

Applications of He's Variational Principle method and the Kudryashov method to nonlinear time-fractional differential equations

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ABSTRACT. In this paper, we establish exact solutions for the time-fractional Klein-Gordon equation, and the time-fractional Hirota-Satsuma coupled KdV system. The Hes semi-inverse and the Kudryashov methods are used to construct exact solutions of these equations. We apply Hes semi-inverse method to establish a variational theory for the time-fractional Klein-Gordon equation, and the time-fractional Hirota-Satsuma coupled KdV system. Based on this formulation, a solitary solution can be easily obtained using the Ritz method. The Kudryashov method is used to construct exact solutions of the time-fractional Klein-Gordon equation, and the time-fractional Hirota-Satsuma coupled KdV system. Moreover, it is observed that the suggested techniques are compatible with the physical nature of such problems.

Keywords: He's semi-inverse method; time-fractional Klein-Gordon equation; time-fractional Hirota-Satsuma coupled KdV system.

2000 Mathematics subject classification: 02.30.Jr; 05.45.Yv.

1. INTRODUCTION

The effort in finding exact solutions of nonlinear equations is very important for understanding most nonlinear physical phenomena [1-6]. For instance, the nonlinear wave phenomena observed in fluid dynamics,

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Received: 23 February 2014
Revised: 09 September 2014
Accepted: 09 September 2014

plasma and optical fibers are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The exact solution [7,8], if available, of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions. In the past few years, many new approaches to nonlinear equations were proposed to search for solitary solutions, among which the variational iteration method [9-11], the homotopy perturbation method [12-14] and the exp function method [15, 16] have been shown to be effective, easy and accurate for a large class of nonlinear problems. Recently, several powerful methods have also been provided to construct the approximate or exact solutions of fractional ordinary differential equations, integral equations and fractional partial differential equations, such as the fractional variational iteration method [17], the homotopy perturbation method [18], the fractional sub-equation method [19] and so on. Using these methods, solutions with various forms for some given fractional differential equations have been established. In this paper we will use Hes semi-inverse and the Kudryashov methods to the time-fractional Klein-Gordon equation, and the time-fractional Hirota-Satsuma coupled KdV system [20-29].

2. THE MODIFIED RIEMANN-LIOUVILLE DERIVATIVE AND HE'S SEMI-INVERSE METHOD

Jumaris modified Riemann-Liouville derivative of order α is defined as

$$D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} [f(\xi) - f(0)] d\xi, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(x))^{(\alpha-n)}, & n \leq \alpha \leq n+1, n \geq 1. \end{cases} \quad (2.1)$$

where $f : R \rightarrow R, x \rightarrow f(x)$ denote a continuous (but not necessarily differentiable) function. For some properties of the above modified derivative, we refer the reader to [30, 31].

We now describe Hes semi-inverse method for exact solution of nonlinear time fractional differential equations as follows.

Step 1. Let us consider the time-fractional differential equation with independent variable $\{t, x, y, z, \dots\}$ and a dependent variable u

$$F(u, D_t^\alpha u, u_x, u_y, u_z, D_t^{2\alpha} u, u_{xy}, u_{yz}, u_{xz}, \dots) = 0, \quad (2.2)$$

where the subscript denotes partial derivative. Using the variable transformation

$$u(t, x, y, z, \dots) = U(\xi), \quad \xi = l_1x + l_2y + l_3z + \dots - \frac{\lambda t^\alpha}{\Gamma(1 + \alpha)} \quad (2.3)$$

where l_i and λ are constant to be determined later; the fractional differential equation (2.2) is reduced to an ordinary differential equation (ODE)

$$H(U(\xi), U'(\xi), U''(\xi), \dots) = 0. \quad (2.4)$$

where $' = \frac{d}{d\xi}$.

Step 2. If possible, integrate Eq. (2.4) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3. According to the Hes semi-inverse method, we construct the following trialfunctional

$$J(U) = \int Ld\xi, \quad (2.5)$$

where L is an unknown function of U and its derivatives.

