Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **4**(1)(2015), 77-85

Solution of the fractional Zakharov-Kuznetsov equations by reduced differential transform method

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Abstract. In this paper an approximate analytical solution of the fractional Zakharov-Kuznetsov equations will be obtained with the help of the reduced differential transform method (RDTM). It is indicated that the solutions obtained by the RDTM are reliable and present an effective method for strongly nonlinear fractional partial differential equations.

Keywords: Fractional Zakharov-Kuznetsov equation, Fractional calculus, Reduced differential transform method.

2000 Mathematics subject classification: 35A22; 26A33.

1. Introduction

In the recent years, fractional differential operators have played a very important role in various fields such as electrical circuits, biology, biomechanics, viscoelasticity, etc. Such operators are the generalization, to real (or complex) order of the classical derivatives and integrals [1–4]. Recently various methods such as the Adomian decomposition method (ADM), the homotopy perturbation method (HPM), the variational

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Received: 29 June 2014
Revised: 13 August 2014
Accepted: 13 August 2014

iteration method (VIM) and the homotopy analysis method (HAM) have been applied for fractional PDEs [5–8]. The differential transform method (DTM) is a powerful mathematical technique which was introduced in 1986 by Zhou [9].
The DTM has been extended to obtain analytical approximate solutions to linear and nonlinear partial differential equations of fractional order [10–12]. Afterwards, the Reduced differential transform method (RDTM) has been used by many authors to obtain analytical and approximate solutions to nonlinear problems [13–17]. This method gives an analytical solution in the form of a polynomial, but, it is different from Taylor series method that requires computation of the high order derivatives.
In the present work, we are concerned with the application of RDTM for the fractional version of the Zakharov-Kuznetsov equations ($FZK(p,q,r)$) [18–22]:

$$D_t^{\alpha}u + a(u^p)_x + b(u^q)_{xxx} + c(u^r)_{yyx} = 0, \tag{1.1}$$

where $u = u(x,y,t)$, α is parameter describing the order of the fractional derivative ($0 < \alpha \leq 1$), a, b and c are arbitrary constants and p, q and r are integers. The Zakharov-Kuznetsov equation was first derived for describing weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma in two dimensions [22].

2. Fractional calculus

In this section, we present a review of the notations, definitions and preliminary of fractional calculus, according to the refferences [1] and [2].

Definition 2.1. A real function $f(x), x > 0$ is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $q(> \mu)$, such that $f(x) = x^q g(x)$, where $g(x) \in C[0,\infty]$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2.2. For a function $f \in C_\mu, \mu \geq -1$, the Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, is defined as

$$\begin{cases} J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, & \alpha > 0, x > 0, \\ J^0 f(x) = f(x). \end{cases} \tag{2.1}$$

For $f \in C_\mu, \mu \geq -1$ and $\forall \alpha, \beta \in \mathbb{R}_+$, the operator J^α has the properties:
$i) J^{\alpha}J^{\beta}f(x) = J^{\alpha+\beta}f(x)$.
$ii) J^{\alpha}J^{\beta}f(x) = J^{\beta}J^{\alpha}f(x)$.

We all know that, the Riemann-Liouville approach leads to initial conditions containing the limit values of the Riemann-Liouville fractional derivatives which there is no known physical interpretation for such types of initial conditions. A modified fractional differential operator D^α which proposed by Caputo in his work on the theory of viscoelasticity [23] is

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \quad (2.2)$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$ and $f \in C_{-1}^m$.
The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations.

Definition 2.3. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in \mathbb{N}. \end{cases} \quad (2.3)$$

3. Reduced differential transform method

In this section, we apply the reduced differential transform method for three variables function $u(x,y,t)$ which has been developed in [16] and [17].
Consider a function of three variables $u(x,y,t)$ which is analytic and differentiated continuously in the domain of interest, and suppose that it can be represented as $u(x,y,t) = f(x,y)g(t)$.

