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On The Convergence Of Modified Noor Iteration For Nearly Lipschitzian Maps In Real Banach Spaces

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Abstract. In this paper, we obtained the convergence of modified Noor iterative scheme for nearly Lipschitzian maps in real Banach spaces. Our results contribute to the literature in this area of research.

Keywords: Fixed point iteration schemes; Uniformly L-Lipschitzian asymptotically pseudocontractive mappings; Banach spaces, nearly uniformly $L-$Lipschitzian mappings.

1. INTRODUCTION

We denote J by the normalized duality mapping from X into 2^{X^*} by

$$J(x) = \{f \in \mathrm{X}^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where X^* denotes the dual space of real normed linear space X and $\langle ., .\rangle$ denotes the generalized duality pairing between elements of X and X^*. We first defined some concepts as follows (see, [7]):
Let C be a nonempty subset of real normed linear space X.
The mapping T is said to be uniformly L- Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

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for any $x, y \in C$ and $\forall n \geq 1$.
The mapping T is said to be asymptotically pseudocontractive if there exists a sequence $(k_n) \subset [1, \infty)$ with $lim_{n\to\infty} k_n = 1$ and for any $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$< T^n x - T^n y, j(x - y) >\leq k_n \|x - y\|^2, \forall n \geq 1.$$

The concept of asymptotically pseudocontractive mappings was introduced through Schu [10].
A mapping $T : C \to X$ is called Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

for all $x, y \in C$ and is called generalized Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + 1),$$

for all $x, y \in C$.
A mapping $T : C \to C$ is called uniformly $L-$ Lipschitzian if for each $n \in N$, there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$.
It is obvious that the class of generalized Lipschitzian map includes the class of Lipschitz map. Moreover, every mapping with a bounded range is a generalized Lipschitzian mapping.
Sahu [11] introduced the following new class of nonlinear mappings which is more general than the class of generalized Lipschitzian mappings and the class of uniformly L- Lipschitzian mappings.
Fix a sequence $\{r_n\}$ in $[0, \infty]$ with $r_n \to 0$.
A mapping $T : C \to C$ is called nearly Lipschitzian with respect to $\{r_n\}$ if for each $n \in N$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + r_n)$$

for all $x, y \in C$.
A nearly Lipschitzian mapping T with sequence $\{r_n\}$ is said to be nearly uniformly $L-$ Lipschitzian if $k_n = L$ for all $n \in N$.
Observe that the class of nearly uniformly $L-$ Lipschitzian mapping is more general than the class of uniformly $L-$ Lipschitzian mappings.
In recent years, many authors have given much attention to iterative methods for approximating fixed points of Lipschitz type pseudocontractive type nonlinear mappings (see, [1-5, 7- 12]).
Ofoedu [7] used the modified Mann iteration process introduced by Schu

[10] ,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \quad n \geq 0, \tag{1.1}$$

to obtain a strong convergence theorem for uniformly Lipschitzian asymptotically pseudo-contractive mapping in real Banach space setting. This result itself is a generalization of many of the previous results (see [7] and the references therein).

Recently, Chang et al. [3] proved a strong convergence theorem for a pair of L- Lipschitzian mappings instead of a single map used in [7]. In fact, they proved the following theorem :

Theorem 1.1 ([3]). Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i : K \to K$, $(i = 1, 2)$ be two uniformly L_i-Lipschitzian mappings with $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ is the set of fixed points of T_i in K and ρ be a point in $F(T_1) \cap F(T_2)$. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences in $[0, 1]$ satisfying the following conditions:
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ (iii) $\sum_{n=1}^{\infty} \beta_n < \infty$
(iv) $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

Proof. See in [3].

For any $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1^n y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T_2^n x_n. \end{aligned} \tag{1.2}$$

If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T_1^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|)$$

for all $j(x-\rho) \in J(x-\rho)$ and $x \in K$, (i = 1, 2), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to ρ.

The result above extends and improves the corresponding results of [7] from one uniformly Lipschitzian asymptotically pseudocontractive mapping to two uniformly Lipschitzian mappings. In fact, if the iteration parameter $\{\beta_n\}_{n=0}^{\infty}$ in Theorem 1.1 above is equal to zero for all n and $T_1 = T_2 = T$ then, we have the main result of Ofoedu [7].

