Approximate mixed additive and quadratic functional in 2-Banach spaces

Shirin Eivani 1 and Saeed Ostadbashi 2
1 Department of Mathematics, Faculty of Sciences, Urmia University, Urmia, Iran
1 shirin.eivani@gmail.com
2 Department of Mathematics, Faculty of Sciences, Urmia University, Urmia, Iran
2 s.ostadbashi@urmia.ac.ir

Abstract. In the paper we establish the general solution of the function equation $f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 2f(2x) - 2f(x)$ and investigate the Hyers-Ulam-Rassias stability of this equation in 2-Banach spaces.

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1. Introduction

In 1940, S. M. Ulam gave a talk before the Mathematics club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1, \ast)$ be a group and let $(G_2, \circ, d)$ be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta(\varepsilon) > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(x \ast y), h(x) \circ h(y)) < \delta$$

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for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with
\[ d(h(x), H(x)) < \varepsilon \]
for all $x \in G_1$.

In 1941, D. H. Hyers [4] considered the case of approximately additive mappings $f : E \rightarrow E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies Hyers inequality
\[ \| f(x + y) - f(x) - f(y) \| \leq \varepsilon \]
for all $x, y \in E$. It was shown that the limit
\[ L(x) = \lim_{n \rightarrow \infty} \frac{f(2^nx)}{2^n} \]
exists for all $x \in E$ and that $L : E \rightarrow E'$ is the unique additive mapping satisfying
\[ \| f(x) - L(x) \| \leq \varepsilon. \]

In 1978, Th. M. Rassias [7] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded.

In this paper, we deal with the next functional equation deriving from additive and quadratic functions:
\[ f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 2f(2x) - 2f(x) \quad (1.1) \]
It is easy to see that the function $f(x) = ax^2 + bx + c$ is a solution of the functional equation (1.1).

The main purpose of this paper is to establish the general solution of Eq. (1.1) and investigate the Hyers- Ulam- Rassias stability for Eq. (1.1).

We recall some basic facts concerning 2-Banach spaces and some preliminary results [2, 3].

**Definition 1.1.** Let $X$ be a linear space over $\mathbb{R}$ with $\dim X > 1$ and let $\| ., . \| : X \times X \rightarrow \mathbb{R}$ be a function satisfying the following properties:
1. $\| x, y \| = 0$ if and only if $x$ and $y$ are linearly dependent;
2. $\| x, y \|=\| y, x \| ;$
3. $\| \alpha x, y \| = \| x, y \| ;$
4. $\| x, y + z \| \leq \| x, y \| + \| x, z \|$
for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$. Then the function $\| ., . \|$ is called a 2-norm on $X$ and the pair $(X, \| ., . \|)$ is called a linear 2-normed spaces. Sometimes the condition (4) called the triangle inequality.

We introduce a basic property of linear 2-normed spaces as follows.

Let $(X, \| ., . \|)$ be a linear 2-normed spaces, $x \in X$ and $\| x, y \|= 0$ for all $y \in X$. Suppose $x \neq 0$ and take $y_1, y_2$ linearly independent (so nonzero) in $X$. The condition (1) implies that $x$ and $y_1$ are linearly
dependent. Thus there exist \( \alpha_1, \beta_1 \in \mathbb{R} \) such that \((\alpha_1, \beta_1) \neq (0, 0)\) and \(\alpha_1 x + \beta_1 y_1 = 0\), if \(\beta_1 = 0\), we get \(\alpha_1 \neq 0\). So we have \(x = -\frac{\beta_1}{\alpha_1} y_1 = 0\), which is a contradiction. Thus we have \(\beta_1 \neq 0\) and \(y_1 = -\frac{\alpha_1}{\beta_1} x\).

Similarly, there exist \(\alpha_2, \beta_2 \in \mathbb{R} \) such that \(\beta_2 \neq 0\) and \(y_2 = -\frac{\alpha_2}{\beta_2} x\). Hence \(y_1\) and \(y_2\) are linearly dependent, which is a contradiction. Therefore we have the following lemma.

**Lemma 1.2.** \((\text{[5]}\) Let \((X, \| .. . \|)\) be a linear 2-normed space. If \(x \in X\) and \(\| x, y \| = 0\) for all \(y \in X\), then \(x = 0\).

**Definition 1.3.** A sequence \(\{x_n\}\) in a linear 2-normed space \(X\) is called a cauchy sequence if there are two points \(y, z \in X\) such that \(y\) and \(z\) are linearly independent,

\[
\lim_{m,n \to \infty} \| x_n - x_m, y \| = 0
\]

and

\[
\lim_{m,n \to \infty} \| x_n - x_m, z \| = 0.
\]

**Definition 1.4.** A sequence \(\{x_n\}\) in a linear 2-normed space \(X\) is called a convergent sequence if there is an \(x \in X\) such that

\[
\lim_{n \to \infty} \| x_n - x, y \| = 0
\]

for all \(y \in X\). If \(\{x_n\}\) converges to \(x\), write \(x_n \to x\) as \(n \to \infty\) and call \(x\) the limit of \(\{x_n\}\). In this case, we also write \(\lim_{n \to \infty} x_n = x\).