Step 4. By Ritz method, we can obtain different forms of solitary wave solutions, such as $U(\xi) = Asech(B\xi)$, $U(\xi) = Acsch(B\xi)$, $U(\xi) = Atanh(B\xi)$, $U(\xi) = Acoth(B\xi)$ and so on. For example in this paper we search a solitary wave solution in the form

$$U(\xi) = Asech(B\xi), \quad (2.6)$$

where A and B are constants to be further determined. Substituting Eq. (2.6) into Eq. (2.5) and making J stationary with respect to A and B , we have

$$\frac{\partial J}{\partial A} = 0, \quad (2.7)$$

$$\frac{\partial J}{\partial B} = 0. \quad (2.8)$$

Solving simultaneously the Eq. (2.7) and Eq. (2.8) we obtain and . Hence, the solitary wave solution Eq. (2.6) is well determined.

3. THE MODIFIED RIEMANN-LIOUVILLE DERIVATIVE AND THE KUDRYASHOV METHOD

The main steps of the Kudryashov method are the following:

Step 1. Determination of the dominant term with highest order of singularity. To find dominant terms, we substitute

$$U = \xi^{-p}, \quad (3.1)$$

to all terms of Eq. (2.4). Then we compare degrees of all terms of Eq. (2.4) and choose two or more with the lowest degree. The maximum value of p is the pole of Eq. (2.4) and we denote it as N . This method can be applied when N is integer. If the value N is non-integer, one can transform the equation studied.

Step 2. We look for exact solution of Eq. (2.4) in the form

$$u(z) = \sum_{i=0}^N b_i Q^i(\xi), \quad (3.2)$$

where b_i ($i = 0, 1, \dots, N$) are constants to be determined later, such that $b_N \neq 0$, while $Q(\xi)$ has the form

$$Q(\xi) = \frac{1}{1 + d \exp(\xi)}, \quad (3.3)$$

which is a solution to the Riccati equation

$$Q'(\xi) = Q^2(\xi) - Q(\xi), \quad (3.4)$$

where d is arbitrary constant.

Step 3. We can calculate necessary number of derivative of function U . It is easy to do using Maple or Mathematica package. Using case $N = 1$ we have some derivatives of function $U(\xi)$ in the form

$$\begin{aligned} U &= b_0 + b_1 Q, \\ U_\xi &= -b_1 Q + b_1 Q^2, \\ U_{\xi\xi} &= b_1 Q - 3b_1 Q^2 + 2b_1 Q^3, \\ U_{\xi\xi\xi} &= -b_1 Q + 7b_1 Q^2 - 12b_1 Q^3 + 6b_1 Q^4. \end{aligned} \quad (3.5)$$

Step 4. We substitute expressions given by Eqs. (3.5) in Eq. (2.4). Then we collect all terms with the same powers of function $Q(\xi)$ and equate expressions to zero. As a result we obtain algebraic system of equations. Solving this system we get the values of unknown parameters.

4. APPLICATIONS

In this section we apply the proposed methods to solve the time-fractional Klein-Gordon equation, and the time-fractional Hirota-Satsuma coupled KdV system.

4.1. Time-fractional Klein-Gordon equation. We consider the non-linear fractional Klein-Gordon equation

$$\frac{\partial^{2\alpha} u(x, t)}{\partial t^{2\alpha}} = \frac{\partial^2 u(x, t)}{\partial x^2} + au(x, t) + cu^3(x, t), \quad t > 0, \quad 0 < \alpha \leq 1, \quad (4.1)$$

where a and c are arbitrary constants. For our purpose, we introduce the following transformations:

$$u(x, t) = U(\xi), \quad \xi = lx - \frac{\lambda t^\alpha}{\Gamma(1 + \alpha)} \tag{4.2}$$

where λ, l are constants.

Substituting (4.2) into equation (4.1), we can know that equation (4.1) is reduced into an ordinary differential equation:

$$\lambda^2 \frac{\partial^2 U}{\partial \xi^2} = l^2 \frac{\partial^2 U}{\partial \xi^2} + aU + cU^3 \tag{4.3}$$

By He's semi-inverse method, we can obtain the following variational formulation

$$J = \int_0^\infty \left[-\frac{1}{2}(l^2 - \lambda^2)(U')^2 + \frac{a}{2}U^2 + \frac{c}{4}U^4 \right] d\xi \tag{4.4}$$

By Ritz-like method, we search for a solitary wave solution in the form

$$U(\xi) = A \operatorname{sech}(B\xi), \tag{4.5}$$

where A and B are unknown constant to be further determined.

