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Nilpotency and solubility of groups relative to an automorphism

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Abstract. In this paper we introduce the concept of α-commutator which its definition is based on generalized conjugate classes. With this notion, α-nilpotent groups, α-solvable groups, nilpotency and solvability of groups related to the automorphism α are defined. $\mathcal{N}(G)$ and $\mathcal{S}(G)$ are the set of all nilpotency classes and the set of all solvability classes for the group G with respect to different automorphisms of the group, respectively. If G is nilpotent or solvable with respect to the all its automorphisms, then is referred as it absolute nilpotent or solvable group. Subsequently, $\mathcal{N}(G)$ and $\mathcal{S}(G)$ are obtained for certain groups. This work is a study of the nilpotency and solvability of the group G from the point of view of the automorphism which the nilpotent and solvable groups have been divided to smaller classes of the nilpotency and the solvability with respect to its automorphisms.

Keywords: Nilpotent group, solvable group, automorphism.

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1. Introduction

Nilpotent and solvable groups in terms of the certain normal series of subgroups are defined in [5]. This approach demonstrates that there is a connection between nilpotent groups, solvable groups and commutators. A solvable group is a group whose derived series terminates in the trivial subgroup. Historically, the word solvable arose from Galois theory and the proof of the general unsolvability of quintic equation. Specifically, a polynomial equation is solvable by radicals if and only if the corresponding Galois group is solvable. The idea of nilpotent groups is motivated by the fact that all of them are solvable. In this paper, we introduce the α-nilpotent and the α-solvable group and we try to verify which properties of the ordinary nilpotent and solvable groups are valid here. This work is organized in five sections. In the next section we explain the concept of the α-commutator and its preliminary properties which are useful in the text. The relative nilpotent groups with respect to a certain automorphism are the subjects discussed in the third section. In the fourth section, We will specify the set $\mathcal{N}(G)$ for dihedral, quaternion, quasi-dihedral and modular $p-$group. In the last section, some properties of α-solvable group are discussed and also the set $\mathcal{S}(G)$ is gained for some groups. Throughout the paper, all notations and terminologies are standard (for instance see [5]).

2. Preliminary results

Definition 2.1. Let G be a group and $\alpha \in \mathrm{Aut(G)}$. For two elements $x, y \in G$, we say x and y commute under the automorphism α whenever $yx = xy^{\alpha}$. Moreover, $x^{-1}y^{-1}xy^{\alpha}$ is called α-commutator of x, y and denoted by $[x, y]_{\alpha}$.

It is clear that if α is the identity automorphism, then we have ordinary commutator. Furthermore, we nominate $[x, y]_{\alpha}$ a α-commutator because of the similar properties which it has in comparison to the usual commutator as $yx[x, y]_{\alpha} = xy^{\alpha}$. So we can think of the α-commutator of x and y as the "difference" between yx and xy^{α}. By the same method, one can define a α-commutator of weight n as follows

$$[x_1, x_2, \cdots, x_n]_{\alpha} = [x_1, [x_2, \cdots, x_n]_{\alpha}]_{\alpha}.$$

By definition, we conclude the following identities for the α-commutator.

Proposition 2.2. *Let G be a group. Then we have*

(i) $[x, x]_{\alpha} = [1, x]_{\alpha} = [x, \alpha]$, *where* $[x, \alpha] = x^{-1}x^{\alpha}$. *Moreover* $[x, 1]_{\alpha} = 1$.

(ii) $[x,y]_\alpha = [x,y][y,\alpha]$, $[x,y]_\alpha^\alpha = [x^\alpha, y^\alpha]_\alpha$ *and* $[x,y^{-1}]_\alpha = [x,y]_\alpha^{-(y^\alpha)^{-1}}$.

(iii) $[x,y]_{\alpha^\beta}^\beta = [x^\beta, y^\beta]_\alpha$, *where* $\alpha^\beta = \beta^{-1}\alpha\beta$.

(iv) $[x,y_1y_2]_\alpha = [x,y_2]_\alpha [x,y_1]_\alpha^{y_2^\alpha}$ *and* $[x_1x_2,y]_\alpha = [x_1,y]_\alpha^{x_2}[x_2,y^\alpha]$.

(v) $[[x,y^{-1}]_\alpha, z]_\alpha^{y^\alpha} = [x,y^{-1},z]^y [z^y,\alpha]$.

Definition 2.3. Let G be a group and X_1, X_2 be two non-empty subsets of G. We define α-commutator of X_1 and X_2 as follows

$$[X_1,X_2]_\alpha = \langle [x_1,x_2]_\alpha : x_1 \in X_1, x_2 \in X_2 \rangle.$$

It is obvious that $[X_1,X_2]_\alpha$ is a subgroup of G and $[X_1,X_2]_\alpha$ is not equal to $[X_2,X_1]_\alpha$ in general.

