in \mathbb{Z}$ we have $\varphi^{r_1\cdots r_k t}(\theta_i) = \theta_i$, moreover if $(y_\alpha)_{\alpha\in\Gamma} = \sigma_\varphi^{r_1\cdots r_k t}((w_\alpha)_{\alpha\in\Gamma}) = (w_{\varphi^{r_1\cdots r_k t}(\alpha)})_{\alpha\in\Gamma}$, then for $i = 1, \ldots, k$ we have $y_{\theta_i} = w_{\varphi^{r_1\cdots r_k t}(\theta_i)} = w_{\theta_i}$, which leads to $(y_\alpha)_{\alpha\in\Gamma} \in \prod\limits_{\alpha\in\Gamma} U_\alpha (\subseteq U)$ and completes the proof. □

Lemma 3.3. *If* $\mathrm{Per}(\varphi) = \Gamma$*, and* B *is a compact subset of* X^Γ*, then for each* $\alpha \in \Gamma$ *there exists finite subset* D_α *of* X *such that* $B \subseteq \prod\limits_{\alpha\in\Gamma} D_\alpha =: D$ *and* $D_\beta = D_{\varphi(\beta)}$ *for all* $\beta \in \Gamma$*, hence* $\sigma_\varphi(D) = D$*.*

Proof. For $\alpha \in \Gamma$ let $\mathrm{orb}(\varphi, \alpha) := \{\varphi^n(\alpha) : n \in \mathbb{Z}\}$ suppose $\pi_\alpha : X^\Gamma \to X$ is the projection map on the αth coordinate. Since $\alpha \in \mathrm{Per}(\varphi)$, the set $\mathrm{orb}(\varphi, \alpha)$ is finite and $\mathrm{orb}(\varphi, \alpha) = \mathrm{orb}(\varphi, \beta)$ for all $\beta \in \mathrm{orb}(\varphi, \alpha)$. If B is a compact nonempty subset of X^Γ and $\alpha \in \Gamma$, then $\pi_\alpha(B)$ is a compact and hence finite subset of X. Thus $D_\alpha := \bigcup\{\pi_\beta(B) : \beta \in \mathrm{orb}(\varphi, \alpha)\}$ is finite and $D_\alpha = D_\beta$ for all $\beta \in \mathrm{orb}(\varphi, \alpha)$ in particular, $D_\alpha = D_{\varphi(\alpha)} = D_{\varphi^{-1}(\alpha)}$. Moreover, $\sigma_\varphi(D) = \prod_{\alpha\in\Gamma} D_{\varphi(\alpha)} = \prod_{\alpha\in\Gamma} D_\alpha = D$. □

Lemma 3.4. *If* $\mathrm{Per}(\varphi) = \Gamma$*, and* B *is a compact subset of* X^Γ*, then* $\overline{\{\sigma_\varphi^n(B) : n \in \mathbb{Z}\}}$ *is compact.*

Proof. Consider $D = \prod_{\alpha\in\Gamma} D_\alpha \supseteq B$ as in Lemma 3.3. By the Tykhonoff theorem D is a compact and hence closed subset of X^Γ. Using $\sigma_\varphi(D) = D$ and $B \subseteq D$ we have $\overline{\{\sigma_\varphi^n(B) : n \in \mathbb{Z}\}} \subseteq \overline{D} = D$, thus $\overline{\{\sigma_\varphi^n(B) : n \in \mathbb{Z}\}}$ is compact. □

Lemma 3.5. *If* $\mathrm{Per}(\varphi) = \Gamma$*, and* B *is a compact invariant subset of* $(X^\Gamma, \sigma_\varphi)$*, then* $(B, \sigma_\varphi \restriction_B)$ *is regularly almost periodic. In particular* $(B, \sigma_\varphi \restriction_B)$ *is almost periodic and recurrent.*

Proof. For $H \subseteq \Gamma$ let

$$\beta_H := \{((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma}) : (x_\alpha, y_\alpha)_{\alpha\in\Gamma} \in \prod_{\lambda\in\Gamma} \alpha_\lambda\} \qquad (*)$$

where $\alpha_\lambda = \{(w, w) : w \in X\}$ for $\lambda \in H$ and $\alpha_\lambda = X \times X$ for $\lambda \in \Gamma \setminus H$. Consider $D = \prod_{\alpha\in\Gamma} D_\alpha \supseteq B$ as in Lemma 3.3. We recall that since D_λs and $\prod_{\lambda\in\Lambda} D_\lambda$ are compact Hausdorff, they admit a unique uniformity. If α is an index of $\prod_{\lambda\in\Lambda} D_\lambda$, then there exist $\lambda_1, \ldots, \lambda_m \in \Gamma$ such that $\beta_{\{\lambda_1,\ldots,\lambda_m\}} \cap (D \times D) \subseteq \alpha$. Since $\mathrm{Per}(\varphi) = \Gamma$, there exists $n \in \mathbb{N}$ such that $\varphi^n(\lambda_i) = \lambda_i$ for all $i \in \{1, \ldots, m\}$. For all $x = (x_\lambda)_{\lambda\in\Gamma} \in D$, $k \in \mathbb{Z}$, and for $y = (y_\lambda)_{\lambda\in\Gamma} = \sigma_\varphi^{kn}(x) = (x_{\varphi^{kn}(\lambda)})_{\lambda\in\Gamma}$ we have:

$$\forall i \in \{1, \ldots, m\} \quad y_{\lambda_i} = x_{\varphi^{kn}(\lambda_i)} = x_{\lambda_i} \in D_{\lambda_i} ,$$

which leads to $(x, \sigma_\varphi^{kn}(x)) = (x, y) \in \beta_{\{\lambda_1,\ldots,\lambda_m\}} \cap (D \times D)$. Hence $(x, \sigma_\varphi^{kn}(x)) \in \alpha$ for all $x \in D, k \in \mathbb{Z}$ and $(D, \sigma_\varphi \restriction_D)$ is regularly almost periodic, therefore $(B, \sigma_\varphi \restriction_B)$ is regularly almost periodic. □

