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Spectrum preserving linear maps between Banach Algebras

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Abstract. In this paper we show that if A is a unital Banach algebra, B is a purely infinite C^*-algebra such that has a non-zero commutative maximal ideal and $\phi : A \to B$ is a unital surjective spectrum preserving linear map, then ϕ is a Jordan homomorphism.

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1. Introduction and Preliminaries

All algebras we deal with in this paper are complex and unital. The identity element of an algebra A will be denoted e, or e_A, for distinction. For a given algebra A and $a \in A$, $\sigma(a)$ and $r(a)$ will denote the *spectrum* and the *spectral radius* of a, respectively. Let A and B be two Banach algebras. A linear map $\phi : A \to B$ is said to be spectrum preserving if $\sigma(\phi(a)) = \sigma(a)$ for all $a \in A$. Furthermore, ϕ is said to be *unital* if $\phi(e_A) = e_B$ and it is called *invertibility preserving* if $\phi(a)$ is invertible in B whenever a is invertible in A. Now, if A is a Banach algebra, the set of all non-zero complex homomorphisms of A is a compact Hausdorff space in its usual (weak*) topology, the so-called Gelfand topology. This

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space will be called the maximal ideal space of A, and it will be denoted by M_A.

Spectrum preserving linear mappings were studied for the first time by G. Frobenius [8]. He proved that a surjective linear mapping $\varphi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ which preserves the spectrum has one of the forms $\varphi(T) = ATA^{-1}$ or $\varphi(T) = A^tTA^{-1}$ for some invertible A. In [12] Jafarian and sourour proved that a surjective linear map preserving spectrum from $B(X)$ onto $B(Y)$ is either an isomorphism or an anti-isomorphism where X and Y are complex Banach spaces and $B(X)$ is the Banach algebra of all bounded linear operators acting on X. The following conjecture seems to be still open:

Any spectrum-preserving linear map from a unital Banach algebra onto a unital semi-simple (non-commutative) Banach algebra that preserves the unit is a Jordan homomorphism, (Kaplansky's conjecture).

The G-K-Z Theorem ([15], [14]) asserts that a unital linear functional defined on a Banach algebra is multiplicative if it is invertibility preserving and the theorem has inspired a number of papers on more general preserver problems. It is a straightforward conclusion of the G-K-Z Theorem that a unital and invertibility preserving linear map from a Banach algebra into a semi-simple commutative Banach algebra is a homomorphism. This conjecture is still unsolved. The most important partial results obtained this direction are [1], [2], [4], [6], [9], [10], [11], [12], [13], [19], [20], [21], [22].

Recently Aupetit [2] showed that a spectrum preserving surjective linear map from a von Neumann algebra onto another is a Jordan isomorphism.

A C^*-subalgebra B of A is hereditary if $0 \leq a \leq b, a \in A, b \in B$ implies $a \in B$. A projection in a C^*-algebra A is called infinite if it is equivalent to a proper subprojection of itself. A C^*-algebra is purely infinite if every hereditary subalgebra contains an infinite projection.

In this paper we show that if A is a unital Banach algebra, B is a purely infinite C^*-algebra such that has a non-zero commutative maximal ideal and $\phi : A \to B$ is a unital surjective spectrum preserving linear map, then ϕ is a Jordan isomorphism (Theorem 2.2).

There are many results on the conjecture of Kaplansky. One of the most important results is [2, Theorem 1.3] of Aupetit. Among other theorems, Larwrence Harris proved the following.

Theorem 1.1. [9] *Let A be a unital Banach algebra, B be a unital semi-simple commutative Banach algebra and $\phi : A \to B$ is a unital invertibility preserving linear map. Then ϕ is a continuous multiplicative.*

Theorem 1.2. [17] *Let A be a C^*-algebra, and $a \in A$. Then there is an irreducible representation π of A such that $\|a\| = \|\pi(a)\|$.*

Remark 1.3. By Theorem 5.1.6 (2) in [17], it follows that if A is a non-commutative C^*-algebra, then irreducible representations π in Theorem 1.2 has dimension greater than 1.

Theorem 1.4. [16] *Let $\phi : A \to B$ be a unital surjective spectrally bounded operator from a unital C^*-algebra A onto a unital semisimple Banach algebra B. If A is a purely infinite C^*-algebra of real rank zero, then ϕ is a Jordan homomorphism.*

2. Main results

Recall that all algebras we deal with have an identity element. Moreover, note that by an ideal we always mean a 2-sided ideal.