Definition 2.4. Let $\alpha \in \mathrm{Aut(G)}$. Consider the action $\psi : G \times G \to G$ such that $(x,g) \mapsto g^{-1}xg^\alpha$. With this action G is partitioned to the classes which are called generalized conjugacy classes or α-conjugacy class. We denote a α-conjugacy class which contains x by x_α^G and it is clear that,

$$x_\alpha^G = \{g^{-1}xg^\alpha : g \in G\}.$$

If $|x_\alpha^G| = 1$, then α is an inner automorphism. Moreover, let us recall a subgroup $C_G^\alpha(x) = \{y \in G : [x,y]_\alpha = 1\}$ which satisfies $|x_\alpha^G| = [G : C_G^\alpha(x)]$ (see [1]). Alternatively, we can write $x[x,y]_\alpha = y^{-1}xy^\alpha$, so $[x,y]_\alpha$ can also be viewed as the "difference" between x and its α-conjugate $y^{-1}xy^\alpha$. It is interesting to know that the number of generalized conjugacy classes is the number of ordinary conjugacy classes which are invariant under α and it is also equal to the number of irreducible characters which are invariant under α (see [3, 4]). We define α-center of the group G as

$$Z^\alpha(G) = \bigcap_{x\in G} C_G^\alpha(x) = \{y \in G : [x,y]_\alpha = 1 \text{ for all } x \in G\}.$$

One can deduce $Z^\alpha(G) = Z(G) \cap \mathrm{Fix}(\alpha)$, where $\mathrm{Fix}(\alpha) = \{g \in G : g^\alpha = g\}$. Moreover, $L(G) = \bigcap_{\alpha \in \mathrm{Aut}(G)} Z^\alpha(G)$, where $L(G)$ is called the absolute center of a group G (see [2]).

Definition 2.5. Let N be a normal subgroup of G and $N^\alpha = N$. Then we define $(xN)\overline{\alpha} = x^\alpha N$; where $\overline{\alpha} : G/N \to G/N$. It is trivial $[\overline{x_1}, \overline{x_2}, \cdots, \overline{x_n}]_{\overline{\alpha}} = [x_1,x_2,\cdots,x_n]_\alpha N$.

3. Relative Nilpotency of Groups

In this section we introduce the nilpotent group with respect to the fixed automorphism. Moreover, we obtain properties which are very similar to the argument about ordinary nilpotent groups (see [5]). Thus we omit some proofs.

Definition 3.1. A α-central series of G is a normal series

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

where $G_i^{\alpha} = G_i$ and $G_{i+1}/G_i \leq Z^{\overline{\alpha}}(G/G_i)$, $0 \leq i \leq n-1$.

If G has a α-central series, then G is α-nilpotent or we call G is nilpotent with respect to α. In this situation the smallest length of α-central series is called relative nilpotency class of the group.

If G is α-nilpotent of class c_{α}, then it is nilpotent of class at most c_{α}. We define the set of all relative nilpotency class of the group with respect to an automorphism of the group as $\mathcal{N}(G) = \{c_{\alpha} : \alpha \in \mathrm{Aut(G)}\}$. Moreover, if G is nilpotent with respect to a non-trivial automorphism, then we call G is a non-trivially nilpotent.

Definition 3.2. We call G is an absolute nilpotent group if it is nilpotent with respect to every automorphism of the group.

Definition 3.3. For the group G and $\alpha \in \mathrm{Aut}(G)$. We define the derived subgroup of G with respect to the automorphism α by

$$\Gamma_2^{\alpha}(G) = \langle [x, y]_{\alpha} : x, y \in G \rangle.$$

Similarly, we have

$$\Gamma_1^{\alpha}(G) = G,\ \Gamma_{n+1}^{\alpha}(G) = [G, \Gamma_n^{\alpha}(G)]_{\alpha}\ (n \geq 1).$$

By induction on n, one can see $\Gamma_{n+1}^{\alpha}(G) \subseteq \Gamma_n^{\alpha}(G)$ and $\Gamma_n^{\alpha}(G)$ is invariant under α.

Easily, we observe that $\Gamma_n^{\alpha}(G) \trianglelefteq G$ and the following series is a normal series,

$$G = \Gamma_1^{\alpha}(G) \trianglerighteq \Gamma_2^{\alpha}(G) \trianglerighteq \cdots .$$

Moreover, $\Gamma_n^{\overline{\alpha}}(G/N) = \Gamma_n^{\alpha}(G)N/N$, where $N^{\alpha} = N$.

Definition 3.4. Put $Z_1^{\alpha}(G) = Z^{\alpha}(G)$. Clearly $Z_i^{\alpha}(G) \trianglelefteq G$, $(Z_i^{\alpha}(G))^{\alpha} = Z_i^{\alpha}(G)$ and

$$Z^{\overline{\alpha}}\left(\frac{G}{Z_{i-1}^{\alpha}(G)}\right) = \frac{Z_i^{\alpha}(G)}{Z_{i-1}^{\alpha}(G)} \text{ for } i \in \mathbb{N}.$$

For a group G, the normal series $\{1\} = Z_0^{\alpha}(G) \trianglelefteq Z_1^{\alpha}(G) \trianglelefteq Z_2^{\alpha}(G) \trianglelefteq \cdots$ is called an upper central α-series and $G = \Gamma_1^{\alpha}(G) \trianglerighteq \Gamma_2^{\alpha}(G) \trianglerighteq \cdots$ is a lower central α-series for G. In general, these two series will not stop, but if so, we will prove that G is α-nilpotent and its converse is valid. Thus we find equivalent definitions for a α-nilpotent group G.