Substituting Eq. (4.4) into Eq. (4.5), we have

$$\begin{aligned} J &= \int_0^\infty \left[-\frac{1}{2}(l^2 - \lambda^2)A^2B^2 \operatorname{sech}^2(B\xi) \operatorname{tanh}^2(B\xi) + \frac{aA^2}{2} \operatorname{sech}^2(B\xi) \right. \\ &\quad \left. + \frac{cA^4}{4} \operatorname{sech}^4(B\xi) \right] d\xi \\ &= -\frac{1}{6}(l^2 - \lambda^2)A^2B + \frac{aA^2}{2B} + \frac{cA^4}{6B}. \end{aligned} \tag{4.6}$$

Making J stationary with A and B results in

$$\frac{\partial J}{\partial A} = -\frac{1}{3}(l^2 - \lambda^2)AB + \frac{aA}{B} + \frac{2cA^3}{3B} = 0, \tag{4.7}$$

$$\frac{\partial J}{\partial B} = -\frac{1}{6}(l^2 - \lambda^2)A^2 - \frac{aA^2}{2B^2} - \frac{cA^4}{6B^2} = 0, \tag{4.8}$$

From Eq. (4.7) and Eq. (4.8), we get

$$A = \sqrt{\frac{2a}{-c}}, \quad B = \sqrt{\frac{a}{\lambda^2 - l^2}} \tag{4.9}$$

The solitary solution is, therefore, obtained as follows

$$u(x, t) = \sqrt{\frac{2a}{-c}} \operatorname{sech}\left(\sqrt{\frac{a}{\lambda^2 - l^2}}\left(lx - \frac{\lambda t^\alpha}{\Gamma(1 + \alpha)}\right)\right).$$

4.2. Time-fractional Hirota-Satsuma coupled KdV system. The HirotaSatsuma system of equations was introduced to describe the interaction of two long waves with different dispersion relations. In this section, we consider the solution of generalized HirotaSatsuma coupled KdV of time-fractional order, which is presented by a system of nonlinear partial differential equations, of the form:

$$\begin{aligned} D_t^\alpha u &= \frac{1}{4}u_{xxx} + 3uu_x + 3(-v^2 + w)_x, \\ D_t^\alpha v &= -\frac{1}{2}v_{xxx} - 3uv_x, \\ D_t^\alpha w &= -\frac{1}{2}w_{xxx} - 3uw_x, \quad 0 < \alpha \leq 1, \end{aligned} \quad (4.10)$$

where $u = u(x, t)$, $v = v(x, t)$ and $w = w(x, t)$.

For our purpose, we introduce the following transformations:

$$\begin{aligned} u(x, t) &= \frac{1}{\lambda}U^2(\xi), \quad v(x, t) = -\lambda + U(\xi), \\ w(x, t) &= 2\lambda^2 - \lambda U(\xi), \quad \xi = x - \frac{\lambda t^\alpha}{\Gamma(1 + \alpha)}, \end{aligned} \quad (4.11)$$

where λ is a constant. Substituting (4.11) into equations (4.10), we can know that equations (4.10) is reduced into an ordinary differential equation:

$$\lambda \frac{\partial^2 U}{\partial \xi^2} + 2U^3 - 2\lambda^2 U = 0, \quad (4.12)$$

A variational formulation should be established using the semi-inverse method:

$$J = \int_0^\infty \left[-\frac{\lambda}{2}(U')^2 + \frac{1}{2}U^4 - \lambda^2 U^2 \right] d\xi \quad (4.13)$$

By Ritz-like method, we search for a solitary wave solution in the form

$$U(\xi) = A \operatorname{sech}(B\xi), \quad (4.14)$$

where A and B are unknown constant to be further determined.