First we will quote some result by the authors in \([5,6]\), which will be applied later on.

**Lemma 1.5.** If an even function \(f : X \to Y\) with \(f(0) = 0\) satisfies (1.1) for all \(x, y \in X\), \(X\) and \(Y\) will be real vector spaces, then \(f\) is quadratic.

**Lemma 1.6.** If an odd function \(f : X \to Y\) satisfies (1.1) for all \(x, y \in X\), \(X\) and \(Y\) will be real vector spaces, then \(f\) is additive.

**Lemma 1.7.** For a convergent sequence \(\{x_n\}\) in a linear 2-normed space \(X\),

\[
\lim_{n \to \infty} \| x_n, y \| = \| \lim_{n \to \infty} x_n, y \|
\]

for all \(y \in X\).

**Lemma 1.8.** Let \(0 < p \leq 1\) and let \(x_1, x_2, \ldots, x_n\) be non-negative real numbers. Then

\[
\left( \sum_{i=1}^{n} x_i \right)^p \leq \sum_{i=1}^{n} x_i^p.
\]
Definition 1.9. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.

In this paper, we investigate approximate mixed additive and quadratic function in 2-Banach spaces.

2. Main result


Theorem 2.1. Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ and $p + q > 2$ and let $f : X \rightarrow Y$ with $f(0) = 0$ be a mapping satisfying

$$\| Df(x, y), z \| = \| f(2x + y) + f(2x - y) - f(x + y) - f(x - y) - 2f(2x) + 2f(x), z \| \leq \theta \| x \| p \| y \| q \| z \| r$$

(2.1)

for all $x, y, z \in X$. Then there is a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\| f(x) - Q(x), y \| \leq \frac{4 + 3^q}{(3^p q)(2^p q - 4)} \theta \| x \| p + q \| y \| r$$

(2.2)

for all $x, y \in X$.

Proof. By replacing $y$ by $x + y$ in (2.1), we get

$$\| f(3x + y) + f(x - y) - f(2x + y) - f(y) - 2f(2x) + 2f(x), z \| \leq \theta \| x \| p + q \| z \| r + \theta \| x \| p \| y \| q \| z \| r$$

(2.3)

for all $x, y, z \in X$. Replacing $y$ by $-y$ in (2.3), we get

$$\| f(3x - y) + f(x + y) - f(2x - y) - f(y) - 2f(2x) + 2f(x), z \| \leq \theta \| x \| p + q \| z \| r + \theta \| x \| p \| y \| q \| z \| r$$

(2.4)

for all $x, y, z \in X$. It follows from (2.1), (2.3) and (2.4),

$$\| f(3x + y) + f(3x - y) - 2f(y) - 6f(2x) + 6f(x), z \| \leq 2\theta \| x \| p \| y \| q \| z \| r + 2\theta \| x \| p + q \| z \| r$$

(2.5)

for all $x, y, z \in X$. By letting $y = 0$ and $y = 3x$ in (2.5), we get the inequalities

$$\| 2f(3x) - 6f(2x) + 6f(x), z \| \leq 2\theta \| x \| p + q \| z \| r$$

(2.6)

$$\| f(6x) - 2f(3x) - 6f(2x) + 6f(x), z \| \leq (2 + 3^q)\theta \| x \| p + q \| z \| r$$

(2.7)
for all \(x, z \in X\). It follows from (2.6) and (2.7),
\[
\| f(6x) - 4f(3x), z \| \leq (4 + 3^q)\theta \| x \|^{p+q} \| z \|^r
\]  
for all \(x, z \in X\). If we replace \(x\) by \(x/3\) in (2.8), we get
\[
\| f(2x) - 4f(x), z \| \leq \frac{4 + 3^q}{3^{p+q}}\theta \| x \|^{p+q} \| z \|^r
\]  
for all \(x, z \in X\). If we replace \(x\) in (2.9) by \(x_{2n+1}^{m}\) and multiply both sides of (2.9) by \(2^n\), then we have
\[
\| 4^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 4^n f\left(\frac{x}{2^n}\right), z \| \leq \frac{(4 + 3^q)2^{n(2-p-q)}}{(3^{p+q})^{2p+q}}\theta \| x \|^{p+q} \| z \|^r
\]  
for all \(x, z \in X\) and all non-negative integers \(n\). For all integer \(m\) and \(n\) with \(n \geq m\), we get
\[
\| 4^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 4^m f\left(\frac{x}{2^m}\right), z \|
\leq \sum_{i=m}^{n} \frac{4 + 3^q}{(3^{p+q})^{2p+q}}2^{i(2-p-q)}\theta \| x \|^{p+q} \| z \|^r
\]  
for all \(x, z \in X\). So we get
\[
\lim_{n,m \to \infty} \| 4^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 4^m f\left(\frac{x}{2^m}\right), z \| = 0
\]  
for all \(x, z \in X\). Thus the sequence \(\{4^n f\left(\frac{x}{2^n}\right)\}\) is a Cauchy sequence in \(Y\). Since \(Y\) is a 2-Banach space, the sequence \(\{4^n f\left(\frac{x}{2^n}\right)\}\) converges. So one can define the mapping \(Q : X \to Y\) by
\[
Q(x) := \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right)
\]  
for all \(x \in X\). That is,
\[
\lim_{n \to \infty} \| 4^n f\left(\frac{x}{2^n}\right) - Q(x), y \| = 0
\]  
for all \(x, y \in X\). Now, we show that \(Q\) is quadratic.

