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T-Rough Sets Based on the Lattices

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Abstract.The aim of this paper is to introduce and study set-valued homomorphism on lattices and T-rough lattice with respect to a sublattice. This paper deals with T-rough set approach on the lattice theory. The result of this study contributes to, T-rough fuzzy set and approximation theory and proved in several papers.
Keywords: approximation space; lattice; prime ideal; rough ideal; T-rough set; set-valued homomorphism; T-rough fuzzy ideal

1. INTRODUCTION and PRELIMINARIES

Lattice theory plays an important role in the rough set theory and fuzzy set theory. Various uncertainties in real world applications can bring difficulties in determining the crisp membership functions of fuzzy sets. They involve not only vagueness (lack of sharp class boundaries), but also ambiguity (lack of information). Hence many extensions have been developed to represent these uncertainties in membership values, such as interval-valued fuzzy sets. The notion of rough sets has been introduced by Pawlak in his papers [19],...,[26] and Pawlak and Skowron

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[27], [28], [29]. It soon invoked a natural question concerning about possible connection between rough sets and algebraic systems. The algebraic approach to rough sets have been given and studied by Bonikowaski in [6], Iwinski in [14], Rosenfeild in [30] and W. Zhang, W. Wu in [34]. Banerjee and Pal in [2],Biswas in [3, 4], Biswas and Nanda in [5], Nanda in [18], introduced the notion of rough set and rough subgroups. Kuroki in[16] introduced the notion of rough ideals in a semigroups. Davvaz [8] introduced the notion of rough subring with respect to an ideal of a ring. Dubois and Prade [9] combined fuzzy sets and rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets. Qi-Mei Xiao and Zhen-Liang Zhang in[31, 34] discussed the lower and the upper approximations of prime ideals and of fuzzy prime ideals in a semigroup with details. Davvaz in [7] introduced T-rough set and T-rough homomorphism in a group. Rough set theory is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. It is common that a partition induces an equivalence relation on a set and vice versa. The properties of rough sets can thus be examined via either partition or equivalence classes. Rough sets are a suitable mathematical model of vague concepts, i.e., concepts without sharp boundaries. Hosseini et al.[12, 13] studied some properties of T-rough set in semigroup and commutative rings. In this paper, T-rough set and ideal based on lattice is defined and some properties are given. We attempt to conduct a further study along this line. In particular, We prove some more general and fundamental properties of the generalized rough sets. We discuss the relations between the upper and lower T-rough prime ideals on lattices and the upper and lower approximations of their homomorphism images and generalize some theorems have been proved.

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief. Some of them were in [8, 19, 20]. Suppose that U is a non-empty set. A partition or classification of U is a family Θ of non-empty subsets of U such that each element of U is contained in exactly one element of Θ. Recall that an equivalence relation θ on a set U is a reflexive, symmetric and transitive binary relation on U. Each partition Θ induces an equivalence relation θ on U by setting

$$x\theta y \Leftrightarrow x \textit{ and } y \textit{ are in the same class of } \Theta.$$

Conversely, each equivalence relation θ on U induces a partition Θ of U whose classes have the form

$$[x]_\theta = \{y \in U \mid x\theta y\}.$$

Definition 1.1. A pair (U, θ) where $U \neq \emptyset$ and θ is an equivalence relation on U is called an approximation space.

Definition 1.2. For an approximation space (U, θ) by a rough approximation in (U, θ) we mean a mapping $Apr : P(U) \longrightarrow P(U) \times P(U)$ defined by for every $X \in P(U), Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where

$$\underline{Apr}(X) = \{x \in U \mid [x]_\theta \subseteq X\}, \overline{Apr}(X) = \{x \in U \mid [x]_\theta \bigcap X \neq \emptyset\}.$$

$\underline{Apr}(X)$ is called a lower rough approximation of X in (U, θ) whereas $\overline{Apr}(X)$ is called upper rough approximation of X in (U, θ).

Definition 1.3. Given an approximation space (U, θ) a pair (A, B) in $P(U) \times P(U)$ is called a rough set in (U, θ) if $(A, B) = (\underline{Apr}(X), \overline{Apr}(X))$ for some $X \in P(U)$.

Definition 1.4. A subset X of U is called definable if $\underline{Apr}(X) = \overline{Apr}(X)$. If $X \subseteq U$ is given by a predicate P and $x \in U$, then
(1)$x \in \underline{Apr}(X)$ means that x certainly has property P,
(2)$x \in \overline{Apr}(X)$ means that x possibly has property P,
(3)$x \in U \setminus \overline{Apr}(X)$ means that x definitely does not have property P.

Proposition 1.5. [14, 18] *Let U be a nonempty set and θ be an equivalence relation on U. For any subsets $A, B \subseteq U$, we have*
(i) $\underline{Apr}(A) \subseteq A \subseteq \overline{Apr}(A)$;
(ii) If $A \subseteq B$, *then* $\underline{Apr}(A) \subseteq \underline{Apr}(A)$ *and* $\overline{Apr}(A) \subseteq \overline{Apr}(B)$;
(iii) $\underline{Apr}(A \cap B) = \underline{Apr}(A) \cap \underline{Apr}(B)$;
(iv) $\overline{Apr}(A) \cup \overline{Apr}(B) = \overline{Apr}(A \cup B)$.