Rafiq [9], introduced a new type of iteration- the modified three-step iteration process, to approximate the common fixed point of three nonlinear mappings in real Banach spaces. It is defined as follows:

Let $T_1, T_2, T_3 : K \to K$ be three mappings. For any given $x_1 \in K$, the modified Noor iteration $\{x_n\}_{n=1}^{\infty} \subset K$ is defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n$$

$$y_n = (1-\beta_n)x_n + \beta_n T_2 z_n$$
$$z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n, \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are three real sequences satisfying some conditions. It is clear that the iteration scheme (1.3) includes iterations defined in (1.1) and (1.2).

It is also worth mentioning that, several authors, (for example, see [8]), have recently used the iteration in equation (1.3) to approximate the common fixed points of some non-linear operators in Banach spaces.

A natural question to ask is whether the results in Ofoedu [7] and Chang et al. [3] can be extend to three nearly uniformly $L-$ Lipschitzian mappings instead of a uniformly $L-$ Lipschitzian asymptotically pseudocontractive map or two uniformly $L-$ Lipschitzian asymptotically pseudocontractive maps employed in [7] and [3]?

It is the purpose of this paper to answer this question. For this, we need the following Lemmas.

Lemma 1.1 [1, 7]. Let E be real Banach Space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) >, \forall j(x+y) \in J(x+y).$$

Lemma 1.2 [6]. Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying

$$\sum_{n=0}^{\infty} b_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n\Phi(a_{n+1}), \quad \forall n \geq N_0$$

where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

2. MAIN RESULTS

Theorem 2.1. Let C be a nonempty closed convex subset of a real Banach space X and $T_1, T_2, T_3 : C \to C$ be three nearly uniformly L_i-Lipschitzian mappings with sequences $\{r_{n_i}\}(i = 1, 2, 3)$ such that $F(T_1) \cap F(T_2) \cap F(T_3) \neq \phi$, where $F(T_i)$ $(i = 1, 2, 3)$ is the set of fixed points of T_1, T_2,T_3 in C and, ρ be a point in $F(T_1) \cap F(T_2) \cap F(T_3)$. Let $k_n \subset [1,\infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$ satisfying (i)$\frac{1}{\alpha_n}$ is bounded (ii)$\sum_{n\geq1} \alpha_n = \infty$ (iii) $\lim_{n\to\infty} \alpha_n, \beta_n, \gamma_n = 0$. For any $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined by (1.3). If there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< T_1^n x_n - \rho, j(x_n - \rho) >\leq k_n\|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|)$$

for all $j(x - \rho) \in J(x - \rho)$ and $x \in C$, (i = 1, 2, 3), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to ρ.

Proof. Since T_1, T_2 and T_3 are nearly uniformly L_i-Lipschitzian mappings with $\{r_{n_i}\}$, we have all $x, y \in C$

$$\|T_i^n x - T_i^n y\| \leq L_i(\|x - y\| + r_{n_i}), (i = 1, 2, 3).$$

For convenience, denote L= $\max\{L_i\}$ and $r_n = sup\{r_{n_i}\} : n \in N$.
And, since T_1 is a nearly uniformly L-Lipschitzian with sequence r_n, then there exists a strictly increasing continuous function $\Phi : [0,\infty) \to [0,\infty)$ $\Phi(0) = 0$ such that

$$\|T_1^n x_n - T_1^n \rho\| \leq L(\|x_n - \rho\| + r_n)$$

and

$$\langle T_1^n x_n - T_1^n \rho, j(x_n - \rho)\rangle \leq k_n\|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|), \tag{2.1}$$

for $x \in C, \rho \in F(T)$, that is

$$\langle (k_n I - T_1^n)x_n - (k_n I - T_1^n)\rho, j(x_n - \rho)\rangle \geq \Phi(\|x_n - \rho\|). \tag{$*$}$$