By lemma (1.7) and (2.1), we get
\[
\| DQ(x, y), z \| = \| Q(2x + y) + Q(2x - y) - Q(x + y) - Q(x - y)
- 2Q(2x) + 2Q(x), z \|
\leq \lim_{n \to \infty} 4^n \| Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), z \|
\leq \theta \| x \|^{p} \| y \|^{q} \| z \|^r \lim_{n \to \infty} 2^{n(2-p-q)} = 0
\]
for all $x, y, z \in X$. By lemma (1.2),

$$Q(2x + y) + Q(2x - y) = Q(x + y) + Q(x - y) + 2Q(2x) - 2Q(x)$$

for all $x, y \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (2.11), we get

$$\| f(x) - Q(x), y \| = \lim_{n \to \infty} \| f(x) - 4^n f(\frac{x}{2^n}), y \|$$

$$\leq \frac{4 + 3q}{(3p+q)(4 - 2p+q - 4)} \theta \| x \|^{p+q} \| y \|^r$$

for all $x, y \in X$.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.2). Then we have

$$\| Q(x) - T(x), y \| = 4^n \| Q(\frac{x}{2^n}) - T(\frac{x}{2^n}), y \|$$

$$\leq 4^n \| Q(\frac{x}{2^n}) - f(\frac{x}{2^n}), y \| + \| f(\frac{x}{2^n}) - T(\frac{x}{2^n}), y \|$$

$$\leq \frac{4 + 3q}{(3p+q)(2p+q-1)} \theta \| x \|^{p+q} \| y \|^r$$

which tends to zero as $n \to \infty$ for all $x, y \in X$. By lemma (1.2), we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $Q$. □

**Theorem 2.2.** Let $\theta \in [0, \infty), p, q, r \in (0, \infty)$ and $p + q < 2$ and let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying

$$\| Df(x), y, z \| = \| f(2x + y) + f(2x - y) - f(x + y) - f(x - y) - 2f(2x) + 2f(x), z \|$$

$$\leq \theta \| x \|^{p} \| y \|^{q} \| z \|^{r} \quad (2.12)$$

for all $x, y, z \in X$. Then there is a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x), y \| \leq \frac{4 + 3q}{(3p+q)(4 - 2p+q)} \theta \| x \|^{p+q} \| y \|^r \quad (2.13)$$

for all $x, y \in X$.

**Proof.** By the same argument as in the proof of Theorem 2.1 we get

$$\| f(2x) - 4f(x), z \| \leq \frac{4 + 3q}{3p+q} \theta \| x \|^{p+q} \| z \|^r \quad (2.13)$$

for all $x, z \in X$. Replacing $x$ by $2^n x$ and dividing $4^{n+1}$ in (2.13), we obtain

$$\| \frac{1}{4^{n+1}} f(2^{n+1} x) - \frac{1}{4^{n}} f(2^{n} x), z \| \leq \frac{4 + 3q}{4(3p+q)} 2^{n(p+q-2)} \theta \| x \|^{p+q} \| z \|^r$$
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for all \( x, z \in X \) and all integer \( n > 0 \). For all integer \( m \) and \( n \) with \( n \geq m \), we get

\[
\| \frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^{m}} f(2^{m}x), z \| \leq \sum_{i=m}^{n} \frac{4 + 3^q}{4(3^p+q)} 2^{i(p+q-2)} \theta \| x \|^{p+q} z \|^{r} \]

for all \( x, z \in X \). So we get

\[
\lim_{n,m \to \infty} \| \frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^{m}} f(2^{m}x), z \| = 0
\]

for all \( x, z \in X \). Thus the sequence \( \{ \frac{1}{4^{n}} f(2^{n}x) \} \) is a Cauchy sequence in \( Y \). Since \( Y \) is a 2-Banach space, the sequence \( \{ \frac{1}{4^{n}} f(2^{n}x) \} \) converges. So one can define the mapping \( Q : X \to Y \) by

\[
Q(x) := \lim_{n \to \infty} \frac{1}{4^{n}} f(2^{n}x)
\]

for all \( x \in X \). That is,

\[
\lim_{n \to \infty} \| \frac{1}{4^{n}} f(2^{n}x) - Q(x), y \| = 0
\]

for all \( x, y \in X \).

The further part of the proof is similar to the proof of Theorem 2.1.

\[\Box\]

References