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# On the Szeged and Eccentric connectivity indices of non-commutative graph of finite groups

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Abstract. Let $G$ be a non-abelian group. The non-commuting graph $\Gamma_G$ of $G$ is defined as the graph whose vertex set is the non-central elements of $G$ and two vertices are joined if and only if they do not commute.

In this paper we study some properties of $\Gamma_G$ and introduce $n$-regular $AC$-groups. Also we then obtain a formula for Szeged index of $\Gamma_G$ in terms of $n$, $|Z(G)|$ and $|G|$ . Moreover, we determine eccentric connectivity index of $\Gamma_G$ for every non-abelian finite group $G$ in terms of the number of conjugacy classes $k(G)$ and the size of the group $G$.

Keywords: non-commuting graph, eccentric connectivity index, Szeged index.

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## 1. Introduction

Let $\Gamma$ be an undirected connected graph without loops or multiple edges. We denote the vertex and the edge sets of $\Gamma$ by $V(\Gamma)$ and $E(\Gamma)$, respectively. For two vertices $x$ and $y$ of $V(\Gamma)$ the distance $d(x, y)$ is defined as the length of any shortest path connecting $u$ and $v$ in $\Gamma$ and $deg(v)$ denotes the degree of vertex $v$, i.e., the number of its neighbors in $\Gamma$.

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For an edge $e(= uv)$ of $\Gamma$, let $n_u(e)$ denote the set of vertices of $\Gamma$ lying closer to $u$ than to $v$ and $n_v(e)$ the set of vertices of $\Gamma$ lying closer to $v$ than to $u$.
The sets $n_u(e)$ and $n_v(e)$ play an important role in metric graph theory. For more information on research in this direction see [5, 8, 9]). Ivan Gutman [7] defined the Szeged index, $Sz(\Gamma)$, of a graph $\Gamma$ as:

$$Sz(\Gamma) = \sum_{uv=e\in E(\Gamma)} |n_u(e)| \cdot |n_v(e)|.$$

For a given $u$ of $V(\Gamma)$ its eccentricity the topological distance i.e., the number of edges on the shortest path, joining the two vertices of $\Gamma$. Since $\Gamma$ is connected, $d(x, y)$ exists for all $x, y \in V(\Gamma)$. The eccentricity of a vertex $v$ in $V(\Gamma)$, denoted by $\varepsilon(v)$, is defined to be

$$\varepsilon(v) = \max\{d(v, w)|w \in V(\Gamma)\}$$

Sharma, Goswami and Madan [13] introduced a distance-based molecular structure descriptor, they named " eccentric connectivity index" and which is defined as

$$\xi(\Gamma) = \sum_{v\in V(\Gamma)} deg(v) \cdot \varepsilon(v),$$

For further information see references [6, 14].

Let $G$ be a non-abelian group and let $Z(G)$ be the center of $G$. Associate with $G$ a graph $\Gamma_G$ as follows: Take $G\backslash Z(G)$ as vertices of $\Gamma_G$ and join two distinct vertices $x$ and $y$ whenever $xy \neq yx$. Graph $\Gamma_G$ is called the non-commutative graph of $G$ and many of graph theoretical properties of $\Gamma_G$ have been studied in [1, 2, 10]. Recently, it has been shown (see [3]), that if $G$ is a finite non-abelian group, then Wiener index of $\Gamma_G$ would be as:

$$W(\Gamma_G) = \frac{(|G| - |Z(G)|)(|G| - 2|Z(G)| - 2) + |G|(k(G) - |Z(G)|)}{2},$$

where $k(G)$ is the number of conjugacy classes of $G$.

In this paper we study some metric properties of $\Gamma_G$ and introduce *$n$-regular $AC$-groups* and obtain a formula of Szeged index of $\Gamma_G$ in terms $n$, $|Z(G)|$ and $|G|$. Also, for a finite non-abelian group $G$, we compute the eccentric connectivity index of $\Gamma_G$ in terms of $k(G)$ and $|Z(G)|$. The main results are

**Theorem 1.1.** *Let $G$ be a finite $n$-regular $AC$- group. Then*

$$Sz(\Gamma_G) = \frac{1}{2}(|G| - n)(|G| - |Z(G)|)(n - |Z(G)|)^2.$$

**Theorem 1.2.** *Let $G$ be a finite non-abelian group. Then*

$$\xi(\Gamma_G) = 2|G|.(|G| - k(G)).$$

## 2. Szeged index of non-commuting graphs of certain groups

In this section we first introduce groups that have regular non-commuting graph and then introduce $AC$-groups. Then we obtain Szeged index of non-commuting graphs of some regular $AC$-groups.

We begin with some basic metric properties of the non-commuting graph of group $G$. Here, a significant result that we seriously need will be presented without proof of [1].