Remark 2.1. We recall that if A, B and D are C^*-algebras, and if homomorphisms $\varphi : A \longrightarrow D$ and $\psi : B \longrightarrow D$ are given, then the C^*-algebra $A \oplus_D B$ is defined as

$$A \oplus_D B = \{(a, b) \in A \oplus_D B : \varphi(a) = \psi(b)\}.$$

Let A be a C^*-algebra, by [18, Lemmas 10 and 11] A has a unique maximal commutative ideal I (I may be obtained as the intersection of the kernels of all irreducible representations of A of dimension greater than 1) and a closed ideal J such that $I \cap J = \{0\}$ and A/J is commutative, furthermore, $A \cong A/J \oplus_{A/(I+J)} A/I$ by $*$-isomorphism $\varphi : A \longrightarrow A/J \oplus_{A/(I+J)} A/I$ such that $\varphi(a) = (a + J, a + I)$.

Theorem 2.2. *Let A be a unital Banach algebra and B be a purely infinite C^*-algebra such that it has a non-zero commutative maximal ideal. Suppose that $\phi : A \to B$ is a unital surjective spectrum preserving linear map, then ϕ is a Jordan homomorphism.*

Proof. By [18, Lemmas 10 and 11] B has a unique maximal commutative ideal I and a closed ideal J with the properties $I \cap J = 0$, B/J is commutative and $B \cong B/J \oplus_{B/(I+J)} B/I$. Define $\phi_1 : A \longrightarrow B/J$ and $\phi_2 : A \longrightarrow B/I$ by $\phi_1(a) = \phi(a) + J$ and $\phi_2(a) = \phi(a) + I$ for every $a \in A$.

We can show that ϕ_1 and ϕ_2 are well-defined and non-zero unital linear maps. For any $a \in A$, if $\phi(a)$ is invertible then $\phi_1(a)$ and $\phi_2(a)$ are invertible. Hence ϕ_1 and ϕ_2 preserve invertibility. Therefore, ϕ_1 is continuous homomorphism by Theorem 1.1. (Note that B/J is commutative C^*-algebra). Since I contains every commutative ideal by the hypothesis I is a commutative maximal ideal in B (see proof of Lemma 10 in [18]). It is clear that B/I is purely infinite simple C^*-algebra, so B/I has real rank zero by [7, Theorem V.7.4].

We prove that ϕ_2 is injective. To prove this, suppose $a \in A$ such that $\phi_2(a) = 0$, so $\phi(a) \in I$, and Theorem 1.2, Remarks 1.3 and 2.1

imply that $\|\phi(a)\| = \|\pi(\phi(a))\| = 0$ for some irreducible representation with dimension greater than 1. On the other hand ϕ is continuous and injective (see [2] and [3]), so ϕ_2 is injective. Also, by Theorem 1.5, ϕ_2^{-1} is Jordan isomorphism and hence ϕ_2 is Jordan isomorphism.

Now, we show that ϕ is a Jordan homomorphism. We have $\phi_1(a) = \phi(a) + J$ and $\phi_2(a) = \phi(a) + I$ for all $a \in A$. Hence for every $a \in A$

$$\phi_1(a)^2 = \phi(a)^2 + J,\ \phi_2(a)^2 = \phi(a)^2 + I. \tag{1}$$

Also we have

$$\phi_1(a^2) = \phi(a^2) + J,\ \phi_2(a^2) = \phi(a^2) + I. \tag{2}$$

Since ϕ_1 and ϕ_2 are Jordan homomorphism, (1) and (2) imply that $\phi(a)^2 - \phi(a^2) \in J$ and $\phi(a)^2 - \phi(a^2) \in I$. But $I \cap J = 0$. Therefore, $\phi(a^2) = \phi(a)^2$ for all $a \in A$, that is, ϕ is a Jordan homomorphism. This completes the proof. $\square$

Corollary 2.3. *Let A be a unital Banach algebra and B be a purely infinite C^*-algebra such that it has a non-zero commutative ideal. Suppose that $\phi : A \to B$ is a surjective spectrum preserving linear map, so ϕ is a Jordan homomorphism multiplied by an invertible element.*

Proof. For $b \in B$, denote L_b the linear map from B into itself defined by multiplying by b from the left hand, that is, $L_b(x) = bx$ for every $x \in B$. Let $\psi = L_{\phi(e)^{-1}} \circ \phi$, then $\psi(e) = e$. As a preserver, ψ has the same property as ϕ has. It is easy to check ψ preserves invertibility. Now by Theorem 2.2, ψ is a Jordan homomorphism and $\phi = L_{\phi(e)} \circ \psi$. This completes the proof. $\square$

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