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Ill-Posed and Linear Inverse Problems

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Abstract. In this paper ill-posed linear inverse problems that arises in many applications is considered. The instability of special kind of these problems and it's relation to the kernel, is described. For finding a stable solution to these problems we need some kind of regularization that is presented. The results have been applied for a singular equation

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1. Introduction

The concept of ill-conditioned problem has been first introduced by Hadamard [1].
Fredholm integral equation of the first kind is a classical example of ill-posed problem that arises frequently in applied problems. A simple case of this equation is as follows:

$$\int_0^1 K(s,t)f(t)dt = g(s). \tag{1.1}$$

The difficulties with the above integral equation inseparably connected with the compactness of the operator which is associated with the kernel

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K [2, Chapter 15]. In physical terms, the integration with K in (1.1) has a "smoothing" effect on f, in the sense that high-frequency components, cusps, and edges in f are "smoothed out" by the integration.

2. Singular value expansion (SVE)

If

$$||K||^2 = \int_0^1 \int_0^1 K(s,t)^2 dtds$$

is bounded then, we have [3]:

$$K(s,t) = \sum_{i=1}^{\infty} \mu_i u_i(s) v_i(t) \tag{2.1}$$

u_i and v_i's are the singular functions of the K, and μ_i's are the singular values of the kernel K. Some properties of these quantities are as follows:

- $(u_i, u_j) = (v_i, v_j) = \delta_{ij}$,
- $||K||^2 = \sum_{i=1}^{\infty} \mu_i^2$,
- $\mu_1 \geq \mu_2 \geq ... \geq \mu_n \geq ...$,
- $\{\mu_i^2, u_i\}$ are the eigensolutions of the symmetric kernel $\int_0^1 K(s,x)K(t,x)dx$,
- $\{\mu_i^2, v_i\}$ are the eigensolutions of the symmetric kernel $\int_0^1 K(x,s)K(x,t)dx$,
- $\int_0^1 K(s,t)v_i(t)dt = \mu_i u_i(s)$,

The second property with $||K||^2 < \infty$ inquires that μ_i must decay faster than $i^{-1/2}$.
By the above relations we can rewrite the equation (1.1) as :

$$\sum_{i=1}^{\infty} \mu_i (v_i, f) u_i(s) = \sum_{i=1}^{\infty} (u_i, g) u_i(s), \tag{2.2}$$

$$\sum_{i=1}^{\infty} \mu_i (v_i, f) u_i(s) = \sum_{i=1}^{\infty} (u_i, g) u_i(s), \tag{2.3}$$

so,

$$f(t) = \sum_{i=1}^{\infty} \frac{(u_i, g)}{\mu_i} v_i(t) \tag{2.4}$$

3. Computation of the SVE

One practical way to compute the SVE of an integral equation is to approximate it by SVD (singular value decomposition) of the linear system of equations, comes from the discretization of the main integral equation. The SVD reveals all the difficulties associated with the ill-conditioning of the matrix A.

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix, Then the SVD of A is a decomposition of the form

$$A = U\Sigma V^T = \sum_{i=1}^{n} u_i \sigma_i v_i^T$$

Where $U = (u_1, ..., u_n) \in \mathbb{R}^{n\times n}$, and $V = (v_1, ..., v_n) \in \mathbb{R}^{n\times n}$ are matrices with orthonormal columns, $U^T U = V^T V = I_n$. And where the diagonal matrix $\Sigma = diag(\sigma_1, ..., \sigma_n)$ has nonnegative diagonal elements appearing in nonincreasing order such that,

$$\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0.$$

The numbers σ_i's are called the singular values of A, while the vectors u_i and v_i's are the left and right singular vectors of A, respectively.
The classical algorithm for computing the SVD of a dense matrix is due to Golub, Kahan, and Reinsch [4], [5]. The algorithm consists of two main stages. In the first stage, A is transformed into upper bidiagonal form B by means of a finite sequence of alternating left and right Householder transformations. In the second, iterative, stage, the shifted QR algorithm is applied implicitly to the matrix $B^T B$, and consequently B converges to Σ. The left and right orthogonal transformations, if accumulated, produce the matrices U and V.
Now we will explain the computation of the approximation of SVE by means of SVD [6]. The algorithm takes the following form. Assume that we choose orthonormal basis functions $\phi_1, \phi_2, ..., \phi_n$, and $\psi_1, \psi_2, ..., \psi_n$, and compute matrix A by $a_{ij} = \int_0^1 K(s,t)\phi_i(s)\psi_j(t)dsdt$, then compute its SVD.
The n singular values of A, $(\sigma_i^{(n)})$ are approximately the n singular values of K, and,

$$u_j(t) \simeq \widetilde{u}_j(t) = \sum_{i=1}^{n} u_{ij}\phi_i(t), \qquad v_j(s) \simeq \widetilde{v}_j(s) = \sum_{i=1}^{n} v_{ij}\psi_i(s)$$

where u_{ij}, and v_{ij}'s are the elements of the singular vectors of A. More precisely we have the following theorem.