Substituting Eq. (4.14) into Eq. (4.13), we have

$$\begin{aligned} J &= \int_0^\infty \left[-\frac{\lambda}{2}A^2 B^2 \operatorname{sech}^2(B\xi) \operatorname{tanh}^2(B\xi) + \frac{A^4}{2} \operatorname{sech}^4(B\xi) \right. \\ &\quad \left. - \lambda^2 A^2 \operatorname{sech}^2(B\xi) \right] d\xi \\ &= -\frac{\lambda}{6}A^2 B + \frac{A^4}{3B} - \frac{\lambda^2 A^2}{B}. \end{aligned} \quad (4.15)$$

Making J stationary with A and B results in

$$\frac{\partial J}{\partial A} = -\frac{\lambda}{3}AB + \frac{4A^3}{3B} - \frac{2\lambda^2 A}{B} = 0, \tag{4.16}$$

$$\frac{\partial J}{\partial B} = -\frac{\lambda}{6}A^2 - \frac{A^4}{3B^2} + \frac{\lambda^2 A^2}{B^2} = 0. \tag{4.17}$$

From Eq. (4.16) and Eq. (4.17), we get

$$A = \sqrt{2}\lambda, \quad B = \sqrt{2}\lambda. \tag{4.18}$$

The solitary solution is, therefore, obtained as follows

$$\begin{aligned} u(x, t) &= 2\lambda \operatorname{sech}^2\left(\sqrt{2}\lambda\left(x - \frac{\lambda t^\alpha}{\Gamma(1 + \alpha)}\right)\right), \\ v(x, t) &= \lambda\left(-1 + \sqrt{2}\operatorname{sech}\left(\sqrt{2}\lambda\left(x - \frac{\lambda t^\alpha}{\Gamma(1 + \alpha)}\right)\right)\right), \\ w(x, t) &= 2\lambda^2\left(1 - \sqrt{2}\operatorname{sech}\left(\sqrt{2}\lambda\left(x - \frac{\lambda t^\alpha}{\Gamma(1 + \alpha)}\right)\right)\right). \end{aligned}$$

5. THE KUDRYASHOV METHOD

In this subsection, we apply the Kudryashov method to solve the time-fractional Klein-Gordon equation, and the time-fractional Hirota-Satsuma coupled KdV system.

5.1. Time-fractional Klein-Gordon equation. The pole order of Eq. (4.3) is $N = 1$. So we look for solution of Eq. (4.3) in the following form

$$U(\xi) = b_0 + b_1 Q. \tag{5.1}$$

Substituting Eq. (5.1) into Eq. (4.3), we obtain the system of algebraic equations in the following form

$$\begin{aligned} Q^0 : ab_0 + cb_0^3 &= 0, \\ Q^1 : (l^2\lambda^2)b_1 + ab_1 + 3cb_0^2b_1 &= 0, \\ Q^2 : -3(l^2\lambda^2)b_1 + 3b_0b_1^2 &= 0, \\ Q^3 : 2(l^2\lambda^2)b_1 + cb_1^3 &= 0, \end{aligned}$$

Solving the algebraic equations above, yields:

Case1.

$$b_0 = \frac{1}{2}\sqrt{\frac{-2(l^2\lambda^2)}{c}}, \quad b_1 = -\frac{1}{2}\sqrt{\frac{-2(l^2\lambda^2)}{c}}, \quad l = \pm\sqrt{\lambda^2 + 2a} \tag{5.2}$$

Substituting (5.2) into (5.1), we have

$$U(\xi) = \frac{1}{2}\sqrt{\frac{-2(l^2\lambda^2)}{c}} - \sqrt{\frac{-2(l^2\lambda^2)}{c}}Q, \tag{5.3}$$

Now, the exact solution of Eq. (4.1) has the form

$$u(x, t) = \left(\frac{1}{2} \sqrt{\frac{-2(l^2\lambda^2)}{c}} - \sqrt{\frac{-2(l^2\lambda^2)}{c}} \frac{1}{1 + de^{\pm\sqrt{\lambda^2+2a}\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}}} \right),$$

where d is arbitrary constant.

Case 2.

$$b_0 = -\frac{1}{2} \sqrt{\frac{-2(l^2\lambda^2)}{c}}, \quad b_1 = +\frac{1}{2} \sqrt{\frac{-2(l^2\lambda^2)}{c}}, \quad l = \pm\sqrt{\lambda^2 + 2a} \quad (5.4)$$

Substituting (5.4) into (5.1), we have

$$U(\xi) = -\frac{1}{2} \sqrt{\frac{-2(l^2\lambda^2)}{c}} + \sqrt{\frac{-2(l^2\lambda^2)}{c}} Q, \quad (5.5)$$

Now, the exact solution of Eq. (4.1) has the form

$$u(x, t) = \left(-\frac{1}{2} \sqrt{\frac{-2(l^2\lambda^2)}{c}} + \sqrt{\frac{-2(l^2\lambda^2)}{c}} \frac{1}{1 + de^{\pm\sqrt{\lambda^2+2a}\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}}} \right),$$

where d is arbitrary constant.