Theorem 3.5. *Suppose G is a group and $\alpha \in \mathrm{Aut}(G)$. Then*

(i) $Z_n^{\alpha}(G) = Z_n^{\alpha^{-1}}(G)$.

(ii) *$x \in Z_n^{\alpha}(G)$ if and only if $[g_1, \cdots, g_n, x]_{\alpha} = 1$ for all $g_1, g_2, \cdots, g_n \in G$.*

(iii) $Z_n^\alpha(G) \subseteq Z_n(G)$.

Proof. (i) and (ii) is easy and for the third part use induction on n. $\square$

An automorphism α of G is said to be central if α commutes with every inner automorphism or equivalently $\alpha(g)g^{-1} \in Z(G)$ for all $g \in G$.

Theorem 3.6. *Let α be a central automorphism. If $Z(G) = Z^\alpha(G)$, then $Z_n^\alpha(G) = Z_n(G)$ for all $n \geq 2$.*

Proof. The proof follows by induction on n. $\square$

Also for an inner automorphism α we have $Z_n^\alpha(G) = Z_n(G)$ for all n.

Lemma 3.7. *Suppose H and N are two subgroups of G. If $N \trianglelefteq G$ and $N^\alpha = N$, then $[G, HN]_\alpha \leq [G, H]_\alpha N$.*

Proof. For $h \in H$, $n \in N$ and $g \in G$ we have

$$[g, hn]_\alpha = [g, h]_\alpha (h^{-1}gh^\alpha)^{-1} n^{-1} (h^{-1}gh^\alpha) n^\alpha.$$

Hence, the result is clear. $\square$

Theorem 3.8. *For a group G the following is equivalent,*

(i) *G is α-nilpotent.*
(ii) *There is an integer r such that $\Gamma_r^\alpha(G) = 1$.*
(iii) *There is an integer s such that $Z_s^\alpha(G) = G$.*

Proof. Suppose (i), we will prove (ii). the group G has a central α-series,

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n.$$

Also $G_{i+1}/G_i \leq Z^\alpha(G/G_i)$, $0 \leq i \leq n-1$. By induction, it follows $\Gamma_{i+1}^\alpha(G) \leq G_{n-i}$. Thus, for $i = n-1$ we have $\Gamma_n^\alpha(G) \leq G_1$. On the other hand $G_1 \leq Z^\alpha(G)$ and

$$\Gamma_{n+1}^\alpha(G) = [G, \Gamma_n^\alpha(G)]_\alpha \leq [G, G_1] = \{1\}.$$

Hence $r = n+1$.

Let us prove (iii) by assuming (ii). By induction and Lemma 3.7, it can be readily observed that $\Gamma_{r-i}^\alpha(G) \leq Z_i^\alpha(G)$, $0 \leq i \leq r-1$. Now for $i = r-1$ we conclude $G = \Gamma_1^\alpha(G) \leq Z_{r-1}^\alpha(G)$ and $G = Z_{r-1}^\alpha(G)$. Thus we imply $s = r-1$. Hence the assertion is clear. $\square$

Theorem 3.9. *Suppose α-nilpotent group G has upper and lower α-series as follows*

$$\{1\} = Z_0^\alpha(G) \trianglelefteq Z_1^\alpha(G) \trianglelefteq \cdots \trianglelefteq Z_m^\alpha(G) = G$$

$$G = \Gamma_1^\alpha(G) \trianglerighteq \Gamma_2^\alpha(G) \trianglerighteq \cdots \trianglerighteq \Gamma_n^\alpha(G).$$

Then $m = n$ is α-nilpotency class of G.

Proof. Assume G has a α-central series of the least length c as follows,

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_c.$$

Since upper α-central series is a central series, we have $c \leq m$. By induction and Lemma 3.7, we deduce that $G_i \leq Z_i^{\alpha}(G)$. Now for $i = c$ we have $G = G_c \leq Z_c^{\alpha}(G)$. Thus $c = m$. Moreover, we observe that $\Gamma_i^{\alpha}(G) \leq Z_{m+1-i}^{\alpha}(G)$. If $i = m+1$, then $\Gamma_{m+1}^{\alpha}(G) \leq Z_0^{\alpha}(G) = \{1\}$. Therefore $m \geq n$ which implies $c = n$. □

By third part of Proposition 2.2, it follows, if α and β are in the same conjugacy class, then the nilpotency class of G with respect to α and β are equal. The proof of the following theorems are obvious, so we omit the demonstration.