Definition 3.1. If function $u(x,y,t)$ is analytic and differentiated continuosly with respect to x, y and t in the domain of interest, then let

$$U_k(x,y) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,y,t) \right]_{t=0}, \quad (3.1)$$

where the t-dimensional spectrum function $U_k(x,y)$ is the transformed function which is called T-function.

The differential inverse transform of $U_k(x,y)$ is defined as

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y) t^{k\alpha}. \quad (3.2)$$

Combining Eqs. (3.1) and (3.2) gives that

$$u(x,y,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,y,t)\right]_{t=0}t^{k\alpha}. \tag{3.3}$$

In real applications, by consideration of $U_0(x,y)=h(x,y)$ as transformaiton of initial condition

$$u(x,y,0)=h(x,y), \tag{3.4}$$

the function $u(x,y,t)$ can be approximated by a finite series of Eq. (3.2) as

$$\widetilde{u}_n(x,y,t)=\sum_{k=0}^{n}U_k(x,y)t^{k\alpha}. \tag{3.5}$$

A straightforward iterative calculations, gives the $U_k(x,y)$ values for $k=1,2,\cdots,n$. Then the inverse transformation of the $\{U_k(x,y)\}_{k=0}^{n}$ gives the approximation solutoin as $\widetilde{u}_n(x,y,t)$, where n is order of approximation solution. Next, the exact solutoin is obtained by

$$u(x,y,t)=\lim_{n\longrightarrow\infty}\widetilde{u}_n(x,y,t).$$

Some basic properties of the reduced differential transformation, obtained from Eqs. (3.1) and (3.2), are summarized in Table 1. Note, in this table

$$\Gamma(z):=\int_0^{\infty}e^{-t}t^{z-1}dt, \qquad z\in\mathbb{C}.$$

Function Form	Transformed Form
$u(x,y,t)=v(x,y,t)+w(x,y,t)$	$U_k(x,y)=V_k(x,y)+W_k(x,y)$
$u(x,y,t)=cv(x,y,t)$	$U_k(x,y)=cV_k(x,y)$ (c *is a constant*)
$u(x,y,t)=v(x,y,t)w(x,y,t)$	$U_k(x,y)=\sum_{k_1=0}^{k}V_{k_1}(x,y)W_{k-k_1}(x,y)$
$u(x,y,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}v(x,y,t)$	$U_k(x,y)=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}V_{k+N}(x,y)$
$u(x,y,t)=\frac{\partial^{m+n}}{\partial x^m\partial y^n}v^n(x,y,t)$	$U_k(x,y)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\partial^{m+n}}{\partial x^m\partial y^n}V_{k_1}(x,y)V_{k_2-k_1}(x,y)\cdots V_{k_{n-1}-k_{n-2}}(x,y)V_{k-k_{n-1}}(x,y)$

Table 1. Some basic reduced differential transformations.

According to the RDTM and Table 1, we can construct the following iteration for the Eq. (1.1) as

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y)$$
$$+\,a\,\frac{\partial}{\partial x}\Big(\sum_{k_{p-1}=0}^{k}\sum_{k_{p-2}=0}^{k_{p-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U_{k_1}(x,y)U_{k_2-k_1}(x,y)\cdots U_{k_{p-1}-k_{p-2}}(x,y)U_{k-k_{p-1}}(x,y)\Big)$$
$$+\,b\,\frac{\partial^3}{\partial x^3}\Big(\sum_{k_{q-1}=0}^{k}\sum_{k_{q-2}=0}^{k_{q-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U_{k_1}(x,y)U_{k_2-k_1}(x,y)\cdots U_{k_{q-1}-k_{q-2}}(x,y)U_{k-k_{q-1}}(x,y)\Big)$$
$$+\,c\,\frac{\partial^3}{\partial y^2\partial x}\Big(\sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U_{k_1}(x,y)U_{k_2-k_1}(x,y)\cdots U_{k_{r-1}-k_{r-2}}(x,y)U_{k-k_{r-1}}(x,y)\Big)=0. \tag{3.6}$$