**Lemma 2.1.** *Let $G$ be a non-abelian group. Then $\Gamma_G$ is a connected graph of diameter 2 and girth 3.*

*Proof.* See [1]. □

By the above Lemma it is clear that:

*Remark* 2.2. Let $G$ be a non-abelian group and $x \in G\backslash Z(G)$. Then

$$d(x,y) = \begin{cases} 1, & \text{if } y \in G\backslash C_G(x)\ ; \\ 2, & \text{if } y \in C_G(x)\backslash Z(G). \end{cases}$$

**Lemma 2.3.** *Let $G$ be a non-abelian group. Then $\Gamma_G$ is regular if and only if there exists natural number $k$, $|C_G(x)| = k$, where $x$ is a non-central element.*

*Proof.* We know that if $x$ is a non-central element, then by definition of non-commuting graph of a group, $deg(x) = |G\backslash C_G(x)|$. Now, $\Gamma_G$ is a regular if and only if $deg(x) = deg(y)$, for all $x, y \in V(G)$. So $\Gamma_G$ is regular if and only if $|C_G(x)| = |C_G(y)|$, for all $x, y \in V(G)$. □

A group $G$ is called an $AC$-group if the centralizer of every non-central element of $G$ is abelian.

**Lemma 2.4.** *The following conditions on a group $G$ are equivalent.*
*(a) $G$ is an $AC$-group.*
*(b) If $[x, y] = 1$, then $C_G(x) = C_G(y)$, where $x, y \in G\backslash Z(G)$.*
*(c) If $[x, y] = [x, z] = 1$, then $[y, z] = 1$, where $x \in G\backslash Z(G)$.*
*(d) If $x, y \in G\backslash Z(G)$ with distinct centralizers, then $C_G(x) \cap C_G(y) = Z(G)$.*

*Proof.* The proof is straightforward. See also [12], Lemma 3.2. □

**Definition 2.5.** A non-abelian group $G$ is called regular $AC$- group if $G$ is $AC$-group and $\Gamma_G$ is a regular graph. Also a non-abelian group $G$ is said to be $n$-regular $AC$- group if $G$ is regular $AC$- group and $|C_G(x)| = n$, where $x \in V(G)$.

Before proving one of the main theorems, we need to present a key lemma of [3] without any proof.

**Lemma 2.6.** *Let $G$ be a non-abelian group and $e = uv \in E(\Gamma_G)$. Then*

$$n_u(e) = ((C_G(v)\backslash C_G(u))\backslash\{v\}) \cup \{u\}.$$

*Proof.* See [3]. □

Now we are ready to prove Theorem1.1.

**Proof of Theroem 1.1**

*Proof.* Since $G$ is a $n$-regular $AC$- group so we have $|C_G(x)| = n$, where $x \in V(\Gamma_G)$. We know that $Sz(G) = \sum_{uv=e\in E(\Gamma)} |n_u(e)| \cdot |n_v(e)|$ and by Lemma 2.6, $n_u(e) = ((C_G(v)\backslash C_G(u))\backslash\{v\}) \cup \{u\}$. Now, $C_G(v)\backslash C_G(u) = C_G(v)\backslash(C_G(v)\cap C_G(u))$. Since $G$ is $AC$-group, $C_G(v)\backslash C_G(u) = C_G(v)\backslash Z(G)$. Hence $|n_u(e)| = |C_G(v)\backslash Z(G)| = n - |Z(G)|$. Therefore

$$Sz(\Gamma_G) = \sum_{uv=e\in E(\Gamma)} (n - |Z(G)|)^2 = |E(\Gamma_G)|(n - |Z(G)|)^2.$$

On the other hand,

$$\begin{aligned}|E(\Gamma_G)| &= \frac{1}{2}\sum_{x\in V(\Gamma)} d(x) = \frac{1}{2}\sum_{x\in V(\Gamma_G)} |G\backslash C_G(x)| \\ &= \frac{1}{2}\sum_{x\in V(\Gamma_G)} (|G| - n) = \frac{1}{2}(|G| - n)(|G| - |Z(G)|).\end{aligned}$$

The proof is completed by replacing the above value of $|E(\Gamma_G)|$. □

As mentioned, in [3] for every non-abelian finite group $G$, the value of Wiener index $W(\Gamma_G)$ is determined. Here, we present a simple and different proof to obtain $W(\Gamma_G)$ when $G$ is a finite $n$-regular $AC$- group.

**Theorem 2.7.** *Let $G$ be a finite $n$-regular $AC$- group. Then*

$$W(\Gamma_G) = \frac{1}{2}(|G| - |Z(G)|)(|G| - 2|Z(G)| + n - 2).$$

*Proof.* Since $G$ is an $n$-regular $AC$- group, $|C_G(x)| = n$, where $x \in V(\Gamma_G)$. We know that

$$W(\Gamma_G) = \frac{1}{2}\sum_{x\in V(\Gamma_G)} d(G, x)$$

and

$$d(G, x) = \sum_{y\in G\backslash C_G(x)} d(x, y) + \sum_{y\in C_G(x)\backslash Z(G)} d(x, y).$$