Theorem 3.10. *If G is an α-nilpotent group, $N \trianglelefteq G$ and $N^{\alpha} = N$, then $N \cap Z^{\alpha}(G) \neq 1$.*

Theorem 3.11. *Let $\alpha \in \mathrm{Aut(G)}$ and $\beta \in \mathrm{Aut(H)}$. Then*

(i) $\Gamma_n^{\alpha \times \beta}(G \times H) = \Gamma_n^{\alpha}(G) \times \Gamma_n^{\beta}(G)$,
(ii) $Z_n^{\alpha \times \beta}(G \times H) = Z_n^{\alpha}(G) \times Z_n^{\beta}(H)$,
(iii) $Z_m^{\overline{\alpha}}(\frac{G}{Z_n^{\alpha}(G)}) = \frac{Z_{m+n}^{\alpha}(G)}{Z_n^{\alpha}(G)}$,

where $(\alpha \times \beta)(g, h) = (\alpha(g), \beta(h))$ for all $g \in G$ and $h \in H$.

Theorem 3.12. *Let $\alpha \in Aut(G)$. Then the group G is α-nilpotent if and only if G is direct product of Sylow p-subgroups which are nilpotent with respect to α.*

Proof. By Theorem 3.11, the assertion follows. □

There is a question here to ask whether or no we could imply three subgroups lemma or the following result? Let G be a group and $\alpha \in$ $\mathrm{Aut(G)}$*. Then*

(i) $[\Gamma_i^{\alpha}(G), \Gamma_j^{\alpha}(G)]_{\alpha} \leq \Gamma_{i+j}^{\alpha}(G)$.
(ii) $\Gamma_i^{\alpha}(\Gamma_j^{\alpha}(G)) \leq \Gamma_{ij}^{\alpha}(G)$.
(iii) *If $i \geq j$, then $[Z_i^{\alpha}(G), \Gamma_j^{\alpha}(G)]_{\alpha} \leq Z_{i+j}^{\alpha}(G)$.*

Now, we conclude the following theorems for abelian groups.

Theorem 3.13. *Let G be an abelian absolute nilpotent group. Then G is a 2-group.*

Proof. Consider the automorphism $\alpha(x) = x^{-1}$ and suppose that the nilpotency class of G with respect to α is c. Therefore $Z_i^{\alpha}(G) = \{x \in G : x^{2i} = 1\}$ and the assertion is clear. □

Theorem 3.14. *Let G be an abelian group. Then for all $\alpha \in \mathrm{Aut(G)}$*

$$\frac{G}{Z_n^\alpha(G)} \cong \Gamma_{n+1}^\alpha(G).$$

Proof. By induction on n, we have $[g_1, g_2, \cdots, g_n, g] = (\alpha - \imath)^n(g)$, where $\imath$ is identity automorphism. Thus $\mathrm{Im}(\alpha - \imath)^n = \Gamma_{n+1}^\alpha(G)$ and $\mathrm{Ker}(\alpha - \imath)^n = Z_n^\alpha(G)$ and the result is clear. □

By the above theorem, an abelian group G is α-nilpotent if and only if $(\alpha - \imath)^n = 0$ and $(\alpha - \imath)^{n-1} \neq 0$. Moreover, for all abelian group G, if $G/Z_n^\alpha(G)$ is finite, then $\Gamma_{n+1}^\alpha(G)$ is finite. Now, we try to prove it in general. In the following theorem we observe that this assertion is valid for $n = 1$.

Theorem 3.15. *Let G be a finite abelian group. Then there are $|Z^\alpha(G)|$ α-conjugacy classes of length $|\Gamma_2^\alpha(G)|$.*

Proof. $x, y \in G$ are $\alpha-$conjugate if and only if the following hold:

$$\begin{aligned} x &= -g + y + g^\alpha \\ x - y &= g^\alpha - g \\ x - y &= (\alpha - \imath)(g) \\ x - y &\in Im(\alpha - \imath) = \Gamma_2^\alpha(G). \end{aligned}$$

By the Theorem 3.15, the assertion follows. □

For instance, $\mathbb{Z}_{27}$ with automorphisms $\alpha : n \mapsto 10n$ and $\beta : n \mapsto 4n$ we have $|Z^\alpha(G)| = 9$ and $|Z^\alpha(G)| = 3$.
By Theorem 3.15, for every abelian group G, if $G/Z_n^\alpha(G)$ is finite, then $\Gamma_{n+1}^\alpha(G)$ is finite. There is a question here to ask whether or no we could imply it, in general?

Theorem 3.16. *Suppose G is a group. If $G/Z^\alpha(G)$ is finite, then $\Gamma_2^\alpha(G)$ is finite.*

Proof. Assume $|G/Z^\alpha(G)| = n$. The function $T : G \to Z^\alpha(G)$ which $T(a) = a^n$ is a transfer homomorphism, for $a \in G$. Therefore, every element of Ker(T) has finite order. Also $T([x, y]_\alpha) = [T(x), T(y)]_\alpha = 1$ which implies $\Gamma_2^\alpha(G) \leq \mathrm{Ker(T)}$. Thus every element of $\Gamma_2^\alpha(G)$ is of finite order. Furthermore $\Gamma_2^\alpha(G)$ is finitely generated. Since, if $\{x_1, \cdots, x_n\}$ is the set of transversal of $Z^\alpha(G)$ in G and $u, v \in G$, then $u = z_1 x_i$ and $v = z_2 x_j$ where $z_1, z_2 \in Z^\alpha(G)$ $1 \leq i, j \leq n$. So we can write $[u, v]_\alpha = [x_i, x_j]_\alpha$. This fact conveys the $\Gamma_2^\alpha(G)$ is finitely generated. On the other hand $\Gamma_2^\alpha(G)/\Gamma_2^\alpha(G) \cap Z^\alpha(G)$ is finite. By Schereir Theorem, $\Gamma_2^\alpha(G) \cap Z^\alpha(G)$ is finitely generated. Since $\Gamma_2^\alpha(G) \cap Z^\alpha(G)$ is abelian and the order of each elements is finite, we conclude that $\Gamma_2^\alpha(G) \cap Z^\alpha(G)$ is finite. Hence the assertion is clear. $\square$

4. Some Interesting Examples

Let us start this section by the following theorem.