4. A test example

We consider the time-fractional $FZK(2,2,2)$ in the form :

$$D_t^{\alpha}u+(u^2)_x+\frac{1}{8}(u^2)_{xxx}+\frac{1}{8}(u^2)_{yyx}=0, \tag{4.1}$$

where $0<\alpha\leq 1$ is order of the fractional time derivative. The exact solution to (4.1) when $\alpha=1$ and subject to the initial condition

$$u(x,y,0)=-\frac{4}{3}\lambda cosh^2(x+y), \tag{4.2}$$

where λ is an arbitrary constant, was derived in [19] and is given as:

$$u(x,y,t)=-\frac{4}{3}\lambda cosh^2(x+y-\lambda t). \tag{4.3}$$

By using the basic properties of the differential transform and Table 1, we can find transformed form of (4.1) and (4.2) as:

$$U_{k+1}(x,y)=-\,\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\Big(\frac{\partial}{\partial x}\sum_{r=0}^{k}U_r(x,y)U_{k-r}(x,y)$$
$$+\frac{1}{8}\frac{\partial^3}{\partial x^3}\sum_{r=0}^{k}U_r(x,y)U_{k-r}(x,y)+\frac{1}{8}\frac{\partial^3}{\partial y^2\partial x}\sum_{r=0}^{k}U_r(x,y)U_{k-r}(x,y)\Big), \tag{4.4}$$

and

$$U_0(x,y)=\frac{4}{3}\,\lambda\,cosh^2(x+y). \tag{4.5}$$

The recurrence relation (4.4) and the transformed initial condition (4.5) yield

$$U_1(x,y) = \frac{8\lambda^2}{9\Gamma(1+\alpha)}\Big(4\sinh[2(x+y)] + 5\sinh[4(x+y)]\Big),$$
$$U_2(x,y) = \frac{512\lambda^4}{81\Big(\Gamma(1+\alpha)\Big)^2\Gamma(1+3\alpha)}\sinh\Big(2(x+y)\Big)\Big[$$
$$\Big(3396 + 7380\cosh[2(x+y)] + 6600\cosh[4(x+y)] + 5100\cosh[6(x+y)]\Big)\Big(\Gamma(1+\alpha)\Big)^2$$
$$+\Big(140 + 465\cosh[2(x+y)] + 300\cosh[4(x+y)] + 425\cosh[6(x+y)]\Big)\Gamma(1+2\alpha)$$
$$\Big], \cdots. \tag{4.6}$$

For a comparison between the exact and approximate soluton, we consider $\alpha = 1$ and $\lambda = 0.001$. Consequently, the next terms of $U_k(x,y)$ become

$$U_1(x,y) = 0.0000248889\cosh^3(x+y)\sinh(x+y)$$
$$+ 0.0000106667\cosh(x+y)\sinh^3(x+y),$$
$$U_2(x,y) = -1.87259\times 10^{-7}\cosh^6(x+y)$$
$$- 1.73274\times 10^{-6}\cosh^4(x+y)\sinh^2(x+y)$$
$$- 9.03111\times 10^{-7}\cosh^2(x+y)\sinh^4(x+y)$$
$$- 2.13333\times 10^{-8}\sinh^6(x+y),$$
$$U_3(x,y) = 8.00658\times 10^{-11}\sinh(2(x+y))$$
$$+ 1.24313\times 10^{-9}\sinh(4(x+y))$$
$$+ 3.79259\times 10^{-9}\sinh(6(x+y)) + 3.13416\times 10^{-9}\sin([8(x+y)), \cdots.$$

The comparision is shown in Table 2 by taking only 5 terms into account, i.e.

$$u(x,y,t) = \sum_{k=0}^{5} U_k(x,y)t^{k\alpha}.$$

In addition, this approximate solution for $\alpha = 0.4$ and $\alpha = 0.75$ are given in Table 3. (See also Fig 1.)