Step 1. We first show that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence.
For this, if $x_{n_1} = T_1 x_{n_1}$, $n \geq 1$ then it clearly holds. So, let if possible, there exists a positive integer $x_{n_1} \in C$ such that $x_{n_1} \neq T_1 x_{n_1}$, thus denote $x_{n_1} = x_1$ and $a_1 = (k_n + L)\|x_1 - \rho\|^2 + L\|x_1 - \rho\|$. Thus by (*) for any $n \geq 1$,

$$\langle k_n(x_1 - \rho) - (T_1^n x_1 - \rho), j(x_1 - \rho)\rangle \geq \Phi(\|x_1 - \rho\|), \tag{2.2}$$

that is, $(k_n + L)\|x_1 - \rho\|^2 + L\|x_1 - \rho\| \geq \Phi(\|x_1 - \rho\|)$. Thus, to simplify, we have

$$\|x_1 - \rho\| \leq \Phi^{-1}(a_1). \tag{2.3}$$

Now, we claim that $\|x_n - \rho\| \leq 2\Phi^{-1}(a_1)$, $n \geq 0$. Clearly, inview of (2.3), the claim holds for $n = 1$. We next assume that $\|x_n - \rho\| \leq 2\Phi^{-1}(a_1)$, for some n and we shall prove that $\|x_{n+1} - \rho\| \leq 2\Phi^{-1}(a_1)$. Suppose this is not true, i.e. $\|x_{n+1} - \rho\| > 2\Phi^{-1}(a_1)$.
Since $\{r_n\} \in [0,\infty]$ with $r_n \to 0$ and $\frac{1}{\alpha_n}$ a bounded sequence, set

$M = sup\{r_n : n \in N\}$ and $M_* = sup\{\frac{1}{\alpha_n} : n \in N\}$. Denote

$$\begin{array}{rcl} \tau_0 & = & min\frac{1}{3}\{1, \frac{\Phi(2(\Phi^{-1}(a_1)))}{18(\Phi^{-1}(a_1))^2}, \\ & & \frac{\Phi(2(\Phi^{-1}(a_1))}{12L[(2+3L)\Phi^{-1}(a_1)+ML+MM_*](\Phi^{-1}(a_1))^2}, \\ & & \frac{\Phi(2(\Phi^{-1}(a_1)))}{6[(2+3L)\Phi^{-1}(a_1)+ML](\Phi^{-1}(a_1))^2}, \\ & & \frac{3\Phi^{-1}(a_1)}{(2+3L)\Phi^{-1}(a_1)+ML}, \frac{3\Phi^{-1}(a_1)}{2(1+L)\Phi^{-1}(a_1)+ML}\}. \end{array} \tag{2.4}$$

Since $lim_{n\to\infty}\alpha_n, \beta_n, \gamma_n = 0$, without loss of generality, let $0 \leq \alpha_n, \beta_n, \gamma_n, k_n - 1 \leq \tau_0$ for any $n \geq 1$. Then, we have the following estimates from (1.3)

$$\begin{array}{rcl} \|z_n - \rho\| & = & \|(1-\gamma_n)x_n + \gamma_n T_3^n x_n - \rho\| \\ & \leq & \|x_n - \rho\| + \gamma_n \|T_3^n x_n - x_n\| \\ & \leq & \|x_n - \rho\| + \gamma_n[(1+L)\|x_n - \rho\| + r_n L \\ & \leq & 2\Phi^{-1}(a_1) + \tau_0[(1+L)2\Phi^{-1}(a_1) + ML] \\ & \leq & 3\Phi^{-1}(a_1). \\ \|y_n - \rho\| & = & \|(1-\beta_n)x_n + \beta_n T_2^n z_n - \rho\| \\ & \leq & \|x_n - \rho\| + \beta_n \|T_2^n z_n - x_n\| \\ & \leq & \|x_n - \rho\| + \beta_n(L(\|z_n - \rho\| + r_n) + \|x_n - \rho\|) \\ & \leq & 2\Phi^{-1}(a_1) + \beta_n[L(2\Phi^{-1}(a_1) + M) + 2\Phi^{-1}(a_1)] \end{array}$$

$$\begin{array}{rl} \leq & 2\Phi^{-1}(a_1) + \beta_n[(2+3L)\Phi^{-1}(a_1) + ML] \\ \leq & 2\Phi^{-1}(a_1) + \tau_0[(2+3L)\Phi^{-1}(a_1) + ML] \\ \leq & 3\Phi^{-1}(a_1). \end{array}$$