Now by Remark 2.2, $\sum_{y\in G\backslash C_G(x)} d(x,y) = |G\backslash C_G(x)|$ and

$$\sum_{y\in C_G(x)\backslash Z(G)} d(x,y) = 2(|C_G(x)| - |Z(G)| - 1).$$

Hence $d(G,x) = |G| - 2|Z(G)| + n - 2$. Therefore
$W(\Gamma_G) = \frac{1}{2}\sum_{x\in V(\Gamma_G)} |G| - 2|Z(G)| + n - 2$. Since $V(G) = G\backslash Z(G)$ we have
$W(\Gamma_G) = \frac{1}{2}(|G| - |Z(G)|)(|G| - 2|Z(G)| + n - 2)$, as desired. □

**Example 2.8.** Let $G$ be a finite non abelian group of order $p^3$. Then $Sz(\Gamma_G) = \frac{1}{2}p^4(p-1)^3(p+1)$ and $W(\Gamma_G) = \frac{1}{2}p(p+1)^2(p-1)(p^2-2)$.

*Proof.* Since $G$ is non-abelian, $|Z(G)| = p$ and if $x \in V(\Gamma_G)$ then $|C_G(x)| = p^2$. Hence $C_G(x)$ is abelian and so $G$ is $p^2$-regular $AC$- group. Therefore $Sz(\Gamma_G) = \frac{1}{2}(p^3-p^2)(p^3-p)(p^2-p)^2 = \frac{1}{2}p^4(p-1)^3(p+1)$ and $W(\Gamma_G) = \frac{1}{2}(p^3-p)(p^3-2p+p^2-2) = \frac{1}{2}p(p+1)^2(p-1)(p^2-2)$. □

## 3. Eccentric connectivity index of non-commutative graph of group

In this section we suppose that $G$ is a finite non-abelian group. The main objective in this section is to present an explicit formula of $\xi(\Gamma_G)$ the eccentric connectivity index in terms of $|G|$ and the number of conjugacy classes $k(G)$. Now, it is time to prove of secondary the main theorems.

**Proof of Theroem 1.2**

*Proof.* Suppose that $u \in V(\Gamma_G)$. If $x \in G\backslash C_G(u)$, then $d(u,x) = 1$ and if $x \in (C_G(u)\backslash Z(G))\backslash\{u\}$ then $d(u,x) = 2$. It follows that $\varepsilon(u) = max\{d(u,x) \mid x \in V(G)\} = 2$. We know, using the definition of non-commuting graph, that $deg(u) = |G\backslash C_G(u)|$. Hence

$$\begin{aligned}\xi(\Gamma_G) &= \sum_{u\in V(G)} deg(u)\cdot\varepsilon(u) = 2\sum_{u\in V(G)} |G\backslash C_G(u)| \\ &= 2\Big(\sum_{u\in V(G)} |G| - \sum_{u\in V(G)} |C_G(u)|\Big) = 2|G||G\backslash Z(G)| - 2\sum_{u\in V(G)} |C_G(u)|. \qquad (*)\end{aligned}$$

On the other hand, we have $\sum_{u\in G} |C_G(u)| = k(G)\cdot|G|$. So $\sum_{u\in Z(G)} |C_G(u)| + \sum_{u\in G\backslash Z(G)} |C_G(u)| = k(G)\cdot|G|$ and so $|Z(G)||G| + \sum_{u\in G\backslash Z(G)} |C_G(u)| = k(G)\cdot|G|$. It follows that

$$\sum_{u\in V(G)} |C_G(u)| = k(G)|G| - |Z(G)|\cdot|G| = |G|(k(G) - |Z(G)|).$$

Now, by using $(*)$, we have

$$\xi(\Gamma_G) = 2|G| \cdot (|G| - |Z(G)) - 2|G| \cdot (k(G) - |Z(G)|) = 2|G|(|G| - k(G)).$$

$\square$

The paper winds up with the application of the above theorem for projective special linear group.

**Example 3.1.** Let $G$ be a projective special linear group $PSL(2, q)$, where $q$ is a power of a prime $p$ and $q \geq 4$. Then

$$\xi(\Gamma_G) = \begin{cases} 2q(q+1)^2(q^2 - q - 1) & \text{if } q \equiv 0 \ (mod\ 4), \\ \frac{1}{2}q(q-1)(q+1)(q^3 - 2q - 5) & \text{if } q > 5 \text{ and } q \not\equiv 0 \ (mod\ 4). \end{cases}$$

*Proof.* We know that if $q \equiv 0 \ (mod\ 4)$, then $|G| = q(q-1)(q+1)$ and if $q \not\equiv 0 \ (mod\ 4)$, then $|G| = \frac{1}{2}q(q-1)(q+1)$. However, by [11, Theorems 5.5, 5.6 and 5.7],

$$k(G) = \begin{cases} q+1 & q \equiv 0 \ (\text{mod } 4); \\ \frac{q+5}{2} & q > 5 \text{ and } q \not\equiv 0 \ (\text{mod } 4). \end{cases}$$

Now, by using of Theorem 1.2 , the result follows. $\square$

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