2. T-ROUGH PRIME IDEAL OF A LATTICE

In this section,we introduce lattices and define the concept of set-valued homomorphism on lattices. some basic properties of generalized lower and upper approximation sets in a lattice are investigated.

Definition 2.1. [11] An order $(L, \leq)$ is a lattice if ***sup***{a,b} and ***inf***{a,b} exist for all $a, b \in$ Ł.

Remark 2.2. We consider $a \vee b$ for ***sup***{a,b} and $a \wedge b$ for ***inf***{a,b}. Supremum and infimum are frequently called ***join*** and ***meet***.

Definition 2.3. [7] Let $(L, \leq)$ and $(K, \leq)$ be two lattices and $A \in P^*(K)$ where $P^*(K)$ denotes the set of all non-empty subsets of K. Let $T : L \to P^*(K)$ be a set-valued mapping. The upper inverse and the lower inverse of A under T are defined by

$$T^{-1}(A) = \{x \in L| \ T(x) \bigcap A \neq \emptyset\}; \ T^+(A) = \{x \in L|T(x) \subseteq A\}.$$

Definition 2.4. The pair $(T^+(A), T^{-1}(A))$ is referred to as the generalized rough set with respect to A, induced by T or T- rough set with respect to A.

Example 2.5. Let (L, θ) be an approximation space and $T : L \to P^*(L)$ be a set-valued mapping where $T(x) = [x]_\theta$ for all $x \in L$, then for any $A \subseteq L$, $T^+(A) = \underline{Apr}(A)$ and $T^{-1}(A) = \overline{Apr}(A)$.

Proposition 2.6. [7] *Let L and K be two lattices and $A, B \in P^*(K)$. Let $T : L \to P^*(K)$ be a set-valued mapping. Then the following points hold:*
(i)] $T^{-1}(A \bigcup B) = T^{-1}(A) \bigcup T^{-1}(B)$;
(ii) $T^+(A \bigcap B) = T^+(A) \bigcap T^+(B)$;
(iii) $A \subseteq B$ *implies* $T^+(A) \subseteq T^+(B)$ *and* $T^{-1}(A) \subseteq T^{-1}(B)$;
(iv) $T^+(A) \bigcup T^+(B) \subseteq T^+(A \bigcup B)$;
(v) $T^{-1}(A \bigcap B) \subseteq T^{-1}(A) \bigcap T^{-1}(B)$.

Definition 2.7. [11] (i) A non-empty subset K of L is a sublattice of the lattice $(L, \bigvee, \bigwedge)$ if $a \vee b, a \wedge b \in K$ for all $a, b \in K$.
(ii) Let L be a lattice and I be a nonempty subset of L. I is called an ideal of L, if $x \wedge a, a \vee b \in I$ for all $x \in L$ and $a, b \in I$.
(iii) A proper ideal P of L is called a prime ideal, if $a, b \in L$ and $a \wedge b \in P$ implies that $a \in P$ or $b \in P$.

Definition 2.8. [1] If A and B are non-empty subsets of L, we define $A \bigwedge B$ and $A \bigvee B$ as follows:

$$A \bigwedge B = \{a \wedge b \mid a \in A, b \in B\}; \ A \bigvee B = \{a \vee b \mid a \in A, b \in B\}.$$

Definition 2.9. Let L and K be two lattices and $T : L \to P^*(K)$ be a set-valued mapping. T is called a set-valued homomorphism if
(i) $T(x \wedge y) = T(x) \bigwedge T(y)$;
(ii) $T(x \vee y) = T(x) \bigvee T(y)$;
for all $x, y \in L$.

Lemma 2.10. *Let L and K be two lattices and $T : L \to P^*(K)$ be a set-valued homomorphism . If S is a sublattice of K and $T^+(S) \neq \emptyset$ and $T^{-1}(S) \neq \emptyset$, then $T^+(S)$ and $T^{-1}(S)$ are sublattices of L.*

Proof. Let $x, y \in T^{+}(S)$, by Definition 2.3 ,$T(x), T(y) \subseteq S$. Since S is a sublattice of K, we have

$$T(x \vee y) = T(x) \bigvee T(y) \subseteq S \text{ and } T(x \wedge y) = T(x) \bigwedge T(y) \subseteq S.$$

It shows that $x \vee y, x \wedge y \in T^{+}(S)$.
Moreover, let $x, y \in T^{-1}(S)$, by Definition 2.3, $T(x) \bigcap S \neq \emptyset$ and $T(y) \bigcap S \neq \emptyset$. Suppose $a \in T(x) \bigcap S$ and $b \in T(y) \bigcap S$. Since S is a sublattice of K, we have $a \vee b \in S$ and $a \vee b \in T(x) \bigvee T(y) = T(x \vee y)$. It implies that $a \vee b \in T(x \vee y) \bigcap S$. Hence $T(x \vee y) \bigcap S \neq \emptyset$. It means that $x \vee y \in T^{-1}(S)$. Again, $a \wedge b \in S$ and $a \wedge b \in T(x) \bigwedge T(y)$. So that $a \wedge b \in T(x \wedge y) \bigcap S$. Therefore $T(x \wedge y) \bigcap S \neq \emptyset$. It means that $x \wedge y \in T^{-1}(S)$.