x, y	t	$\alpha = 1$	$Exact(\alpha = 1)$
x=0.3	t=0.2	-1.86792072E-3	-1.87336795E-3
y=0.3	t=0.6	-1.85668352E-3	-1.87256368E-3
	t=1	-1.84599651E-3	-1.87176018E-3
x=0.6	t=0.2	-4.31581061E-3	-4.36984075E-3
y=0.6	t=0.6	-4.21637816E-3	-4.36692779E-3
	t=1	-4.12395983E-3	-4.36401721E-3
x=0.9	t=0.2	-1.23373501E-2	-1.28703109E-2
y=0.9	t=0.6	-1.08457601E-2	-1.28605665E-2
	t=1	-5.71066844E-2	-1.28508299E-2

Table 2. The exact solutions and the approximate solutions for $\alpha = 1$ when $\lambda = 0.001$.

x, y	t	$\alpha = 0.4$	$\alpha = 0.75$
x=0.3	t=0.2	-1.85729531E-3	-1.86438816-3
y=0.3	t=0.6	-1.84889497E-3	-1.85308338E-3
	t=1	-1.84362452E-3	-1.84417536E-3
x=0.6	t=0.2	-4.21312814E-3	-4.28450478E-3
y=0.6	t=0.6	-4.01901448E-3	-4.18406080E-3
	t=1	-4.70477241E-3	-4.07471651E-3
x=0.9	t=0.2	1.03971416E-2	-1.19571008E-2
y=0.9	t=0.6	2.00072564E-2	-1.23552425E-2
	t=1	5.89035854E-1	6.33400868E-2

Table 3. The approximate solutions for different values of α when $\lambda = 0.001$ and $y = 0.9$.

5. CONCLUSION

In this paper, the reduced differential transform method (RDTM), has been successfully applied for the fractional Zakharov-Kuznetsov equation. It can be concluded that, RDTM is a very powerful and efficient technique for finding approximate solutions for wide classes of problems and can be applied to many complicated linear and non-linear problems, and does not require linearization, discretization or perturbation.

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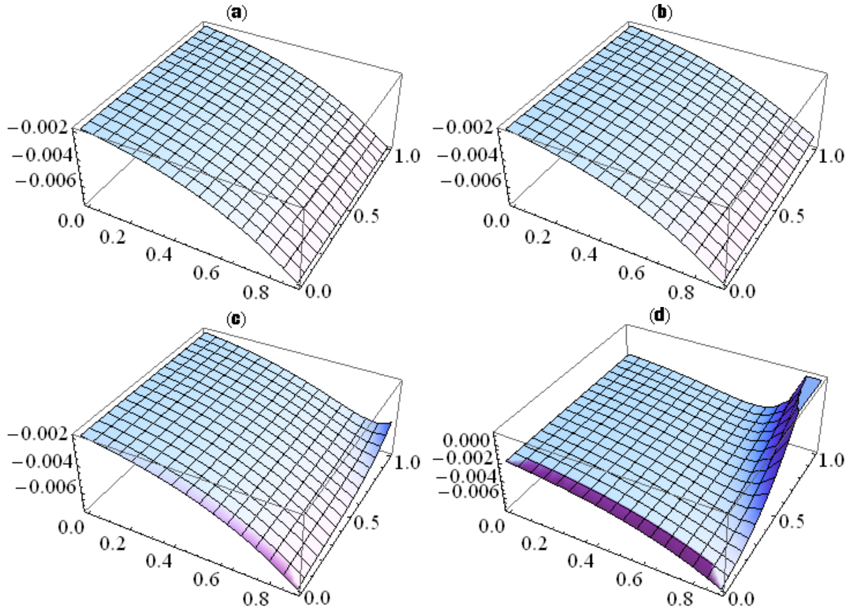

FIGURE 1. Solutions using the $u_5(x, y, t)$ for different values of α when $\lambda = 0.001$ and $y = 0.6$: (a) exact($\alpha = 1$), (b)($\alpha = 1$), (c)($\alpha = 0.75$) and (d)($\alpha = 0.4$).

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