Theorem 4.1. *Suppose $\mathbb{Z}_n$ is the cyclic group of order n and α the automorphism with multiplication by u, where u is an invertible element in $\mathbb{Z}_n$. Then $\Gamma_{c+1}^\alpha(G) = (u-1)^c G$ and $Z_c^\alpha(G) = \{s \in G : (u-1)^c s = 0\}$.*

The above theorem implies that a cyclic group G is α-nilpotent of class c if and only if $(u-1)^c \overset{n}{\equiv} 0$. By this fact, we can verify the relative nilpotency of cyclic groups.

Example 4.2. In this example in each cases we find the nilpotency class of the group with respect to the automorphism which is defined as in the hypothesis of the Theorem 4.1

(i) For $\mathbb{Z}_8$ we have

u	1	3	5	7
c	1	3	2	3

Therefore, $\mathbb{Z}_8$ is absolute nilpotent group.

(ii) For the cyclic group of order 9, $\mathbb{Z}_9$ we conclude that

u	1	2	4	5	7	8
c	1	-	2	-	2	-

It is clear that $\mathbb{Z}_9$ is not absolute nilpotent.

(iii) $\mathbb{Z}_p$ is nilpotent with respect to the trivial automorphism.

Remark 4.3. Let s be a real number. Then $\lceil s \rceil$ and $\lfloor s \rfloor$ are the smallest integer greater than s and the greatest integer less or equal than s, respectively.

Theorem 4.4. *Suppose $\mathfrak{B}_n = \{\lceil n/s \rceil : s = 1, 2, \cdots, n\}$ and $f(n) = \lfloor (1+\sqrt{4n+1})/2 \rfloor$, where $n \in \mathbb{N}$. Then*

$$|\mathfrak{B}_n| = \begin{cases} f(n) + \lfloor \frac{n}{f(n)} \rfloor, & f(n) \nmid n \\ \\ f(n) + \frac{n}{f(n)} - 1, & f(n) \mid n. \end{cases}$$

Proof. It is clear that $\lceil n/s \rceil = k$ if and only if $n/k \leq s < n/(k-1)$. This means $k \in \mathfrak{B}_n$ if and only if an integer s in $[n/k, n/(k-1))$ exists. We complete the proof by discussing about the length of the segment. Initially, if $n/k(k-1) \geq 1$, then $k \leq (1+\sqrt{4n+1})/2$. Thus the number of such k is $f(n)-1$, where $f(n) = \lfloor (1+\sqrt{4n+1})/2 \rfloor$. Secondly, assume the length of segment is strictly less than one. Then

$k = f(n)+1, \cdots, n-1$ and each segment has at most one integer. Hence the assertion is clear. $\square$

Theorem 4.5. *Let $n = p_1^{n_1} \cdots p_k^{n_k}$. Then $\mathcal{N}(\mathbb{Z}_n) = \bigcup_{i=1}^{k} \mathfrak{B}_{n_i}$.*

Proof. Assume α is an automorphism of the group $\mathbb{Z}_n$ by multiplying by u. By Theorem 4.1, $Z_c^{\alpha}(G) = G$ if and only if $(u-1)^c \overset{n}{\equiv} 0$. Therefore, $u = p_1 \cdots p_k l + 1$ and $m_i c \geq n_i$, where $(l, p_i) = 1$ and $1 \leq i \leq k$. Thus $c = \mathrm{Max}\{\lceil \frac{n_i}{m_i} \rceil : \ 1 \leq i \leq k, \ 1 \leq m_i \leq n_i\}$ which implies $\mathcal{N}(\mathbb{Z}_n) \subseteq \cup_{i=1}^{k} \mathfrak{B}_{n_i}$. If $u_i = (n/p_i^{n_i - s}) + 1$, then $c_i = \lceil n_i/s \rceil$, where $1 \leq i \leq k$ and $1 \leq s \leq n_i$. Hence the result follows. $\square$

Now by Theorems 3.13 and 4.5, we conclude the following corollary.