Corollary 2.11. *Let L and K be two lattices and $T : L \to P^{*}(K)$ be a set-valued homomorphism. If S is a sublattice of K and $T^{+}(S) \neq \emptyset \neq T^{-1}(S)$, then $(T^{+}(S), T^{-1}(S))$ is a T-rough sublattice of L.*

Proposition 2.12. *Let L and K be two lattices and $T : L \to P^{*}(K)$ be a set-valued homomorphism. If A, B be non-empty subsets of K , then*
(1) $T^{+}(A) \bigvee T^{+}(B) \subseteq T^{+}(A \bigvee B)$;
(2) $T^{+}(A) \bigwedge T^{+}(B) \subseteq T^{+}(A \bigwedge B)$.

Proof. (1). Suppose z be any element of $T^{+}(A) \bigvee T^{+}(B)$. Then $z = a \vee b$ for some $a \in T^{+}(A)$ and $b \in T^{+}(B)$. By definition, $T(a) \subseteq A$ and $T(b) \subseteq B$. Since

$$\begin{aligned} T(a \vee b) = T(a) \bigvee T(b) &= \{x \vee y \mid x \in T(a), y \in T(b)\} \\ &\subseteq \{x \vee y \mid x \in A, y \in B\} = A \bigvee B, \end{aligned}$$

we imply that $a \vee b \in T^{+}(A \bigvee B)$ and so $z \in T^{+}(A \bigvee B)$.
(2). The proof is similar to the proof of (1).

The following examples show that the converse of above proposition is not true.

Example 2.13. (1). Let $L = \{0, 1, 2, ..., 8\}$. Let $a \vee b = max\{a, b\}$ and $a \wedge b = min\{a, b\}$ for all $a, b \in L$. Then $(L, \vee, \wedge)$ is a lattice. If we consider equivalence classes $[0] = \{0, 1, 2\}$, $[3] = \{3, 4\}$, $[5] = \{5, 6, 7, 8\}$ and $T : L \to P^{*}(L)$ be a set-valued homomorphism with $T(x) = [x]$ for all $x \in L$. Let $A = \{3, 4, 5, 7\}$, $B = \{0, 1, 2, 3, 6, 8\}$. Then $A \bigvee B = \{3, 4, 5, 6, 7, 8\}$, $T^{+}(A \bigvee B) = \{3, 4, 5, 6, 7, 8\}$, $T^{+}(A) = \{3, 4\}$, $T^{+}(B) = \{0, 1, 2\}$ and $T^{+}(A) \bigvee T^{+}(B) = \{3, 4\}$. And so $T^{+}(A \bigvee B) \nsubseteq T^{+}(A) \bigvee T^{+}(B)$.

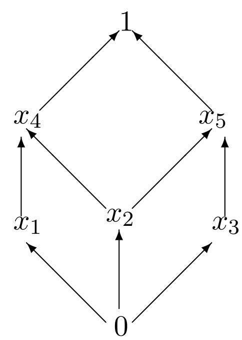
(2). Let $L = [0, 1]$ and $T : L \to P^{*}(L)$ be a set-valued homomorphism with $T(x) = [0, x]$ for all $x \in L$. And let $A = \{0, \frac{1}{2}\}$, $B = \{\frac{1}{3}, \frac{1}{2}\}$. Then $T^{+}(A) = \{0\}$, $T^{+}(B) = \emptyset$, $T^{+}(A \bigwedge B) = \{0\}$, $T^{+}(A) \bigwedge T^{+}(B) = \emptyset$. Therefore $T^{+}(A \bigwedge B) \nsubseteq T^{+}(A) \bigwedge T^{+}(B)$.

Proposition 2.14. *Let L and K be two lattices and $T : L \to P^{*}(K)$ be a set-valued homomorphism. If A, B be non-empty subsets of K, then*
(1) $T^{-1}(A) \bigvee T^{-1}(B) \subseteq T^{-1}(A \bigvee B)$;
(2) $T^{-1}(A) \bigwedge T^{-1}(B) \subseteq T^{-1}(A \bigwedge B)$.

Proof. (1). Let $z \in T^{-1}(A) \bigvee T^{-1}(B)$. Then $z = a \vee b$ for some $a \in T^{-1}(A)$ and $b \in T^{-1}(B)$. Hence $T(a) \bigcap A \neq \emptyset$ and $T(b) \bigcap B \neq \emptyset$ and so there exist $x \in T(a) \bigcap A$ and $y \in T(b) \bigcap B$. Therefore $x \vee y \in A \bigvee B$ and $x \vee y \in T(a) \bigvee T(b) = T(a \vee b)$. Thus $x \vee y \in T(a \vee b) \bigcap (A \bigvee B)$ which implies that $T(a \vee b) \bigcap (A \bigvee B) \neq \emptyset$. So $z = a \vee b \in T^{-1}(A \bigvee B)$.
(2). The proof is similar to the proof of (1).

The following examples show that the converse of above noted proposition is not true.

Example 2.15. (i) Let $L = \{0, x_1, x_2, x_3, x_4, x_5, 1\}$ be the following lattice and $T : L \to P^{*}(L)$ be a set-valued homomorphism with $T(x) = \{x_5\}$ for all $x \in L$. And let $A = \{x_2, x_3\}$, $B = \{x_1, x_2\}$. Then $T^{-1}(A) = \emptyset$, $T^{-1}(B) = \emptyset$, $T^{-1}(A \bigvee B) = L$, $T^{-1}(A) \bigvee T^{-1}(B) = \emptyset$. Therefore
$T^{-1}(A \bigvee B) \nsubseteq T^{-1}(A) \bigvee T^{-1}(B)$.



