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# LI-YORKE CHAOTIC GENERALIZED SHIFT DYNAMICAL SYSTEMS

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Abstract. In this text we prove that in generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$ for finite discrete $X$ with at least two elements, infinite countable set $\Gamma$ and arbitrary map $\varphi : \Gamma \to \Gamma$, the following statements are equivalent:

- the dynamical system $(X^\Gamma, \sigma_\varphi)$ is Li-Yorke chaotic;
- the dynamical system $(X^\Gamma, \sigma_\varphi)$ has an scrambled pair;
- the map $\varphi : \Gamma \to \Gamma$ has at least one non-quasi-periodic point.

Keywords: Generalized shift, Li-Yorke chaos, Scrambled pair.

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## 1. Introduction

In this text we deal with "topological dynamical systems", "generalized shifts" and "Li-Yorke chaos". Our main aim is to find out the common property under which a generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$ for finite discrete space $X$ with at least two elements, infinite countable set $\Gamma$, and arbitrary map $\varphi : \Gamma \to \Gamma$ is Li-Yorke chaotic.

We call $a \in A$ a quasi-periodic point of $f : A \to A$ if there exist $n > m \geq 1$ such that $f^n(x) = f^m(x)$. If $a \in A$ is not a quasi-periodic point of $f : A \to A$, we call $a$ a non-quasi-periodic point of $f$.

By $\mathbb{Z}$ we mean the set of all integers $\{0, \pm 1, \pm 2, \ldots\}$, and by $\mathbb{N}$ we mean

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the set of positive integers $\{1, 2, 3, \dots\}$.
By a *(topological) dynamical system* $(Z, h)$ we simply mean a topological space $Z$ (*phase space*) and a continuous map $h : Z \to Z$. For nonempty arbitrary sets $\Gamma$, $X$ and map $\varphi : \Gamma \to \Gamma$, we call $\sigma_\varphi : X^\Gamma \to X^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ for $(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$, a generalized shift [3]. Whenevere $\Gamma = \mathbb{N}\cup\{0\}$ and $\varphi(n) = n+1$ $(n \in \Gamma)$, $\sigma_\varphi : X^{\mathbb{N}\cup\{0\}} \to X^{\mathbb{N}\cup\{0\}}$ is the familair one sided shift, also whenevere $\Gamma = \mathbb{Z}$ and $\varphi(n) = n + 1$ $(n \in \Gamma)$, $\sigma_\varphi : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ is the well-known two sided shift. For instance, if $X$ is a group, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is a homomorphism, also if $X$ is a topological space and $X^\Gamma$ is considered under product (pointwise convergence) topology, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is continuous, so we may study dynamical properties of generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$. In several texts dynamical and non-dynamical properties of generalized shifts has been studied, for example algebraic entropy has been studied in [1] and [7], topological entropy in [2], and Devaney chaos in [4].
Regarding [4] in the generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$ with $X$ is a discrete topological space with at least two elements, $\Gamma$ is an infinite countable set, $\varphi : \Gamma \to \Gamma$ is an arbitrary map and consider $X^\Gamma$ with product (pointwise convergence) topology, the following statements are equivalent:

- the map $\varphi : \Gamma \to \Gamma$ is one to one without periodic points;
- the dynamical system $(X^\Gamma, \sigma_\varphi)$ is topological transitive;
- the dynamical system $(X^\Gamma, \sigma_\varphi)$ is Devaney chaotic.

Also the authors establish in [5] that in the generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$ with $X$ is a finite discrete topological space with at least two elements, $\Gamma$ is an infinite countable set, $\varphi : \Gamma \to \Gamma$ is an arbitrary map and consider $X^\Gamma$ with product (pointwise convergence) topology, the following statements are equivalent:

- the dynamical system $(X^\Gamma, \sigma_\varphi)$ has uncountable scambled pairs;
- the dynamical system $(X^\Gamma, \sigma_\varphi)$ has an scrambled pair;
- the map $\varphi : \Gamma \to \Gamma$ has at least one non-quasi-periodic point.