Corollary 4.6. *The cyclic group $\mathbb{Z}_n$ is absolute nilpotent if and only if $n = 2^m$ and in this case $\mathcal{N}(\mathbb{Z}_n) = \mathfrak{B}_m$.*

Theorem 4.7. *Let $G = D_{2^{n+1}} = \langle x, y : x^{2^n} = y^2 = (xy)^2 = 1 \rangle$ be dihedral group of order 2^{n+1}. Then $D_{2^{n+1}}$ is absolute nilpotent group and $\mathcal{N}(D_{2^{n+1}}) = \{n, n+1\}$.*

Proof. Suppose $\alpha_{s,t}$ is an automorphism of $D_{2^{n+1}}$ such that $x \mapsto x^t$ and $y \mapsto x^s y$, where $(t, 2^n) = 1$, $1 \leq t \leq 2^n - 1$ and $0 \leq s \leq 2^n - 1$. Then $[x^i, x^j]_{\alpha_{s,t}} = x^{(t-1)j}$, $[x^i y, x^j]_{\alpha_{s,t}} = x^{(t+1)j}$, $[x^i, x^j y]_{\alpha_{s,t}} = x^{-2i-s-(t-1)j}$ and $[x^i y, x^j y]_{\alpha_{s,t}} = x^{2i-s-(t+1)j}$. Let s be an odd number. Since $x^2, x^s \in \Gamma_2^{\alpha_{s,t}}(G)$ and (2,s)=1, $x \in \Gamma_2^{\alpha_{s,t}}(G)$. Therefore, $\Gamma_2^{\alpha_{s,t}}(G) = \langle x \rangle$ and hence $\Gamma_{m+1}^{\alpha_{s,t}}(G) = \langle x^{2^{m-1}} \rangle$. Now let s be an even number. Then $\Gamma_2^{\alpha_{s,t}}(G) = \langle x^2 \rangle$ and hence $\Gamma_{m+1}^{\alpha_{s,t}}(G) = \langle x^{2^m} \rangle$, which complete the proof. $\square$

By similar method, we can deduce Theorems 4.7 and 4.8.

Theorem 4.8. *If $G = Q_{2^{n+2}} = \langle x, y : x^{2n} = y^2, yxy^{-1} = x^{-1} \rangle$, then $Q_{2^{n+2}}$ is absolute nilpotent and $\mathcal{N}(Q_{2^{n+2}}) = \{n+1, n+2\}$.*

Theorem 4.9. *If $G = SD_{2^{n+1}} = \langle x, y : x^{2n} = y^2 = 1, xy = yx^{2^{n-1}-1} \rangle$, $n \geq 3$, then G is absolute nilpotent and $\mathcal{N}(SD_{2^{n+1}}) = \{n\}$.*

Theorem 4.10. *Let $G = M_n(p) = \langle x, y : x^{p^{n-1}} = y^p = 1, xy = yx^{p^{n-2}+1} \rangle$. Then G is not absolute nilpotent group and $\mathcal{N}(G) = \mathfrak{B}_{n-1} \backslash 1$ if $n \geq 4$ and $\mathcal{N}(G) = \{2, 3\}$ if $n = 3$.*

Proof. The properties of the group $M_n(P)$ is as follows:

(i) $|M_n(p)| = p^n$, $M_n'(p) = \langle x^{p^{n-2}} \rangle$, $Z(M_n(p)) = \langle x^p \rangle$.
(ii) $M_n(p) = \{x^i y^j : 0 \leq i \leq p^{n-1}, 0 \leq j \leq p-1\}$.
(iii) $(x^a y^b)(x^c y^d) = x^{a+c-bcp^{n-2}} y^{b+d}$, $(x^a y^b)^{-1} = x^{-a(1+bp^{n-2})} y^{-b}$, $(x^a y^b)^m = x^{ma - ab(\frac{m(m-1)}{2})p^{n-2}} y^{mb}$.

(iv) $\mathrm{Aut}(M_n(p)) = \{\alpha_{ijk} : 0 \leq i \leq p^{n-1}-1,\ i \not\equiv 0 \pmod{p},\ 0 \leq j,k \leq p-1\}$, where α_{ijk} is an automorphism of the group such that $x \mapsto x^i y^j$ and $y \mapsto x^{kp^{n-2}}y$.

We consider two cases. Initially, let p be an odd prime number and $n \geq 3$. Therefore, by the $M_n(P)$ properties, we have

$$\begin{aligned} Z^{\alpha_{ijk}}(M_n(P)) &= \{x^{pt} \in Z(M_n(p)) : (x^{pt})^{\alpha_{ijk}} = x^{pt}\} \\ &= \{x^{pt} \in Z(M_n(p)) : (x^i y^j)^{pt} = x^{pt}\} \\ &= \{x^{pt} \in Z(M_n(p)) : (x^{pt})^i = x^{pt}\} \\ &= \{z \in Z(M_n(P)) : z^{i-1} = 1\}. \end{aligned}$$