(ii) Let L be the above noted lattice and $T : L \to P^{*}(L)$ be a set-valued homomorphism with $T(x) = \{x_2\}$ for all $x \in L$. And let $A = \{x_4\}$, $B = \{x_5\}$. Then $T^{-1}(A) = \emptyset$, $T^{-1}(B) = \emptyset$, $T^{-1}(A \bigwedge B) = L$, $T^{-1}(A) \bigwedge T^{-1}(B) = \emptyset$. Therefore
$T^{-1}(A \bigwedge B) \nsubseteq T^{-1}(A) \bigwedge T^{-1}(B)$.

Lemma 2.16. *Let L and K be two lattices and $T : L \to P^{*}(K)$ be a set-valued homomorphism. If I is an ideal of K and $T^{+}(I) \neq \emptyset$, then $T^{+}(I)$ is an ideal of L.*

Proof. Suppose $r \in L$ and $x, y \in T^{+}(I)$, by Definition 2.3 provided, $T(x), T(y) \subseteq I$. Since I is an ideal of K, so we have $T(x \vee y) =$

$T(x)\bigvee T(y)\subseteq I$. Therefore $x\vee y\in T^{+}(I)$ and $T(r\wedge x)=T(r)\bigwedge T(x)\subseteq I$. Thus $r\wedge x\in T^{+}(I)$. Hence $T^{+}(I)$ is an ideal of L.

Lemma 2.17. *Let L and K be two lattices and $T:L\to P^{*}(K)$ be a set-valued homomorphism. If I is an ideal of K and $T^{-1}(I)\neq\emptyset$, then $T^{-1}(I)$ is an ideal of L.*

Proof. Let $x,y\in T^{-1}(I)$. By Definition 2.3 provided, $T(x)\bigcap I\neq\emptyset$ and $T(y)\bigcap I\neq\emptyset$. Let $a\in T(x)\bigcap I$ and $b\in T(y)\bigcap I$. Since I is an ideal of K, therefore $a\vee b\in I$ and $a\vee b\in T(x)\bigvee T(y)=T(x\vee y)$. Thus $a\vee b\in T(x\vee y)\bigcap I$. Hence $T(x\vee y)\bigcap I\neq\emptyset$ and so $x\vee y\in T^{-1}(I)$.
Now let $r\in L$ and $x\in T^{-1}(I)$. By Definition 2.3, we have $T(x)\bigcap I\neq\emptyset$. Let $a\in T(x)\bigcap I$. Since I is an ideal of K, therefore $T(r)\bigwedge a\subseteq I$ and

$$T(r)\bigwedge a\subseteq T(r)\bigwedge T(x)=T(r\wedge x).$$

Thus $T(r\wedge x)\bigcap I\neq\emptyset$ and so $r\wedge x\in T^{-1}(I)$. Therefore $T^{-1}(I)$ is an ideal of L.

Corollary 2.18. *Let L and K be two lattices and $T:L\to P^{*}(K)$ be a set-valued homomorphism. If I is an ideal of K and $T^{+}(I)\neq\emptyset$ and $T^{-1}(I)\neq\emptyset$, then $(T^{+}(I),T^{-1}(I))$ is a T-rough ideal of L.*

Theorem 2.19. *Let L and K be two lattices and $T:L\to P^{*}(K)$ be a set-valued homomorphism. If P is a prime ideal of K and $L\neq T^{+}(P)\neq\emptyset$, then $T^{+}(P)$ is a prime ideal of L.*

Proof. Let $x,y\in L$ and $x\wedge y\in T^{+}(P)$. By Definition 2.3, we have $T(x)\bigwedge T(y)=T(x\wedge y)\subseteq P$. And so, for any $a\in T(x)$ and $b\in T(y)$, $a\wedge b\in P$. Now we show that $T(x)\subseteq P$ or $T(y)\subseteq P$. Suppose $T(x)\nsubseteq P$ and $T(y)\nsubseteq P$. Then there is $a\in T(x)$ such that $a\notin P$ and there is $b\in T(y)$ such that $b\notin P$. Since P is a prime ideal of K, we deduce that $a\wedge b\notin P$ as which is a contradiction. Hence $T(x)\subseteq P$ or $T(y)\subseteq P$. It means that $x\in T^{+}(P)$ or $y\in T^{+}(P)$.

Theorem 2.20. *Let L and K be two lattices and $T:L\to P^{*}(K)$ be a set-valued homomorphism. If P is a prime ideal of K and $L\neq T^{-1}(P)\neq\emptyset$, then $T^{-1}(P)$ is a prime ideal of L .*

Proof. Let $x,y\in L$ and $x\wedge y\in T^{-1}(P)$, then $T(x\wedge y)\bigcap P\neq\emptyset$. On the other hand, $\{a\wedge b\mid a\in T(x),b\in T(y)\}=T(x\wedge y)$. So there are $a\in T(x),b\in T(y)$ such that $a\wedge b\in P$. Since P is a prime ideal of K, we have $a\in P$ or $b\in P$. Hence $a\in T(x)\bigcap P$ or $b\in T(y)\bigcap P$. It means that $T(x)\bigcap P\neq\emptyset$ or $T(y)\bigcap P\neq\emptyset$. Hence $x\in T^{-1}(P)$ or $y\in T^{-1}(P)$. Therefore $T^{-1}(P)$ is a prime ideal of L.