In [5] we just proved that if $(X^\Gamma, \sigma_\varphi)$ has an scrambled pair, then it has uncountable scrambled pairs and we didn't prove that under the same condition, $(X^\Gamma, \sigma_\varphi)$ has an uncoutable scrambled set (i.e., $(X^\Gamma, \sigma_\varphi)$ is Li-Yorke chaotic). In the present paper we bring a sronger result than [5], i.e. we prove that the following statements are equivalent:

- the dynamical system $(X^\Gamma, \sigma_\varphi)$ is Li-Yorke chaotic;
- the dynamical system $(X^\Gamma, \sigma_\varphi)$ has an scrambled pair;
- the map $\varphi : \Gamma \to \Gamma$ has at least one non-quasi-periodic point.

Li-Yorke chaos has been introduced for the first time in [8].

## 2. Preliminaries

Let's bring some priliminary definitions. Following [6], in dynanmical system $(Z,h)$ with metric phase space $(Z,d)$ we call distinct points $x,y\in Z$, an *scrambled pair* if

$$\liminf_{n\to\infty} d(f^n(x),f^n(y))=0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x),f^n(y))>0\,.$$

In dynamical system $(X,f)$ with metric phase space $X$, we call $A(\subseteq X)$ with at least two elements an *scrambled set* if for all distinct $x,y\in A$, $(x,y)$ is an scrambled pair. We call $(X,f)$ *Li-Yorke chaotic* if $X$ contains an uncountable scrambled set.

*Remark* 2.1. We recall that in dynamical system $(Z,h)$ with compact metric phase space $Z$ if two metrics $d$ and $D$ induce the same original compact topology on $Z$, then for all $x,y\in Z$ we have:
• $\liminf\limits_{n\to\infty} d(f^n(x),f^n(y))=0$ if and only if $\liminf\limits_{n\to\infty} D(f^n(x),f^n(y))=0$ (if and only if there exists $z\in Z$ and a sequence $\{n_k:k\in\mathbb{N}\}$ in $\mathbb{N}$ with $\lim\limits_{k\to\infty} f^{n_k}(x)=\lim\limits_{k\to\infty} f^{n_k}(y)=z$);
• $\limsup\limits_{n\to\infty} d(f^n(x),f^n(y))>0$ if and only if $\limsup\limits_{n\to\infty} D(f^n(x),f^n(y))>0$ (if and only if there exist distinct points $z,w\in Z$ and a sequence $\{n_k:k\in\mathbb{N}\}$ in $\mathbb{N}$ with $\lim\limits_{k\to\infty} f^{n_k}(x)=z$ and $\lim\limits_{k\to\infty} f^{n_k}(y)=w$).
In particular $(x,y)$ is an scrambled pair with respect to $d$ if and only if it is an scrambled pair with respect to $D$.

*Remark* 2.2. For metric space $Z$ with at least two elements and arbitrary nonempty set $\Lambda$, $Z^\Lambda$ is metrizable under product topology if and only if $\Lambda$ is countable [10].

*Remark* 2.3. For metric space $(Z,\rho)$ with at least two elements and arbitrary nonempty countable set $\Lambda$, suppose

$$\Lambda=\{\lambda_n:n\geq 0\}=\{\theta_n:n\geq 0\}\cup\{\mu_n:n\geq 0\}\,,$$

then for $(x_\alpha)_{\alpha\in\Lambda},(y_\alpha)_{\alpha\in\Lambda}\in Z^\Lambda$ let

$$F((x_\alpha)_{\alpha\in\Lambda},(y_\alpha)_{\alpha\in\Lambda})=\sum_{n\geq 0}\frac{\overline{\rho}(x_{\theta_n},y_{\theta_n})+\overline{\rho}(x_{\mu_n},y_{\mu_n})}{2^n}\text{and}G((x_\alpha)_{\alpha\in\Lambda},(y_\alpha)_{\alpha\in\Lambda})=\sum_{n\geq 0}\frac{\overline{\rho}(x_{\lambda_n},y_{\lambda_n})}{2^n},$$

where for $u,v\in Z$ we have $\overline{\rho}(u,v)=\min(1,\rho(u,v))$. Then two metrics $F$ and $G$ on $Z^\Lambda$ are compatible with its product topology (use similar methods described in [10]).