Thus, $Z^{\alpha_{ijk}}(M_n(P)) \neq 1$ if and only if $i = 1 + lp^s$, where $(l,p) = 1$ and $s = 1, 2, \cdots, n-2$. This means $M_n(P)$ is not absolute nilpotent and we are going to verify such i. It is clear that $M_n'(p) \leq Z^{\alpha_{ijk}}(M_n(P))$. Moreover,

$$\begin{aligned} [x^a y^b, x^c y^d]_{\alpha_{ijk}} &= [x^a y^b, x^c y^d][x^c y^d, \alpha_{ijk}] \\ &= (x^{i-1} y^j)^c x^{(kd - ij\frac{c(c-1)}{2} + ad - bc)p^{n-2}}, \end{aligned}$$

clearly $[y,y]_{\alpha_{ijk}} = x^{kp^{n-2}}$, $[x,y]_{\alpha_{ijk}} = x^{(k+1)p^{n-2}}$ and $x^{i-1}y^j \in \Gamma_2^{\alpha_{ijk}}(M_n(P))$ which imply $\Gamma_2^{\alpha_{ijk}}(M_n(P)) = \langle x^{i-1}y^j \rangle M_n'(p)$. In the sequel for $\Gamma_3^{\alpha_{ijk}}(M_n(P))$, we have

$$[x^a y^b, x^{(i-1)^2 c}]_{\alpha_{ijk}} = x^{(i-1)^2 c + (kjc - \frac{ijc(i-1)((i-1)c-1)}{2} + ajc)p^{n-2}}.$$

In particular, if $j = 0$, then $\Gamma_3^{\alpha_{ijk}}(M_n(P)) = \langle x^{(i-1)^2} \rangle$ which implies $c_{\alpha_{10k}} = 2$. Also if $i = j = 1$ and $k = 0$, then $\Gamma_3^{\alpha_{ijk}}(M_n(P)) = M_n(P)'$ and hence $c_{\alpha_{110}} = 3$. Therefore, $\{2,3\} \subseteq \mathcal{N}(M_n(P))$. Since $i-1$ is a multiple of number p for $\Gamma_4^{\alpha_{ijk}}(M_n(P))$, we have

$$\begin{aligned} [x^a y^b, x^{(i-1)^2 c}]_{\alpha_{ijk}} &= (x^{i-1} y^j)^{(i-1)^2 c} x^{-(\frac{i(i-1)^2 jc((i-1)^2 c - 1)}{2} + bc(i-1)^2)p^{n-2}} \\ &= x^{(i-1)^3 c}. \end{aligned}$$

Therefore, $\Gamma_4^{\alpha_{ijk}}(M_n(P)) = \langle x^{(i-1)^3} \rangle$ and similarly $\Gamma_{m+1}^{\alpha_{ijk}}(M_n(P)) = \langle x^{(i-1)^m} \rangle$ for $m \geq 4$. If $i = 1$, then $c_{\alpha_{ijk}} = 2$ or 3. So, we Suppose that $i \neq 1$. By an easy computation the least number for power m, for which $x^{(i-1)^m} = 1$ is $\lceil \frac{n-1}{s} \rceil$. Hence the nilpotency class of $M_n(P)$, is $\lceil \frac{n-1}{s} \rceil$.

Now, let $p = 2$. If $j = 0$, then similar to the previous case we deduce $\mathfrak{B}_{n-1} \backslash \{1\} \subseteq \mathcal{N}(M_n(p))$. Now, let $j = 1$ and $n \geq 4$. Then

$$Z^{\alpha_{i1k}}(M_n(P)) = \{z \in Z(G) : z^{i(1+2^{n-3})} = z\}.$$

Thus $Z^{\alpha_{i1k}}(M_n(P)) \neq 1$ if and only if $i(1+2^{n-3}) = 1 + 2^s l$, where $s \in \{1, \cdots, n-2, \cdots\}$ and $(l, 2) = 1$, since $x^{2^{n-1-s}} \in Z^{\alpha_{i1k}}(M_n(P))$, $1 \leq s \leq n-1$ and $Z^{\alpha_{i1k}}(M_n(P)) = Z(G)$, $s \geq n-2$. Similarly, $\Gamma^{\alpha_{i1k}}_{m+1}(M_n(P)) = \langle x^{(i-1)^m} \rangle$, where $m \geq 3$. It is obvious that for $s \geq n-3$, the nilpotency class is 2 or 3 with respect to α_{i1k}. Now, let $s \leq n-4$. Then $i-1 = 2^s t$, where $t = l - i2^{n-s-3}$. By computation, we conclude the least number for power m is $\lceil \frac{n-1}{s} \rceil$. Hence the result is clear. □

5. Relative Solvable Group

Definition 5.1. Let G be a group and $\alpha \in \mathrm{Aut(G)}$. We define the derived subgroup of G with respect to automorphism α as follows

$$D_\alpha(G) = \langle [x, y]_\alpha : x, y \in G \rangle.$$

Inductively, we introduce the subgroup $D^i_\alpha(G)$ defined as $D^1_\alpha(G) = D_\alpha(G)$, $D^i_\alpha(G) = D_\alpha(D^{i-1}_\alpha(G))$. It is clear that

$$\cdots \trianglelefteq D^3_\alpha(G) \trianglelefteq D^2_\alpha(G) \trianglelefteq D^1_\alpha(G) \trianglelefteq G.$$

Lemma 5.2. *Let N and H be subgroups of G. If $N \trianglelefteq G$ and $N^\alpha = N$, then $D_\alpha(HN) \leq D_\alpha(H)N$.*