Corollary 2.21. *Let L and K be two lattices and $T:L\to P^{*}(K)$ be a set-valued homomorphism. If P is a prime ideal of K and $L\neq T^{+}(P)\neq$*

$\emptyset$ and $L \neq T^{-1}(P) \neq \emptyset$, Then $(T^{+}(P), T^{-1}(P))$ is a T-rough prime ideal of L.

3. T-ROUGH QUOTIENT IDEAL IN LATTICES

In this section, we define T-rough quotient sets with respect to a set-valued homomorphism and investigate some their properties. Let L and K be two lattices and $T : L \to P^{*}(K)$ be a set-valued homomorphism . Let us set $\frac{L}{T} = \{T(x) \mid x \in L\}$. It is clear that $\frac{L}{T}$ is a lattice.

Definition 3.1. Let L and K be two lattices and $T : L \to P^{*}(K)$ be a set-valued homomorphism. The lower T-rough quotient and the upper T-rough quotient set with respect to $A \in P^{*}(K)$ are

$$\frac{T^{+}(A)}{T} = \{T(x) \mid T(x) \subseteq A\}; \ \frac{T^{-1}(A)}{T} = \{T(x) \mid T(x) \cap A \neq \emptyset\}.$$

Lemma 3.2. *Let L and K be two lattices and $T : L \to P^{*}(K)$ be a set-valued homomorphism. If $A \in P^{*}(K)$ be an ideal of K and $\frac{T^{+}(A)}{T} \neq \emptyset$, then $\frac{T^{+}(A)}{T}$ is an ideal of $\frac{L}{T}$.*

Proof. Let $T(x), T(y) \in \frac{T^{+}(A)}{T}$. Since $A \in P^{*}(K)$ is an ideal of K, so $T(x \vee y) = T(x) \bigvee T(y) \subseteq A$. Therefore $T(x) \bigvee T(y) \in \frac{T^{+}(A)}{T}$. Now suppose $T(r) \in \frac{L}{T}$ and $T(x) \in \frac{T^{+}(A)}{T}$, By Definition 3.1, $T(x) \subseteq A$. Since A is an ideal of K, thus $T(x \wedge r) = T(x) \bigwedge T(r) \subseteq A$. It means that $T(x) \bigwedge T(r) \in \frac{T^{+}(A)}{T}$. Therefore $\frac{T^{+}(A)}{T}$ is an ideal of $\frac{L}{T}$.

Lemma 3.3. *Let L and K be two lattices and $T : L \to P^{*}(K)$ be a set-valued homomorphism. If $A \in P^{*}(K)$ be an ideal of K and $\frac{T^{-1}(A)}{T} \neq \emptyset$, then $\frac{T^{-1}(A)}{T}$ is an ideal of $\frac{L}{T}$.*

Proof. Suppose $T(x), T(y) \in \frac{T^{-1}(A)}{T}$. By Definition 3.1, we have $T(x) \bigcap A \neq \emptyset$ and $T(y) \bigcap A \neq \emptyset$. Let $a \in T(x) \bigcap A$ and $b \in T(y) \bigcap A$. Since $A \in P^{*}(K)$ is an ideal of K, so $a \vee b \in T(x) \bigvee T(y) = T(x \vee y)$ and $a \vee b \in A$. Therefore $a \vee b \in T(x \vee y) \bigcap A$. It means that $T(x) \bigvee T(y) \in \frac{T^{-1}(A)}{T}$. Now suppose $T(r) \in \frac{L}{T}$ and $T(x) \in \frac{T^{-1}(A)}{T}$, By Definition 3.1 provided, $T(x) \bigcap A \neq \emptyset$. Let $a \in T(x) \bigcap A$. Since A is an ideal of K, $T(r) \bigwedge a \subset A$ and

$$T(r) \bigwedge a \subseteq T(r) \bigwedge T(x) = T(x \wedge r).$$

Therefore $T(x \wedge r) \bigcap A \neq \emptyset$. It meas that $T(x) \bigwedge T(r) \in \frac{T^{-1}(A)}{T}$.

Corollary 3.4. *Let L and K be two lattices and $T : L \to P^{*}(K)$ be a set-valued homomorphism. If $A \in P^{*}(K)$ is an ideal of K and $\frac{T^{+}(A)}{T} \neq \emptyset$ and $\frac{T^{-1}(A)}{T} \neq \emptyset$, Then $(\frac{T^{+}(A)}{T}, \frac{T^{-1}(A)}{T})$ is a T-rough quotient ideal of $\frac{L}{T}$.*

Proposition 3.5. *Let L and K be two lattices and $T : L \to P^*(K)$ be a set-valued homomorphism. If $P \in P^*(K)$ be a T-lower rough prime ideal of K and $\frac{L}{T} \neq \frac{T^+(P)}{T} \neq \emptyset$, Then $\frac{T^+(P)}{T}$ is a lower T- rough quotient prime ideal of $\frac{L}{T}$.*