Moreover suppose

$$\delta(a,b)=\begin{cases} 0 & a=b\,,\\ 1 & a\neq b\,.\end{cases}$$

*Remark* 2.4. [9] There exists a collection $\mathcal{H}$ of infinite subsets of $\mathbb{N}\cup\{0\}$ such that for all distinct $A, B \in \mathcal{H}$ the set $A\cap B$ is finite, also $\mathcal{H}$ has continuum cardinal. For this aim consider bijection $\psi : \mathbb{Q} \to \mathbb{N}\cup\{0\}$. For each $x \in \mathbb{R}$ there exists a one to one sequence $\{q_n^x\}_{n\in\mathbb{N}}$ of rational numbers (i.e., for $n \neq m$, $q_n^x \neq q_m^x$) such that $\lim\limits_{n\to\infty} q_n^x = x$. For every distinct $x, y \in \mathbb{R}$ the set $\{q_n^x : n \in \mathbb{N}\} \cap \{q_n^y : n \in \mathbb{N}\}$ is finite. Hence $\mathcal{K} = \{\{q_n^x : n \in \mathbb{N}\} : x \in \mathbb{R}\}$ is a collection of infinite subsets of $\mathbb{Q}$ such that for all distinct $A, B \in \mathcal{K}$ the set $A \cap B$ is finite, in addition $\mathcal{K}$ has continuum cardinal. The collection $\mathcal{H} = \{\psi(A) : A \in \mathcal{K}\}$ is the desired collection of subsets of $\mathbb{N}\cup\{0\}$.

In the following text suppose $X$ is a finite discrete topological space with at least two elements, $\Gamma$ is an infinite countable set, $\varphi : \Gamma \to \Gamma$ is an arbitrary map, and consider $X^\Gamma$ with product (pointwise convergence) topology.

## 3. Li-Yorke chaotic generalized shift dynamical systems and its equivalences

Now we are ready to prove our main result, step by step. We bring the prove of Lemma 3.1 from [5].

**Lemma 3.1.** [5] *Suppose all points of* $\Gamma$ *are quasi-periodic under* $\varphi : \Gamma \to \Gamma$*, then* $(X^\Gamma, \sigma_\varphi)$ *does not have any scrambled pair.*

*Proof.* We may suppose $\Gamma = \mathbb{N}\cup\{0\}$, for $(x_n)_{n\geq 0}, (y_n)_{n\geq 0} \in X^{\mathbb{N}\cup\{0\}}$ let

$$D((x_n)_{n\geq 0}, (y_n)_{n\geq 0}) = \sum_{n\geq 0} \frac{\delta(x_n, y_n)}{2^n} .$$

Metric topology induced from $D$ on $X^{\mathbb{N}\cup\{0\}}$ is the same as product topology on $X^{\mathbb{N}\cup\{0\}}$. Now we have the following claim:
**Claim.** For $x = (x_n)_{n\geq 0}, y = (y_n)_{n\geq 0} \in X^{\mathbb{N}\cup\{0\}}$ if $\liminf\limits_{n\to\infty} D(\sigma_\varphi^n(x), \sigma_\varphi^n(y)) = 0$, then

$$\lim_{n\to\infty} D(\sigma_\varphi^n(x), \sigma_\varphi^n(y)) = 0 .$$

*Proof of Claim.* Suppose $\{n_k : k \in \mathbb{N}\}$ is an strictly increasing sequnce in $\mathbb{N}$ with $\lim\limits_{k\to\infty} D(\sigma_\varphi^{n_k}(x), \sigma_\varphi^{n_k}(y)) = 0$. For $p \in \mathbb{N}\cup\{0\}$, $\{\varphi^n(p) : n \geq 0\}$ is finite since $p$ is quasi-periodic under $\varphi$, and there exist $s_p > t_p \geq 1$ with $\varphi^{s_p}(p) = \varphi^{t_p}(p)$, for all $n > t_p$, we have

$$\{\varphi^i(p) : i \geq n\} = \{\varphi^i(p) : t_p + 1 \leq i \leq s_p\}. \tag{3.1}$$