Proof. Let $[h_1n_1, h_2n_2]_\alpha \in D_\alpha(G)$, where $h_1, h_2 \in H$ and $n_1, n_2 \in N$. We have

$$[h_1n_1, h_2n_2]_\alpha = n_1^{-1}(n_2^{-1})^{h_1}[h_1, h_2]_\alpha n_1^{h_2^\alpha} n_2^\alpha.$$

So $[h_1n_1, h_2n_2]_\alpha \in D_\alpha(H)N$ and assertion deduced. □

Theorem 5.3. *Let N be a normal subgroup of G. If $N^\alpha = N$, then $D^n_{\overline{\alpha}}(G/N) = D^n_\alpha(G)N/N$.*

Proof. The proof follows by induction on n, Definition 5.1 and Lemma 5.2. □

Definition 5.4. The group G is called α-solvable or solvable with respect to α if there is a subnormal series,

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G \tag{5.1}$$

such that $G_i^\alpha = G_i$ and $D_{\overline{\alpha}}(G_{i+1}/G_i) = \{1\}$. The length of the shortest series with this property is called solvability class of G with respect to the automorphism α which is denoted by r_α.

It is clear that every α-solvable group is solvable. If we denote the solvability class by r, then $r \leq r_\alpha$. Let us define $\mathcal{S}(G) = \{r_\alpha : \alpha \in \mathrm{Aut}(G)\}$. If G is solvable with respect to a non-trivial automorphism, then it is called non-trivial solvable group. Moreover, if G is solvable with respect to the all automorphisms of the group, then it is called absolute solvable group.

Theorem 5.5. *Let G be a group. Then G is α-solvable if and only if there is an integer r such that $D^r_\alpha(G) = \{1\}$.*

Proof. Suppose G has a normal series (5.1). Since $D_{\overline{\alpha}}(G_{i+1}/G_i) = \{1\}$, we conclude $D_\alpha(G_{i+1}) \leq G_i$. If $i = n-1$, then $D_\alpha(G) \leq G_{n-1}$. Thus $D^2_\alpha(G) \leq D_\alpha(G) \leq G_{n-2}$. By continuing this process, we have $D^r_\alpha(G) \leq G_0 = \{1\}$. The converse of the theorem is obvious. □

Corollary 5.6. *Let G be a solvable group with respect to α and r the least positive integer such that $D^r_\alpha(G) = \{1\}$. Then r is the length of solvability of G with respect to the automorphism α.*

The following theorem is a direct result of Theorem 5.3.

Theorem 5.7. *Let N be a normal subgroup of G and $N^\alpha = N$.*

(i) *If G is α-solvable, then G/N is $\overline{\alpha}$-solvable.*
(ii) *If N and G/N is solvable with respect to α and $\overline{\alpha}$, then G is α-solvable.*

The following corollary is a direct result of Theorem 5.7.

Corollary 5.8. *Suppose M and N are normal subgroups of G and $M^\alpha = M$, $N^\alpha = N$. If M and N are solvable with respect to α, then MN is α-solvable.*

Theorem 5.9. *Let G be an abelian group.*

(i) *If $\Gamma^\alpha_{i+1}(G) = D^i_\alpha(G)$ and consequently $\mathcal{N}(G) = \mathcal{S}(G)$.*
(ii) *If G is absolute solvable, then G is a 2-group.*

Proof. The proof of the first part is clear by the definition. For the second part it is enough to consider the automorphism $\alpha : x \mapsto -x$. □

Theorem 5.10. *Let G be a finite group.*

(i) *If $G = D_{2^{n+1}}$, then $\mathcal{S}(G) = (\mathfrak{B}_n \cup \mathfrak{B}_{n-1}) + 1$.*
(ii) *If $G = Q_{2^{n+2}}$, then $\mathcal{S}(G) = (\mathfrak{B}_n \cup \mathfrak{B}_{n+1}) + 1$.*
(iii) *If $G = SD_{2^{n+1}}$, then $\mathcal{S}(G) = \mathfrak{B}_{n-1} + 1$.*

Proof. The proof is clear by Theorems 4.7, 4.8 and 4.9. □

Open problems. Let G be a group and $\alpha, \beta \in \mathrm{Aut}(G)$. Then

(i) If G is nilpotent (solvable) with respect to α and β, then is G nilpotent (solvable) with respect to $\alpha\beta$ or α^n ? And also if G is solvable with respect to α, then is G solvable with respect to α^{-1}?
(ii) Each group with non-trivial nilpotency (solvability) is nilpotent (solvable), is this proposition is invertible?
(iii) If G is a finite $p-$group, in particular G be a extra special $p-$group, then what can we say about its relative nilpotency?

(iv) With restricting $\mathcal{N}(G)$ or $\mathcal{S}(G)$ to a given order such as order 1 or 2, what can we say about group G? Moreover if G is an abelian group, what can we say about $\mathcal{N}(G)$ or $\mathcal{S}(G)$?

(v) Is it possible to find upper bound for $\mathcal{N}(G)$ or $\mathcal{S}(G)$ with respect to the order of the group?

(vi) For what automorphisms, group G has most nilpotency (solvability) class? Do these automorphisms have basically special properties?

(vii) What can we say about the groups which have just trivial nilpotency (solvability)?

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