

## Use of the Sturm-Liouville problems in the seismic response of earth dams and embankments

A. Neamaty<sup>1</sup> and Y. Khalili

Department of Mathematics, University of Mazandaran, Babolsar, Iran

**ABSTRACT.** In this paper, we obtain a suitable mathematical model for the seismic response of dams. By using the shear beam model (SB model), we give a mathematical formulation that it is a partial differential equation and transform it to the Sturm-Liouville equation.

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### 1. INTRODUCTION

Embankments have three-dimensional structures. For an idealized situation, the seismic problems of dams are often represented in the one-dimensional model that this model has been used extensively to estimate the response of earth dams and embankments. Many researchers have studied the seismic problem of dams (see [1,2,5,9]). The shear beam model is a useful tool for a survey of vibrations of dams during an earthquake. The different versions of the shear stress in a SB model give various boundary value problems. One of these problems was investigated by Rahman [12]. On the other hand, Sturm-Liouville problems are boundary value problems and often appear in engineering.

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<sup>1</sup> Corresponding author: [namaty@umz.ac.ir](mailto:namaty@umz.ac.ir)  
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The Sturm-Liouville problems are also studied in [3,4,7,8,10,11,13] extensively. These have excited us to take these equations for a study of vibrations of dams.

In this paper, a shear beam model is used for a survey of the seismic response of dams by taking two shear stresses. In a present study, we will be focused on the shear stress in the following forms:

$$\begin{aligned} i_1) \tau_{yz}(z; t) &= G \left( \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \right), \\ i_2) \tau_{yz}(z; t) &= G \left( \frac{1}{z-a} \frac{\partial u}{\partial z} \right), \end{aligned}$$

where  $G$  is a average shear modulus,  $a$  is a point in the interval  $(h, H)$  for constants  $h, H$  and  $u$  is a displacement of the dam.

The main goal of this paper is to investigate the seismic response of dams for two different stresses. First the governing equations are presented by physical rules and then the Sturm-Liouville model of these equations is given.

## 2. MAIN RESULT

In this section, considering the net force acting on the elements and the inertia force, we formulate the seismic problem of dams. The shear stress plays a central role in this work. The Sturm-Liouville equations are taken here that this can show use of these equations in engineering.

**Sturm-Liouville equation 1.** First we obtain the following proposition.

**Proposition 2.1.** Let the functions  $q_1$  and  $q_0$  be complex-valued. Consider the partial differential equation

$$-u_{xx} + q_1(x)u_t + q_0(x)u = u_{tt}. \quad (2.1)$$

By taking a separation of variables, the equation (2.1) turns to the ordinary differential equation in the following form

$$-y'' + (\rho q_1(x) + q_0(x))y = \rho^2 y, \quad (2.2)$$

for constant  $\rho$ .

**Proof.** By substituting  $u(x; t) = y(x)\phi(t)$  in (2.1), we have

$$-\frac{d^2 y}{dx^2} \phi + q_1(x)y \frac{d\phi}{dt} + q_0(x)y\phi = y \frac{d^2 \phi}{dt^2}.$$

Therefore

$$-\frac{y''}{y} + q_1(x) \frac{\dot{\phi}}{\phi} + q_0(x) = \frac{\ddot{\phi}}{\phi} = \rho^2. \quad (2.3)$$

Thus, we get

$$\begin{cases} -y'' + \left(q_1(x)\frac{\dot{\phi}}{\phi} + q_0(x)\right)y = \rho^2 y, \\ \ddot{\phi} - \rho^2 \phi = 0. \end{cases} \quad (2.4)$$

The second equation has a solution  $\phi(t) = \exp\left(\sqrt{\rho^2 t}\right)$ , and consequently  $\frac{\dot{\phi}}{\phi} = \rho$ . Substituting this result in the first equation (2.4), we have

$$-y'' + (\rho q_1(x) + q_0(x))y = \rho^2 y. \quad (2.5)$$

The proof is completed.

Now we express first boundary value problem.