Proof. By Lemma 3.2 provided, we have $\frac{T^+(P)}{T}$ is an ideal of $\frac{L}{T}$. Therefore we assume that $T(x), T(y) \in \frac{L}{T}$ such that $T(x) \bigwedge T(y) \in \frac{T^+(P)}{T}$. Thus $T(x \wedge y) \in \frac{T^+(P)}{T}$. Hence $T(x \wedge y) = T(x) \bigwedge T(y) \subseteq P$. It follows that $x \wedge y \in T^+(P)$. Since P is a lower T- rough prime ideal of K,$x \in T^+(P)$ or $y \in T^+(P)$. It means that $T(x) \subseteq P$ or $T(y) \subseteq P$. Thus $T(x) \in \frac{T^+(P)}{T}$ or $T(y) \in \frac{T^+(P)}{T}$. Hence $\frac{T^+(P)}{T}$ is a lower T- rough quotient prime ideal of $\frac{L}{T}$.

Proposition 3.6. *Let L and K be two lattices and $T : L \to P^*(K)$ be a set-valued homomorphism. If $P \subseteq K$ be an upper T-rough prime ideal of K and $\frac{L}{T} \neq \frac{T^{-1}(P)}{T} \neq \emptyset$, Then $\frac{T^{-1}(P)}{T}$ is an upper T-rough quotient prime ideal of $\frac{L}{T}$.*

Proof. By Lemma 3.3 provided, we have $\frac{T^{-1}(P)}{T}$ is an ideal of $\frac{L}{T}$. Then we assume that $T(x), T(y) \in \frac{L}{T}$ such that $T(x \wedge y) = T(x) \bigwedge T(y) \in \frac{T^{-1}(P)}{T}$. Hence $T(x \wedge y) \bigcap P \neq \emptyset$. It shows that $x \wedge y \in T^{-1}(P)$. Since P is an upper T-rough prime ideal of K, thus we have $x \in T^{-1}(P)$ or $y \in T^{-1}(P)$. It means that $T(x) \bigcap P \neq \emptyset$ or $T(y) \bigcap P \neq \emptyset$. Thus $T(x) \in \frac{T^{-1}(P)}{T}$ or $T(y) \in \frac{T^{-1}(P)}{T}$. Therefore $\frac{T^{-1}(P)}{T}$ is an upper T-rough quotient prime ideal of $\frac{L}{T}$.

Corollary 3.7. *Let L and K be two lattices and $T : L \to P^*(K)$ be a set-valued homomorphism. If $(T^+(P), T^{-1}(P))$ be a T-rough prime ideal of K and $\frac{L}{T} \neq \frac{T^+(P)}{T} \neq \emptyset$ and $\frac{L}{T} \neq \frac{T^{-1}(P)}{T} \neq \emptyset$, Then $\frac{T^+(P)}{T}$ and $\frac{T^{-1}(P)}{T}$ are the lower T- rough and the upper T- rough quotient prime ideal of $\frac{L}{T}$,respectively.*

4. SET-VALUED HOMOMORPHISM INDUCED BY A HOMOMORPHISM

In this section, by using a homomorphism on a lattice and a set-valued homomorphism, we define a set-valued homomorphism and check out some relations between of them.

Definition 4.1. [11] Let L and K be two lattices. A mapping $f : L \to K$ is called a homomorphism if
(i) $f(x \wedge y) = f(x) \bigwedge f(y)$;
(ii) $f(x \vee y) = f(x) \bigvee f(y)$; for all $x, y \in L$.

A surjection(onto) homomorphism is called an epimorphism and an injection(one to one) homomorphism is a monomorphism. An isomorphism is a bijection homomorphism.

Lemma 4.2. *Let L and K be two lattices and $f : L \to K$ be an isomorphism and let $T_2 : K \to P^*(K)$ be a set-valued homomorphism. Then $T_1(x) = \{u \in L \mid f(u) \in T_2(f(x))\}$ is a set-valued homomorphism from L to $P^*(L)$.*

Proof. First, we show that T_1 is a well-defined mapping. Suppose $x_1, x_2 \in L$, and $x_1 = x_2$, then

$$\begin{aligned} y_1 \in T_1(x_1) &\Leftrightarrow f(y_1) \in T_2(f(x_1)) = T_2(f(x_2)) \\ &\Leftrightarrow y_1 \in T_1(x_2). \end{aligned}$$

So, $T_1(x_1) = T_1(x_2)$. Now we show that $T_1(x \vee y) = T_1(x) \bigvee T_1(y)$. From the definition of T_1, we have

$$\begin{aligned} T_1(x \vee y) &= \{u \in L \mid f(u) \in T_2(f(x \vee y))\} \\ &= \{u \in L \mid f(u) \in T_2(f(x) \vee f(y))\}. \end{aligned}$$

Suppose $f(u) \in T_2(f(x) \vee f(y)) = \{c \vee d \mid c \in T_2(f(x)), d \in T_2(f(y)\}$. Since f is onto, then there are $s, t \in L$ such that $f(s) = c, f(t) = d$. It follows that

$$\{f(s) \vee f(t) \mid f(s) \in T_2(f(x)), f(t) \in T_2(f(y))\} = \{f(s \vee t) \mid s \in T_1(x), t \in T_1(y)\}.$$

So, $f(u) = f(w \vee z)$ for some $w \in T_1(x), z \in T_1(y)$. Since f is one to one, we deduce that $u = w \vee z$. Hence $T_1(x \vee y) = T_1(x) \bigvee T_1(y)$.
Also to similar reason, we have $T_1(x \wedge y) = T_1(x) \bigwedge T_1(y)$. Hence T_1 is a set-valued homomorphism from L to $P^*(L)$.