5.2. Time-fractional Hirota-Satsuma coupled KdV system. Next, the pole order of Eq. (4.12) is $N = 1$. So we look for solution of Eq. (4.12) in the following form

$$U(\xi) = b_0 + b_1 Q. \quad (5.6)$$

Substituting Eq. (5.6) into Eq. (4.12), we obtain the system of algebraic equations in the following form

$$\begin{aligned} Q^0 : 2b_0 - 2\lambda^2 b_0^3 &= 0, \\ Q^1 : \lambda b_1 + 6b_0^2 b_1 - 2\lambda^2 b_1 &= 0, \\ Q^2 : -3\lambda b_1 + 6b_0 b_1^2 &= 0, \\ Q^3 : 2\lambda^2 b_1 + 2b_1^3 &= 0, \end{aligned}$$

Solving the algebraic equations above, yields:

Case1.

$$b_0 = \frac{1}{4}, \quad b_1 = -\frac{1}{2}, \quad \lambda = -\frac{1}{4} \quad (5.7)$$

Substituting (5.7) into (5.6), we have

$$U(\xi) = \frac{1}{4} - \frac{1}{2} Q, \quad (5.8)$$

Now, the exact solution of Eq. (4.10) has the form

$$\begin{aligned} u(x, t) &= -4\left(\frac{1}{4} - \frac{1}{2} \frac{1}{1 + de^{(x - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)})}}\right)^2, \\ v(x, t) &= \frac{1}{4} + \left(\frac{1}{4} - \frac{1}{2} \frac{1}{1 + de^{(x - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)})}}\right), \\ w(x, t) &= \frac{1}{8} + \frac{1}{2} \left(\frac{1}{4} - \frac{1}{2} \frac{1}{1 + de^{(x - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)})}}\right). \end{aligned}$$

where d is arbitrary constant.

Case 2.

$$b_0 = -\frac{1}{4}, \quad b_1 = +\frac{1}{2}, \quad \lambda = -\frac{1}{4} \tag{5.9}$$

Substituting (5.9) into (5.6), we have

$$U(\xi) = -\frac{1}{4} + \frac{1}{2}Q, \tag{5.10}$$

Now, the exact solution of Eq. (4.10) has the form

$$\begin{aligned} u(x, t) &= -4\left(-\frac{1}{4} + \frac{1}{2} \frac{1}{1 + de^{(x - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)})}}\right)^2, \\ v(x, t) &= \frac{1}{4} + \left(-\frac{1}{4} + \frac{1}{2} \frac{1}{1 + de^{(x - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)})}}\right), \\ w(x, t) &= \frac{1}{8} + \frac{1}{2} \left(-\frac{1}{4} + \frac{1}{2} \frac{1}{1 + de^{(x - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)})}}\right). \end{aligned}$$

where d is arbitrary constant.

6. RESULTS AND DISCUSSION

It is well known that the nonlinear KleinGordon equation has many applications in physics and the HirotaSatsuma system of equations was introduced to describe the interaction of two long waves with different dispersion relations. Hes semi-inverse method and the Kudryashov method are used for constructing exact soliton solutions of the time-fractional Klein-Gordon equation, and the time-fractional Hirota-Satsuma coupled KdV system. Solutions obtained are potentially significant physical problems. The results show that these methods are efficient in finding the exact solutions of fractional partial differential equations.

7. CONCLUSIONS

We established variational formulations for the time-fractional Klein-Gordon equation, and the time-fractional Hirota-Satsuma coupled KdV system by Hes semi-inverse method. It is obvious that the employed

approach is useful and manageable and remarkably simple to find various kinds of solitary solutions. Also the Kudryashov method was used to conduct an analytic study on the time-fractional Klein-Gordon equation, and the time-fractional Hirota-Satsuma coupled KdV system. Moreover, the methods are capable of greatly minimizing the size of computational work compared to other existing techniques.

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