Also, we have the following estimates:

$$\begin{array}{lrcl} (i) & \|T_1^n y_n - x_n\| & \leq & \|x_n - \rho\| + \|T_1^n y_n - \rho\| \\ & & \leq & \|x_n - \rho\| + L(\|y_n - \rho\| + r_n) \\ & & \leq & 2\Phi^{-1}(a_1) + L(3\Phi^{-1}(a_1) + M) \\ & & \leq & (2+3L)\Phi^{-1}(a_1) + ML. \end{array}$$

$$(ii) \quad \|x_{n+1} - \rho\| \leq 3\Phi^{-1}(a_1).$$

$$(iii) \quad \|x_{n+1} - x_n\| \leq \tau_0[(2+3L)\Phi^{-1}(a_1) + ML].$$

$$(iv) \quad \|x_n - T_2^n z_n\| \leq (2+3L)\Phi^{-1}(a_1) + ML.$$

$$\begin{array}{lrcl} (v) & \|y_n - x_{n+1}\| & \leq & \|y_n - x_n\| + \|x_{n+1} - x_n\| \\ & & \leq & \beta_n \|T_2^n z_n - x_n\| + \|x_{n+1} - x_n\| \\ & & \leq & \beta_n[(2+3L)\Phi^{-1}(a_1) + ML] \\ & & & +\alpha_n[(2+3L)\Phi^{-1}(a_1) + ML] \\ & & \leq & 2\tau_0[(2+3L)\Phi^{-1}(a_1) + ML]. \end{array} \tag{2.5}$$

Using Lemma 1.1 and the above estimates, we have

$$\begin{aligned}
\|x_{n+1}-\rho\|^2 &= \|(1-\alpha_n)x_n+\alpha_n T_1^n y_n-\rho\|^2\\
&= \|x_n-\rho+\alpha_n(T_1^n y_n-x_n)\|^2\\
&\leq \|x_n-\rho\|^2-2<x_n-T_1^n y_n, j(x_{n+1}-\rho)>\\
&= \|x_n-\rho\|^2+2\alpha_n<T_1^n x_{n+1}-\rho, j(x_{n+1}-\rho)>\\
&\quad -2\alpha_n<x_{n+1}-\rho, j(x_{n+1}-\rho)>\\
&\quad +2\alpha_n<T_1^n y_n-T_1^n x_{n+1}, j(x_{n+1}-\rho)>\\
&\quad +2\alpha_n<x_{n+1}-x_n, j(x_{n+1}-\rho)>\\
&\leq \|x_n-\rho\|^2+2\alpha_n(k_n\|x_{n+1}-\rho\|^2-\Phi(\|x_{n+1}-\rho\|))\\
&\quad -2\alpha_n\|x_{n+1}-\rho\|^2+2\alpha_n\|T_1^n y_n-T_1^n x_{n+1}\|\|x_{n+1}-\rho\|\\
&\quad +2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\|\\
&= \|x_n-\rho\|^2+2\alpha_n(k_n-1)\|x_{n+1}-\rho\|^2\\
&\quad -2\alpha_n\Phi(\|x_{n+1}-\rho\|))\\
&\quad +2\alpha_n L(\|y_n-x_{n+1}\|+r_n)\|x_{n+1}-\rho\|\\
&\quad +2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\|\\
&\leq \|x_n-\rho\|^2-2\alpha_n\Phi(2(\Phi^{-1}(a_1))+2\alpha_n(k_n-1)\|x_{n+1}-\rho\|^2\\
&\quad +2\alpha_n L[2\tau_0((2+3L)\Phi^{-1}(a_1)+ML)+M]\|x_{n+1}-\rho\|\\
&\quad +2\alpha_n\tau_0[(2+3L)\Phi^{-1}(a_1)+ML]\|x_{n+1}-\rho\|\\
&\leq \|x_n-\rho\|^2-2\alpha_n\Phi(2(\Phi^{-1}(a_0))+18\alpha_n\tau_0(\Phi^{-1}(a_1))^2\\
&\quad +6\alpha_n L[2\tau_0((2+3L)\Phi^{-1}(a_1)+ML)+M)](\Phi^{-1}(a_1))^2\\
&\quad +6\alpha_n\tau_0[(2+3L)\Phi^{-1}(a_0)+ML](\Phi^{-1}(a_0))^2\\
&\leq \|x_n-\rho\|^2-2\alpha_n\Phi(2(\Phi^{-1}(a_1))+18\alpha_n\tau_0(\Phi^{-1}(a_1))^2\\
&\quad +12\alpha_n\tau_0 L[(2+3L)\Phi^{-1}(a_1)+ML+\tfrac{M}{2\alpha_n}](\Phi^{-1}(a_1))^2\\
&\quad +6\alpha_n\tau_0[(2+3L)\Phi^{-1}(a_1)+ML](\Phi^{-1}(a_1))^2
\end{aligned}$$