A vibration causes a shear force on the horizontal face of the form

$$F_H = \tau_{yz}(z; t)A_{xy}(z), \quad h \leq z \leq H, \quad t \geq 0.$$

Here

$$\tau_{yz}(z; t) = G \left( \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \right), \quad (2.6)$$

is a shear stress on the plan  $xy$ , and for constants  $B$  and  $L$ ,

$$A_{xy}(z) = b(z)L, \quad \rightarrow \quad A_{xy}(z) = \frac{BL}{H}z,$$

is area of the plan  $xy$ . Also we consider  $G = G_b \left(\frac{z}{H}\right)^m$  for  $0 \leq m \leq 1$ , as a average shear modulus of the soil that  $G_b$  is the average shear modulus on the base of the dam. Therefore the net force is equal to

$$dF_H = -\frac{G_b BL}{H^{m+1}} \frac{\partial}{\partial z} \left( z^{m+1} \left( \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \right) \right) dz. \quad (2.7)$$

On the other hand, the inertia force on the element is

$$I = \rho_s V(z)\ddot{u},$$

wherein  $\rho_s$  and  $V(z)$  are the mass density of the soil and the volume of the body of the dam, respectively. Hence, we have

$$V(z) = b(z)Ldz, \quad \rightarrow \quad V(z) = \frac{BL}{H}zdz.$$

Therefore

$$I = \rho_s \frac{BL}{H} z dz \ddot{u}. \quad (2.8)$$

The equation of a motion is derived by considering the dynamic equilibrium of an infinitesimal body. For a dynamic equilibrium of the elements,

the net force acting on the elements must be equal to the inertia force on the elements. In other words

$$dF_H + I = 0. \quad (2.9)$$

Substituting (2.7) and (2.8) in (2.9), we arrive at the following equation

$$\frac{G_b}{H^m} \frac{\partial}{\partial z} \left( z^{m+1} \left( \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \right) \right) = \rho_s z \frac{\partial^2 u}{\partial t^2}.$$

Therefore

$$\frac{\partial^2 u}{\partial z^2} + \left( \frac{m+1}{z} + \rho \right) \frac{\partial u}{\partial z} + \rho \frac{m+1}{z} u = \frac{\rho_s H^m}{G_b} z^{-m} \frac{\partial^2 u}{\partial t^2}. \quad (2.10)$$

Setting

$$u(z; t) = v(z) \phi(t), \quad (2.11)$$

and substituting in (2.10), we obtain the ODE in the following form

$$\frac{d^2 v}{dz^2} + \left( \frac{m+1}{z} + \rho \right) \frac{dv}{dz} + \rho \frac{m+1}{z} v = \rho^2 \frac{\rho_s H^m}{G_b} z^{-m} v.$$

The boundary conditions in this problem are  $\tau_{yz}(h; t) = 0$ , i.e., the top surface is stress free and  $u(H; t) = 0$ , i.e., the dam is rigidly connected to the base.

**Corollary 2.2.** The stress (2.6) gives the following boundary value problem for the seismic response of dams

$$\frac{d^2 v}{dz^2} + \left( \frac{m+1}{z} + \rho \right) \frac{dv}{dz} + \rho \frac{m+1}{z} v = \rho^2 \frac{\rho_s H^m}{G_b} z^{-m} v, \quad (2.12)$$

$$U(v) := v'(h) + \rho v(h) = 0, \quad V(v) := v(H) = 0. \quad (2.13)$$

**Theorem 2.3.** We know from [6] that the change of a variable

$$y = \exp \left( -\frac{1}{2} \int_a^x p(t) dt \right) w, \quad x \in [a, b], \quad (2.14)$$

reduces the differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

to the form

$$w'' + \left( q(x) - \frac{1}{4} p^2(x) - \frac{1}{2} p'(x) \right) w = 0. \quad (2.15)$$

Now, by using this transformation, we transform (2.12) to the Sturm-Liouville equation. Thus taking

$$v(z) = h^{\frac{m+1}{2}} \exp \left( \frac{1}{2} \rho h \right) z^{\frac{m+1}{-2}} \exp \left( \frac{-1}{2} \rho z \right) s(z), \quad (2.16)$$

we have

$$\begin{aligned} -s'' + \left( \rho \frac{m+1}{-2z} + \frac{(m+1)^2 - 2(m+1)}{4z^2} \right) s \\ = \rho^2 \left( \frac{-\rho_s H^m}{G_b z^m} - \frac{1}{4} \right) s. \end{aligned} \quad (2.17)$$

Also by using this transformation for the boundary conditions (2.13), we obtain the following corollary.