Now, since $\lim\limits_{k\to\infty} D(\sigma_\varphi^{n_k}(x), \sigma_\varphi^{n_k}(y)) = 0$, for $p \geq 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$ we have

$$\sum_{m\geq 0}\frac{\delta(x_{\varphi^{n_k(m)}},y_{\varphi^{n_k(m)}})}{2^m}=D(\sigma_\varphi^{n_k}(x),\sigma_\varphi^{n_k}(y))<\min\left\{\frac{1}{2^i}:i=\varphi^{t_p+1}(p),\varphi^{t_p+2}(p),\ldots,\varphi^{s_p}(p)\right\}$$

therefore,

$$x_{\varphi^{n_k(m)}}=y_{\varphi^{n_k(m)}}\ \forall k\geq N,m\in\{\varphi^{t_p+1}(p),\varphi^{t_p+2}(p),\ldots,\varphi^{s_p}(p)\}. \quad (3.2)$$

Considering 3.1 and 3.2 we have $x_{\varphi^n(p)}=y_{\varphi^n(p)}$ for all $n\geq\max\{t_p,n_N\}=:$ $M_p$.

Consider $\varepsilon>0$ there exists $q\in\mathbb{N}$ with $\frac{1}{2^q}<\varepsilon$. For all $n\geq\max\{M_0,M_1,\ldots,M_q\}$ and $j\in\{0,\ldots,q\}$ we have $x_{\varphi^n(j)}=y_{\varphi^n(j)}$ therefore

$$D(\sigma_\varphi^n(x),\sigma_\varphi^n(y))=\sum_{m\geq 0}\frac{\delta(x_{\varphi^n(m)},y_{\varphi^n(m)})}{2^m}=\sum_{m>q}\frac{\delta(x_{\varphi^n(m)},y_{\varphi^n(m)})}{2^m}\leq\frac{1}{2^q}<\varepsilon$$

which leads to $\lim_{n\to\infty}D(\sigma_\varphi^n(x),\sigma_\varphi^n(y))=0$ and completes the proof of our Claim.

Using the Claim for all $x,y$ if $\liminf_{n\to\infty}D(\sigma_\varphi^n(x),\sigma_\varphi^n(y))=0$, then $\limsup_{n\to\infty}D(\sigma_\varphi^n(x),\sigma_\varphi^n(y))=0$ and $(x,y)$ is not an scrambled pair, so $(X^{\mathbb{N}\cup\{0\}},\sigma_\varphi)$ does not have any scrambled pair. □

**Lemma 3.2.** *If $\varphi:\Gamma\to\Gamma$ has a non-quasi-periodic point, then $(X^\Gamma,\sigma_\varphi)$ is Li-Yorke chaotic.*

*Proof.* Suppose $\beta\in\Gamma$ is a non-quasi-periodic point of $\varphi$. Also suppose $\Gamma\setminus\{\varphi^n(\beta):n\geq 0\}=\{\theta_0,\theta_1,\ldots\}$ (if $\Gamma=\{\varphi^n(\beta):n\geq 0\}$, then replace $\beta$ with $\varphi(\beta)$ and note to the fact that $\varphi(\beta)$ is non-quasi-periodic point of $\varphi$ and $\Gamma\setminus\{\varphi^n(\beta):n\geq 0\}\neq\varnothing$). Equip $X^\Gamma$ with the following metric ($(x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}\in X^\Gamma$):

$$D((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})=\sum_{n\geq 0}\frac{\delta(x_{\varphi^n(\beta)},y_{\varphi^n(\beta)})+\delta(x_{\theta_n},y_{\theta_n})}{2^n}\,.$$

Metric topology induced from $D$ on $X^\Gamma$ is the same as product topology on $X^\Gamma$. Choose distinct $p,q\in X$.