Theorem 4.3. *Let L and K be two lattices and $f : L \to K$ be an isomorphism and let $T_2 : K \to P^*(K)$ be a set-valued homomorphism. If $T_1(x) = \{u \in L \mid f(u) \in T_2(f(x))\}$ and A is a nonempty subset of K, then*
(1) $f(T_1^+(A)) = T_2^+(f(A))$;

(2) $f(T_1^{-1}(A)) = T_2^{-1}(f(A))$.

Proof. (1). If $y \in f(T_1^+(A))$, then there exists $x \in T_1^+(A)$ such that $y = f(x)$. But if $x \in T_1^+(A)$, then $T_1(x) \subseteq A$. Now, if $w \in T_2(f(x))$, since f is onto, then there exists $z \in L$ such that $w = f(z)$.

So,

$$\begin{aligned} f(z) \in T_2(f(x)) &\Rightarrow z \in T_1(x) \Rightarrow z \in A, \\ &\Rightarrow f(z) \in f(A) \Rightarrow w \in f(A), \\ &\Rightarrow T_2(f(x)) \subseteq f(A) \Rightarrow f(x) \in T_2^+(f(A)), \\ &\Longrightarrow y \in T_2^+(f(A)). \end{aligned}$$

Therefore $f(T_1^+(A)) \subseteq T_2^+(f(A))$.

Conversely, if $y \in T_2^+(f(A))$, then $T_2(y) \subseteq f(A)$. On the other hand, f is onto, then there is $x \in L$ such that $y = f(x)$. Hence, we have $T_2(f(x)) \subseteq f(A)$.

Let $u \in T_1(x)$, then $f(u) \in f(A)$, therefore there exists $a \in A$ such that $f(u) = f(a)$. But f is one to one, so $u = a$. Then $T_1(x) \subset A$. Therefore, $x \in T_1^+(A)$. It implies that$y \in f(T_1^+(A))$.

So, $T_2^+(f(A)) \subseteq f(T_1^+(A))$.

(2). If $y \in f(T_1^{-1}(A))$, then there exists $x \in T_1^{-1}(A)$ such that $y = f(x)$. But if $x \in T_1^{-1}(A)$, then $T_1(x) \cap A \neq \varnothing$. Let $a \in T_1(x) \cap A$, therefore

$$\begin{aligned} f(a) \in T_2(f(x)) \cap f(A) &\Rightarrow T_2(f(x)) \cap f(A) \neq \varnothing, \\ &\Rightarrow f(x) \in T_2^{-1}(f(A)), \\ &\Rightarrow y \in T_2^{-1}(f(A)). \end{aligned}$$

It means that $f(T_1^{-1}(A)) \subseteq T_2^{-1}(f(A))$.

Conversely, if $y \in T_2^{-1}(f(A))$, since f is onto , then there exists $x \in L$ such that $y = f(x)$, and $T_2(y) \cap f(A) \neq \varnothing$. So, we have $T_2(f(x)) \cap f(A) \neq \varnothing$. Hence there is $z \in T_2(f(x)) \cap f(A)$. It means that there exists $a \in A$ such that $z = f(a) \in T_2(f(x))$. Then $a \in T_1(x) \cap A \neq \varnothing$. It obtains that $x \in T_1^{-1}(A)$. Then $y = f(x) \in f(T_1^{-1}(A))$. It follows that $T_2^{-1}(f(A)) \subseteq f(T_1^{-1}(A))$.

5. T-ROUGH FUZZY PRIME IDEAL WITH RESPECT TO A FUZZY PRIME IDEAL

Theory of fuzzy sets initiated by Zadeh [33]. As a natural need, Dubois and Prade [9, 10] combined fuzzy sets and rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets. Rough fuzzy sets and fuzzy rough sets are also studied by Nakamura [17], Nanda [18], Biswas [3, 4] and by Banerjee and Pal [2]. Several research directions have been suggested on fuzzy rough sets and rough fuzzy sets. In this section, we introduce the T- rough fuzzy ideal and T-rough fuzzy prime ideal in a lattice and give some properties of such ideals and then extended some theorems in which have been proved in [7, 8, 12, 30].

Definition 5.1. Let (L,θ) be an approximation space. A subset fuzzy is a mapping μ from L to $[0,1]$. For every $x\in L$, we define ,

$$\underline{Apr}(\mu)(x)=\bigwedge_{a\in[x]_\theta}\mu(a);\ \overline{Apr}(\mu)(x)=\bigvee_{a\in[x]_\theta}\mu(a).$$

They are called , respectively, the lower and upper approximation of the fuzzy subset μ. $Apr(\mu)=(\underline{Apr}(\mu),\overline{Apr}(\mu))$ is called a rough fuzzy set with respect to θ if $\underline{Apr}(\mu)\neq\overline{Apr}(\mu))$. Let μ be a fuzzy subset of L, $\lambda\in[0,1]$. Then the sets

$$\mu_\lambda=\{x\in L\mid \mu(x)\geq\lambda\};\ \mu_\lambda^s=\{x\in L\mid \mu(x)>\lambda\}$$

are called, respectively, λ-levelest and λ-strong levelest of the fuzzy set μ.