Corollary 3.6. *By Lemmas 3.4 and 3.5, if* $\mathrm{Per}(\varphi) = \Gamma$*, then* $(X^\Gamma, \sigma_\varphi)$ *is compactly almost periodic and compactly recurrent.*

Theorem 3.7 (Main Theorem)**.** *The following statements are equivalent:*

- $(X^\Gamma, \sigma_\varphi)$ *is pointwise recurrent;*
- $(X^\Gamma, \sigma_\varphi)$ *is pointwise almost periodic;*

- $(X^\Gamma, \sigma_\varphi)$ *is pointwise regularly almost periodic;*
- $(X^\Gamma, \sigma_\varphi)$ *is compactly almost periodic;*
- $(X^\Gamma, \sigma_\varphi)$ *is compactly recurrent;*
- $\mathrm{Per}(\varphi) = \Gamma$ *($\varphi : \Gamma \to \Gamma$ is pointwise periodic).*

Proof. Use Lemmas 3.1, 3.2, Corollary 3.6 and the fact that if $(X^\Gamma, \sigma_\varphi)$ is compactly almost periodic, then it is pointwise almost periodic and if $(X^\Gamma, \sigma_\varphi)$ is pointwise almost periodic or pointwise regularly almost periodic, then it is pointwise recurrent. □

Uniformly almost periodicity in generalized shift dynamical systems. Let X is finite, then X^Γ is compact, hence by Lemma 3.5 and Theorem 3.7 the following statements are equivalent:

- $(X^\Gamma, \sigma_\varphi)$ is pointwise recurrent;
- $(X^\Gamma, \sigma_\varphi)$ is pointwise almost periodic;
- $(X^\Gamma, \sigma_\varphi)$ is pointwise regularly almost periodic;
- $(X^\Gamma, \sigma_\varphi)$ is recurrent;
- $(X^\Gamma, \sigma_\varphi)$ is almost periodic;
- $(X^\Gamma, \sigma_\varphi)$ is regularly almost periodic;
- $\mathrm{Per}(\varphi) = \Gamma$.

However even for infinite X if we equip X^Γ with uniformity generated by basis $\{\beta_H : H \text{ is a finite subset of } \Gamma\}$ (β_Hs are defined in (*) in Lemma 3.5), then the above statements are equivalent, using a similar method described in this section and Lemma 3.5.

Example 3.8. Define $\varphi : \mathbb{N} \to \mathbb{N}$ with $\varphi(1) = 1$, $\varphi(k) = k + 1$ whenever $2^n \leq k \leq 2^{n+1} - 1$ and $\varphi(2^{n+1} - 1) = 2^n$ for $n \in \mathbb{N}$. Then φ is pointwise periodic and it is not periodic, hence $(X^\mathbb{N}, \sigma_\varphi)$ is pointwise almost periodic and satisfies all equivalent conditions of Theorem 3.7, but $(X^\mathbb{N}, \sigma_\varphi)$ is not pointwise periodic and does not satisfy any of equivalent conditions of Theorem 2.3.

For more details on generalized shifts' properties see [5] (dynamical properties), [1], [6] (algebraic entropy of generalized shifts) and [2] (topological entropy).

4. Acknowledgement

The authors are grateful to the research division of the University of Tehran, for the grant which supported this research under the ref. no. 6103027/1/07.
Moreover this paper is a part of 2nd author's MSc thesis under the supervision of the first author, in the summer of 2013, when he was a student in the University of Tehran. Moreover the primary form of this article has been presented in "The 10th Seminar on Differential Equations

and Dynamic Systems 6-7 November 2013, University of Mazandaran, Babolsar, Iran" [4].

References

1. M. Akhavin, A. Giordano Bruno, D. Dikranjan, A. Hosseini, F. Ayatollah Zadeh Shirazi. Algebraic entropy of shift endomorphisms on abelian groups. *Quaestiones Mathematicae*. 32, 529–550 (2009)
2. F. Ayatollah zadeh Shirazi, D. Dikranjan. Set-theoretical entropy: A tool to compute topological entropy. *Proceedings ICTA2011, Islamabad, Pakistan, July 410, 2011, Cambridge Scientific Publishers*. 11–32 (2012)
3. F. Ayatollah Zadeh Shirazi, N. Karami Kabir, F. Heydari Ardi. A note on shift theory. *Mathematica Pannonica, Proceedings of ITES-2007*. 19/2, 187–195 (2008).
4. F. Ayatollah Zadeh Shirazi; M. Miralaei. Pointwise almost periodicity in a generalized shift dynamical system. *Proceedings The 10th Seminar on Differential Equations and Dynamic Systems 6-7 November 2013, University of Mazandaran, Babolsar, Iran*. 45–48 (2013).
5. F. Ayatollah Zadeh Shirazi; J. Nazarian Sarkooh; B. Taherkhani. On Devaney chaotic generalized shift dynamical systems. *Studia Scientiarum Mathematicarum Hungarica*. 50 (4), 509–522 (2013).
6. A. Giodano Bruno. Algebraic entropy of generalized shifts on direct products. *Communications in Algebra*. 38, 4155–4175 (2010)
7. J. Dugundji. Topology. Allyn and Bacon, Inc., Boston (1966).
8. R. Ellis. Lectures on Topological Dynamics. W. A. Benjamin, New York. (1969).
9. P. Fabel. Characterizations of compactly almost periodic homeomorphisms of metrizable spaces. *Topology and its Applications*. 142, 1–12 (2004).