Theorem 3.1. *Let $||K||$ denote the norm of K, and define [6],*

$$\delta_n^2 = ||K||^2 - ||A||_F^2,$$

then $\sum_{i=1}^{n}(\mu_i - \sigma_i^{(n)})^2 \leq \delta_n^2$,
and for $i = 1, 2, ...$ we have:

$$0 \leq \mu_i - \sigma_i^{(n)} \leq \delta_n$$

$$\sigma_i^{(n)} \leq \sigma_i^{(n+1)} \leq \mu_i$$

$$max\{||u_i - \widetilde{u}_i||_2, ||v_i - \widetilde{v}_i||_2\} \leq (\frac{2\delta_n}{\mu_i - \mu_{i+1}})^{1/2}$$

4. THE SMOOTHING PROPERTIES OF THE KERNEL AND PICARD CONDITION

The overall behavior of the singular values μ_i and the singular functions u_i and v_i is by no means "arbitrary"; their behavior is strongly connected with the properties of the kernel K. The following holds.

- The "smoother" the kernel K, the faster the singular values μ_i decay to zero (where "smoothness" is measured by the number of continuous partial derivatives of K). If the derivatives of order $0, ..., p$ exist and are continuous, then μ_i is approximately $O(i^{-p-1/2})$. The precise result is proved in [7] and summarized in [8].

- The smaller the μ_i the more oscillations (or zero-crossings) there will be in the singular functions u_i and v_i. This property is perhaps impossible to prove in general, but it is often observed in practice. It is related to the Riemann-Lebesgue lemma.

In order that there exist a square integrable solution f to the integral equation (1.1), the right-hand side g must satisfy [9, chapter 2],

$$\sum_{i=1}^{\infty}(\frac{(u_i, g)}{\mu_i})^2 < \infty, \tag{4.1}$$

that is called the *picard condition*.
Picard condition is equal to $g \in Range(K)$.
Consider that g does not belong to $Range(K)$, and let g_k denote the approximation to g obtained from truncating its SVE expansion after k terms, $g_k(s) = \sum_{i=1}^{k}(u_i, g)u_i(s)$.
This g_k is clearly satisfies the Picard condition. The corresponding approximate solution is,

$$f_k(t) = \sum_{i=1}^{k} \frac{(u_i, g)}{\mu_i} v_i(t)$$

We conclude that as $k \to \infty$, we have $g_k \to g$, but,

$$||f_k||_2 \to \infty \qquad as \qquad k \to \infty.$$

It is exactly this lack of stability of f that makes the integral equation (1.1) ill posed.

5. REGULARIZATION

As we have seen in the previous section, the primary difficulty with ill-posed problems is that they are practically underdetermined due to the cluster of small singular values of K. Hence, it is necessary to incorporate further information about the desired solution in order to stabilize the problem and to single out a useful and stable solution. This is the purpose of *regularization.* Although many types of additional information about the solution f to (1.1) are possible in principle see, e.g., the survey in [10] the dominating approach to regularization is to allow a certain residual associated with the regularized solution, with residual norm,

$$\rho(f) = || \int_0^1 K(s,t)f(t)dt - g(s)||$$

and then use one of the following four schemes.

- Minimize $\rho(f)$ subject to the constraint that f belongs to a specified subset, $f \in S$.
- Minimize $\rho(f)$ subject to the constraint that a measure $w(f)$ of the "size" of f is less than some specified upper bound δ, i.e., $w(f) \leq \delta$.
- Minimize $w(f)$ subject to the constraint $\rho(f) \leq \alpha$.
- Minimize a linear combination of $w(f)^2$ and $\rho(f)^2$:

$$Min\{\rho(f)^2 + \lambda^2 w(f)^2\}.$$

The last one is the well-known Tikhonov regularization method. In practice we usually discretize the integral equation and then apply some kind of regularization. For more details one can see [3].

6. Numerical examples

In the previous section we present the regularization scheme for Fredholm integral equation of the first kind. But we can use it for the discrete form of other equations that the matrix of linear system is ill-posed. Consider the singular boundary value problem of the form [11].

$$(p(x)y')' = p(x)f(x,y),\ x \in (0,1] \tag{6.1}$$

with boundary conditions,

$$y'(0) = 0,\ \alpha y(1) + \beta y'(1) = \gamma, \tag{6.2}$$

or

$$y(0) = A,\ \alpha y(1) + \beta y'(1) = \gamma, \tag{6.3}$$

where
$p(x) = x^b g(x),\ x \in [0,1]$.

Here $\alpha > 0$, $\beta \geq 0$, and A and γ are finite constants. Also,the following resrictions are imposed on $p(x)$ and $f(x,y)$.