Definition 5.2. [28] A fuzzy subset μ of a lattice L is called a fuzzy ideal if
(i) $\mu(x\vee y)\geq\mu(x)\wedge\mu(y)$ for all $x,y\in L$;
(ii) $\mu(x\wedge y)\geq\mu(x)\wedge\mu(y)$ for all $x,y\in L$.
It is a fuzzy prime ideal if $\mu(x\wedge y)=\mu(x)$ or $\mu(x\wedge y)=\mu(y)$ for all $x,y\in L$.

Definition 5.3. Let L and K be two lattices and $T:L\to P^*(K)$ be a set-valued homomorphism. Let μ be a fuzzy ideal of K . For every $x\in L$, we define

$$T^+(\mu)(x)=\bigwedge_{a\in T(x)}\mu(a)\ ;\ T^{-1}(\mu)(x)=\bigvee_{a\in T(x)}\mu(a).$$

$T^+(\mu)$ and $T^{-1}(\mu)$ are called , respectively, the lower T-rough and the upper T-rough fuzzy subsets of L. $(T^+(\mu),T^{-1}(\mu))$ is said to be T-rough fuzzy set of L. If $T^+(\mu)$ and $T^{-1}(\mu)$ are fuzzy prime ideals, $(T^+(\mu),T^{-1}(\mu))$ is said to be T-rough fuzzy prime ideal of L.

The following theorem and lemma have been proved in [7, 13]:

Theorem 5.4. *Let μ be a fuzzy subset of a lattice L. Then μ is a fuzzy ideal(fuzzy prime ideal) of L iff $\mu_\lambda,\mu_\lambda^s$ are, if they are nonempty, ideals [prime ideals] of L for every $\lambda\in[0,1]$.*

Lemma 5.5. *Let L and K be two lattices and $T:L\to P^*(K)$ be a set-valued homomorphism. If μ is a fuzzy ideal of K, then for all $\lambda\in[0,1]$,*
(1) $(T^+(\mu))_\lambda=T^+(\mu_\lambda)$;
(2) $(T^{-1}(\mu))_\lambda=T^{-1}(\mu_\lambda)$;
(3) $(T^+(\mu))_\lambda^s=T^+(\mu_\lambda^s)$;
(4) $(T^{-1}(\mu))_\lambda^s=T^{-1}(\mu_\lambda^s)$.

The following theorems are straightforward.

Theorem 5.6. *Let L and K be two lattices and $T : L \to P^*(K)$ be a set-valued homomorphism. If μ is a fuzzy ideal of K, then $T^+(\mu)$ is a fuzzy ideal of L.*

Theorem 5.7. *Let L and K be two lattices and $T : L \to P^*(K)$ be a set-valued homomorphism. If μ is a fuzzy ideal of K, then $T^{-1}(\mu)$ is an fuzzy ideal of L.*

If θ is a complete congruence relation(that is, equivalence relation and $[x]_\theta \bigvee [y]_\theta = [x \vee y]_\theta$ and $[x]_\theta \bigwedge [y]_\theta = [x \wedge y]_\theta$ for $x, y \in L$) on L and define $T : L \to P^*(L)$ where $T(x) = [x]_\theta$ for every $x \in L$, we generalized theorems proved in [8].

Theorem 5.8. *If μ is a fuzzy prime ideal of K and $T : L \to P^*(K)$ be a set-valued homomorphism, then $T^+(\mu), T^{-1}(\mu)$ are fuzzy prime ideals of L.*

Definition 5.9. Let f be a mapping from a set X to a set Y. Let μ be a fuzzy set of X and λ be a fuzzy set of Y. Then the inverse image $f^{-1}(\lambda)$ of λ defined by

$$f^{-1}(\lambda)(x) = \lambda(f(x)) \ for \ all \ x \in X.$$

The image $f(\mu)$ of μ is the fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \{\mu(x)\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

for all $y \in Y$.

Theorem 5.10. *Let L and K be two lattices and $f : L \to K$ be an isomorphism and $T_2 : K \to P^*(K)$ be a set-valued homomorphism. Let μ be a fuzzy subset of L. If $T_1(x) = \{u \in L \mid f(u) \in T_2(f(x))\}$, then*
(1) $T_1^+(\mu)$ is a fuzzy(prime) ideal of L if and only if $T_2^+(f(\mu))$ is a fuzzy (prime)ideal of K;
(2) $T_1^{-1}(\mu)$ is a fuzzy (prime)ideal of L if and only if $T_2^{-1}(f(\mu))$ is a fuzzy (prime)ideal of K.

6. CONCLUSION

The rough sets theory is regarded as a generalization of the classical sets theory. A key notion in rough set is an equivalence relation. An equivalence is sometime difficult to be obtained in rearward problems due to vagueness and incompleteness of human knowledge. In the present paper, we substituted a universe set by a lattice and introduced the set- valued homomorphism, T- rough ideals and T-rough fuzzy ideals in a lattice based on definitions in [7]. We discussed the relations between the upper(lower)T- rough ideals and the upper (lower) approximations of their homomorphism images. We generalized some ideas are presented by Davvaz [7, 8]. Further, we studied and investigated some of their interesting properties of a set- valued homomorphism induced by a lattice homomorphism.

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