**Corollary 2.4.** Considering (2.6), the Sturm-Liouville problem is of the form

$$-s'' + (\rho q_1(z) + q_0(z)) s = \rho^2 r(z) s, \quad (2.18)$$

where  $q_1(z) = \frac{m+1}{-2z}$ ,  $q_0(z) = \frac{(m+1)^2 - 2(m+1)}{4z^2}$  and  $r(z) = \frac{-\rho_s H^m}{G_b z^m} - \frac{1}{4}$ . Also

$$U(s) := s'(h) + (\beta_1 \rho + \beta_0) s(h) = 0, \quad V(s) := s(H) = 0, \quad (2.19)$$

where  $\beta_1 = h - \frac{1}{2} h^{\frac{m+1}{2}}$  and  $\beta_0 = \frac{m+1}{-2} h^{\frac{m+3}{2}}$ .

Therefore this problem is a spectral problem with the differential pencil (2.18) and spectral boundary conditions (2.19).

**Sturm-Liouville equation 2.** Here we will survey the response of dams with shear stress

$$\tau_{yz}(z; t) = G \left( \frac{1}{z-a} \frac{\partial u}{\partial z} \right), \quad h \leq z \leq H, \quad t \geq 0, \quad (2.20)$$

for  $G = G_b$  and  $a \in (h, H)$ . Using this stress, the net force is equal to

$$dF_H = \frac{G_b B L}{H} \left( \frac{-a}{(z-a)^2} \frac{\partial u}{\partial z} + \frac{z}{z-a} \frac{\partial^2 u}{\partial z^2} \right) dz. \quad (2.21)$$

As it is told, for the dynamic equilibrium of the elements, the net force acting on the elements must be equal to the inertia force on the elements. Therefore taking the inertia force calculated in the previous section i.e., (2.8) and (2.21), we have

$$G_b \left( \frac{-a}{(z-a)^2} \frac{\partial u}{\partial z} + \frac{z}{z-a} \frac{\partial^2 u}{\partial z^2} \right) = \rho_s z \frac{\partial^2 u}{\partial t^2}.$$

So

$$\frac{\partial^2 u}{\partial z^2} - \frac{a}{z(z-a)} \frac{\partial u}{\partial z} = \frac{\rho_s}{G_b} (z-a) \frac{\partial^2 u}{\partial t^2}. \quad (2.22)$$

Using the separation of variables technique  $u(z; t) = v(z)\phi(t)$ , we get

$$\frac{d^2 v}{dz^2} - \frac{a}{z(z-a)} \frac{dv}{dz} = \rho^2 \frac{\rho_s}{G_b} (z-a) v.$$

The boundary conditions in this problem are also  $\tau_{yz}(h; t) = 0$  and  $u(H; t) = 0$ .

**Corollary 2.5.** Applying the stress (2.20), we obtain the following boundary value problem for the seismic response of dams

$$\frac{d^2v}{dz^2} - \frac{a}{z(z-a)} \frac{dv}{dz} = \rho^2 \frac{\rho_s}{G_b} (z-a)v, \quad (2.23)$$

$$U(v) := v'(h) = 0, \quad V(v) := v(H) = 0. \quad (2.24)$$

Taking Theorem 2.3, we have the following change of a variable

$$v(z) = \left(\frac{h-a}{h}\right)^{\frac{-1}{2}} \left(\frac{z}{z-a}\right)^{\frac{-1}{2}} s(z). \quad (2.25)$$

Substituting (2.25) in (2.23), we obtain

$$-s'' + \frac{az - \frac{1}{4}a^2}{(z-a)^2} s = \rho^2 \left(\frac{-\rho_s}{G_b} (z-a)\right) s. \quad (2.26)$$

Also using this transformation for the boundary conditions (2.24), we have the following corollary.

**Corollary 2.6.** Considering (2.20), the Sturm-Liouville problem is in the following form

$$-s'' + q(z)s = \rho^2 r(z)s, \quad (2.27)$$

where  $q(z) = \frac{az - \frac{1}{4}a^2}{(z-a)^2}$  and  $r(z) = \frac{-\rho_s}{G_b} (z-a)$ . Also

$$U(s) := s'(h) + \beta s(h) = 0, \quad V(s) := s(H) = 0, \quad (2.28)$$

where  $\beta = \frac{a}{2H(H-a)}$ .

Therefore this problem is a spectral problem for Sturm-Liouville equations with a turning point and singularity.

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