$(I)p(x) > 0$ on $[0,1]$, $p(x) \in C^1(0,1]$, and $1/g(x)$ is analytic in $\{z\ s.t.\ |z| < r\}$ for some $r > 1$.

$(II)f(x,y) \in [0,1] \times R$, is continuous, $\frac{\partial f}{\partial y}$ exists, continuous and non-negative for all $(x,y) \in [0,1] \times R$.

The existence-uniqueness of eq.(6.1) has been established for BCs $y(0) = A$ and $y(1) = B$, with $0 \leq b < 1$. and BC's $y'(0) = 0$ and $y(1) = B$ with $b \geq 0$, provided that $\frac{xp'}{p}$ is analytic in $\{zs.t.|z| < r\}$ for some $r > 1$ [12,13].

The eq.(6.1), arises in the study of tumor growth problems, steady-state oxygen diffusion in a cell with Michaleis-Menten uptake kinetics, and distribution of heat sources in the human head [11].

We have discretize this equation using Legendre wavelets, the result for the coefficients (solution of the linear system obtained by discretzation) was,
$c_0 = -4.6358693076092977194 * 10^6$, $c_1 = -6.7582264983241763273 * 10^6$,
$c_2 = -5.9099142767810083574 * 10^6$, $c_3 = -3.1770021176532618076 * 10^6$,
$c_4 = 2.1188760718345176772 * 10^5$, $c_5 = 3.0419453143786254232 * 10^6$,
$c_6 = 4.4978182063968003491 * 10^6$, $c_7 = 4.3729982946561054292 * 10^6$,
$c_8 = 3.0178295836150515531 * 10^6$, $c_9 = 1.0918111655220634394 * 10^6$,
$c_{10} = -7.3825057261862819434$, $c_{11} = -2.0216877298043717294 * 10^6$,
$c_{12} = -2.6041990921200130978 * 10^6$, $c_{13} = -2.5783384180672618095 * 10^6$,
$c_{14} = -2.1664432264550479477 * 10^6$, $c_{15} = -1.6038311314732019976 * 10^6$,
$c_{16} = -1.0646835980362944770 * 10^6$, $c_{17} = -6.3943235427383833427 * 10^5$,
$c_{18} = -3.4890121013572833342 * 10^5$, $c_{19} = -1.7316778052680677547 * 10^5$,
$c_{20} = -78099.00000139694805$, $c_{21} = -31912.487693990200350$,
$c_{22} = -11757.36500885325238$, $c_{23} = -3878.575888516679465$,
$c_{24} = -1134.7567159384624654$, $c_{25} = -290.64966939574863$,
$c_{26} = -64.022020715699201532$, $c_{27} = -11.825148046192491$,

$c_{28} = -1.763743121252254, \quad c_{29} = -.199859762088155,$
$c_{30} = -0.15358571058327e-1, \quad c_{31} = -0.6035246206578086981e-3.$

By applying the regularization method we obtain the following result with approximately the same residual,
$c_0 = -.89023366514234935536, \quad c_1 = .54799351644467802312,$
$c_2 = .22987869633682692909, \quad c_3 = -.14654224228351210740,$
$c_4 = -.19880287660064923473, \quad c_5 = 0.02385558285009152128,$
$c_6 = 0.15688511320033616955, \quad c_7 = 0.04260445545695083522 8,$
$c_8 = -.10699218169413489446, \quad c_9 = -0.07867195106771863393 9,$
$c_{10} = 0.05400813723112207527 9, \quad c_{11} = 0.09055131651035545733 6,$
$c_{12} = -0.00424055919158032434, \quad c_{13} = -0.08017636560392272455,$
$c_{14} = -0.03681827705766728097, \quad c_{15} = 0.05236514881053306866,$
$c_{16} = 0.06114904944238824772, \quad c_{17} = -0.01245121070830249877,$
$c_{18} = -0.06372770902085218976, \quad c_{19} = -0.02806763699295286968,$
$c_{20} = 0.04048107925096862155, \quad c_{21} = 0.05527295912277616750,$
$c_{22} = 0.00308355940655701700, \quad c_{23} = -0.05026157850544944020,$
$c_{24} = -0.04830911919951919830, \quad c_{25} = 0.00171251540542644326,$
$c_{26} = 0.05026641780697775711, \quad c_{27} = 0.06421752678679124589,$
$c_{28} = 0.04746665224675341727, \quad c_{29} = 0.02334795610035148928 0,$
$c_{30} = 0.00733876044854531235 00, \quad c_{31} = 0.00123384792989866633.$

7. CONCLUSIONS

In this paper we considered regularization for singular differential equations equations. Regularization could be applied to linear systems of equations that the coefficients matrix is ill-posed, so it is not restricted to integral equations. As we have shown in this paper regularization could be applied to singular boundary value problem. It is clear that it can be used for other type of equations that are somehow ill-posed.

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