$$\begin{aligned}
&\leq \|x_n-\rho\|^2-2\alpha_n\Phi(2(\Phi^{-1}(a_1))+18\alpha_n\tau_0(\Phi^{-1}(a_1))^2\\
&\quad +12\alpha_n\tau_0 L[(2+3L)\Phi^{-1}(a_1)+ML+\tfrac{M}{2\alpha_n}](\Phi^{-1}(a_1))^2\\
&\quad +6\alpha_n\tau_0[(2+3L)\Phi^{-1}(a_1)+ML](\Phi^{-1}(a_1))^2\\
&\leq \|x_n-\rho\|^2-2\alpha_n\Phi(2(\Phi^{-1}(a_1))+18\alpha_n\tau_0(\Phi^{-1}(a_1))^2\\
&\quad +12\alpha_n\tau_0 L[(2+3L)\Phi^{-1}(a_1)+ML+MM_*](\Phi^{-1}(a_1))^2 \qquad (2.6)\\
&\quad +6\alpha_n\tau_0[(2+3L)\Phi^{-1}(a_1)+ML](\Phi^{-1}(a_1))^2\\
&\leq \|x_n-\rho\|^2-2\alpha_n\Phi(2(\Phi^{-1}(a_1))+\alpha_n\Phi(2(\Phi^{-1}(a_1))\\
&\leq \|x_n-\rho\|^2-\alpha_n\Phi(2(\Phi^{-1}(a_1))\\
&\leq \|x_n-\rho\|^2\\
&\leq (2(\Phi^{-1}(a_1)))^2,
\end{aligned}$$

which is a contraction. Hence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. So $\{y_n\},\{z_n\}$,
$\{T_1^n y_n\},\{T_2^n z_n\}$ are all bounded sequences.

Step 2. We want to prove $\|x_n-\rho\|\to 0$.

Since $\alpha_n,\beta_n,\gamma_n,(k_n-1)\to 0$ as $n\to\infty$ and $\{x_n\}_{n=1}^{\infty}$ is bounded. From (2. 6), we observed that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\lim_{n\to\infty}\|T_1^n y_n-T_1^n x_{n+1}\|=0, \lim_{n\to\infty}(k_n-1)=0$.

So from (2.5), we have

$$\begin{array}{rcl} \|x_{n+1}-\rho\|^2 & \leq & \|x_n-\rho\|^2 - 2 < x_n - T_1^n y_n, j(x_{n+1}-\rho) > \\ & = & \|x_n-\rho\|^2 + 2\alpha_n < T_1^n x_{n+1} - \rho, j(x_{n+1}-\rho) > \\ & & -2\alpha_n < x_{n+1}-\rho, j(x_{n+1}-\rho) > \\ & & +2\alpha_n < T_1^n y_n - T_1^n x_{n+1}, j(x_{n+1}-\rho) > \\ & & +2\alpha_n < x_{n+1} - x_n, j(x_{n+1}-\rho) > \\ & \leq & \|x_n-\rho\|^2 + 2\alpha_n(k_n-1)\|x_{n+1}-\rho\|^2 \\ & & -2\alpha_n \Phi(\|x_{n+1}-\rho\|) \\ & & +2\alpha_n \|T_1^n y_n - T_1^n x_{n+1}\| \|x_{n+1}-\rho\| \\ & & +2\alpha_n \|x_{n+1}-x_n\| \|x_{n+1}-\rho\| \\ & = & \|x_n-\rho\|^2 - 2\alpha_n \Phi(\|x_{n+1}-\rho\|) + o(\alpha_n), \end{array}$$