For $A\subseteq\mathbb{N}\cup\{0\}$ define $\xi_A:\mathbb{N}\cup\{0\}\to\{p,q\}$ with

$$\xi_A(n)=\begin{cases}p & n\in A\\ q & n\notin A\end{cases}$$

also define $x_A=(x_\alpha^A)_{\alpha\in\Gamma}\in X^\Gamma$ with $x_\alpha^A=q$ for $\alpha\in\{\theta_0,\theta_1,\ldots\}$ and $(x_\beta^A,x_{\varphi(\beta)}^A,x_{\varphi^2(\beta)}^A,\cdots)=(\xi_A(0),q,\xi_A(1),q,q,\xi_A(2),q,q,q,\xi_A(3),q,q,q,q,\cdots)$.

**Claim.** For infinite subsets $A$ and $B$ of $\mathbb{N}\cup\{0\}$, if $A\cap B$ is finite, then $(x_A,x_B)$ is an scrambled pair of $(X^\Gamma,\sigma_\varphi)$.

*Proof of Claim.* Consider $A = \{n_i : i \geq 1\} \subseteq \mathbb{N} \cup \{0\}$ and $B = \{m_i : i \geq 1\} \subseteq \mathbb{N} \cup \{0\}$ with $n_1 < n_2 < \cdots$ and $m_1 < m_2 < \cdots$ also suppose $A \cap B$ is finite. There exists $N \in \mathbb{N}$ such that $n_i \notin B$ and $m_i \notin A$ for all $i \geq N$, since $A \cap B$ is finite. In particular $\xi_A(n_i) \neq \xi_B(n_i)$ for $i \geq N$. For $r \geq 1$ let $s_r := n_{N+r} - 1$, then

$$D(\sigma_\varphi^{s_r}(x_A), \sigma_\varphi^{s_r}(x_B)) \geq \delta(p,q) = 1 \ ,$$

hence

$$\limsup_{n\to\infty} D(\sigma_\varphi^{n}(x_A), \sigma_\varphi^{n}(x_B)) \geq \limsup_{r\to\infty} D(\sigma_\varphi^{s_r}(x_A), \sigma_\varphi^{s_r}(x_B)) \geq 1 \ . \quad (3.3)$$

For $k \geq 1$ let $l_k = 1 + 2 + \cdots + k$, then

$$D(\sigma_\varphi^{l_k}(x_A), \sigma_\varphi^{l_k}(x_B)) \leq \sum_{n \geq k+1} \frac{1}{2^n} = 2^{-k}$$

which leads to $\lim\limits_{k\to\infty} D(\sigma_\varphi^{l_k}(x_A), \sigma_\varphi^{l_k}(x_B)) = 0$ and

$$\liminf_{n\to\infty} D(\sigma_\varphi^{n}(x_A), \sigma_\varphi^{n}(x_B)) = 0 \ . \quad (3.4)$$

Using 3.3 and 3.4 , $(x_A, x_B)$ is an scrambled pair.
Using Remark 2.4 there exists an uncountable collection $\mathcal{H}$ of infinite subsets of $\mathbb{N} \cup \{0\}$ such that for all distinct $A, B \in \mathcal{H}$ the set $A \cap B$ is finite. By the above Claim, $\{x_A : A \in \mathcal{H}\}$ is an uncountable scrambled subset of $X^\Gamma$, which completes the proof. □

**Theorem 3.3** (Main Theorem)**.** *For discrete topological space $X$ with at least two elements, infinite countable set $\Gamma$ and arbitrary map $\varphi : \Gamma \to \Gamma$, in the generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$ the following statements are equivalent:*

- *the map $\varphi : \Gamma \to \Gamma$ has at least one non-quasi-periodic point;*
- *the dynamical system $(X^\Gamma, \sigma_\varphi)$ is Li-Yorke chaotic;*
- *the dynamical system $(X^\Gamma, \sigma_\varphi)$ has at least one scrambled pair.*

*Proof.* Use Lemmas 3.2 and 3.1. □

**Example 3.4.** For $\varphi_1 : \mathbb{Z} \to \mathbb{Z}$ with $\varphi_1(n) = n^2$ and $\varphi_2 : \mathbb{Z} \to \mathbb{Z}$ with $\varphi_2(n) = -n$, $(\{1,2\}^{\mathbb{Z}}, \sigma_{\varphi_1})$ is Li-Yorke chaotic and $(\{1,2\}^{\mathbb{Z}}, \sigma_{\varphi_2})$ does not have any scrambled pair.

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