where

$$\begin{array}{l} 2\alpha_n(k_n-1)\|x_{n+1}-\rho\|^2 + 2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\ +2\alpha_n\|T_1^n y_n - T_1^n x_{n+1}\|\|x_{n+1}-\rho\| \\ = o(\alpha_n). \end{array} \tag{2.7}$$

By Lemma 1.2, we obtain that

$$\lim_{n\to\infty} \|x_n-\rho\| = 0.$$

This completes the proof.

Remark 2.2. In Theorem 2.1, if $\beta_n = \gamma_n = 0$, then, the conclusions are as follows.

Corollary 2.3. Let C be a nonempty closed convex subset of a real Banach space X and $T_1, T_2 : C \to C$ be two nearly uniformly L_i-Lipschitzian mappings with sequences $\{r_{n_i}\} (i = 1, 2)$ such that $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ $(i = 1, 2)$ is the set of fixed points of T_1, T_2 in C and, ρ be a point in $F(T_1) \cap F(T_2)$. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$ satisfying (i)$\frac{1}{\alpha_n}$ is bounded (ii)$\sum_{n\geq 0} \alpha_n = \infty$ (iii) $\lim_{n\to\infty} \alpha_n, \beta_n = 0$. For any $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined by (1.2). If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T_1^n x_n - \rho, j(x_n-\rho) >\leq k_n\|x_n-\rho\|^2 - \Phi(\|x_n-\rho\|)$$

for all $j(x-\rho) \in J(x-\rho)$ and $x \in C$, (i = 1, 2), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to ρ.

Corollary 2.4. Let C be a nonempty closed convex subset of a real Banach space X and $T_1 : C \to C$ be nearly uniformly L-Lipschitzian mapping with sequence $\{r_n\}$ such that $F(T_1) \neq \phi$, where $F(T_1)$ is the fixed point of T_1 in C and, ρ be a point in $F(T_1)$. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0,1]$ satisfying (i)$\frac{1}{\alpha_n}$ is bounded (ii)$\sum_{n\geq 0} \alpha_n = \infty$ (iii) $\lim_{n\to\infty} \alpha_n = 0$. For any $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined by (1.1). If there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< T_1^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|)$$

for all $j(x - \rho) \in J(x - \rho)$ and $x \in C$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to ρ.

Application 2.5. Let $X = R$, $C = [0,1]$ and $T_1 : C \to C$ be a map defined by

$$T_1 x = \frac{x}{4}.$$

Clearly, T_1 is nearly uniformly Lipschitzian ($r_n = \frac{1}{4^n}$) with $F(T_1) = 0$. Define $\Phi : [0, +\infty) \to [0, +\infty)$ by

$$\Phi(t) = \frac{t^2}{4},$$

then Φ is a strictly increasing function with $\Phi(0) = 0$. For all $x \in C, \rho \in F(T_1)$, we get

$$\begin{aligned} < T_1^n x - T_1^n \rho, j(x - \rho) > &= < \tfrac{x^n}{4^n} - 0, j(x - 0) > \\ &= < \tfrac{x^n}{4^n} - 0, x > \\ &= \tfrac{x^{n+1}}{4^n} \\ &\leq x^2 - \tfrac{x^2}{4} \\ &\leq x^2 - \Phi(x). \end{aligned}$$

Obviously, T_1 completes (2.1) with sequence $\{k_n\} = 1$. If we take $\alpha_n = \beta_n = \gamma_n = \frac{1}{n+1}$ for all $n \geq 1$. For arbitrary $x_1 \in C$, the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by (1.3) converges strongly to the unique fixed point $\rho \in T_1$.

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