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positive solutions for nonlinear systems of third-order generalized sturm-liouville boundary value problems with $(p_1, p_2, \ldots, p_n)$-laplacian

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Abstract. In this work, by employing the Leggett-Williams fixed point theorem, we study the existence of at least three positive solutions of boundary value problems for system of third-order ordinary differential equations with $(p_1, p_2, \ldots, p_n)$-Laplacian

$$\begin{cases} (\phi_{p_i}(u_i''(t)))' + a_i(t) f_i(t, u_1(t), u_2(t), \ldots, u_n(t)) = 0 & 0 \le t \le 1, \\ \alpha_i u_i(0) - \beta_i u_i'(0) = \mu_{i1} u_i(\xi_i), \quad \gamma_i u_i(1) + \delta_i u_i'(1) = \mu_{i2} u_i(\eta_i), & u_i''(0) = 0, \end{cases}$$

where $\phi_{p_i}(s) = |s|^{p_i - 2} s,$, are p_i-Laplacian operators, $p_i > 1, 0 < \xi_i < 1, 0 < \eta_i < 1$ and $\mu_{i1}, \mu_{i2} > 0$ for $i = 1, 2, \ldots, n$.

Keywords: Positive solution; Third-order ordinary differential equation; Fixed point theorem, $(p_1, p_2, \ldots, p_n)$-Laplacian.

1. INTRODUCTION

In this paper, we will study the existence of at least three positive solutions of boundary value problems for system of third-order ordinary differential equations with $(p_1, p_2, \ldots, p_n)$-Laplacian

$$\begin{cases} (\phi_{p_i}(u_i''(t)))' + a_i(t) f_i(t, u_1(t), u_2(t), \ldots, u_n(t)) = 0 & 0 \le t \le 1, \\ \alpha_i u_i(0) - \beta_i u_i'(0) = \mu_{i1} u_i(\xi_i), \quad \gamma_i u_i(1) + \delta_i u_i'(1) = \mu_{i2} u_i(\eta_i), & u_i''(0) = 0, \end{cases} \tag{1.1}$$

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where $\phi_{p_i}(s) = |s|^{p_i-2}s,$, are p_i-Laplacian operators, $p_i > 1, 0 < \xi_i < 1, 0 < \eta_i < 1$ and $\mu_{i1}, \mu_{i2} > 0$ for $i = 1, 2, \ldots, n$.
In [4], Il'in and Moiseev studied the existence of solutions for a linear multi-point boundary value problem. Gupta [3] studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors because multi-point boundary value problems describe many phenomena of applied mathematics and physics (see [2, 5, 12, 11]). There is much current interest in questions of positive solutions of boundary value problems for ordinary differential equations, you may see [1, 8-10, 13-15] and references therein. Motivated by the works [7, 16], in this paper we will show the existence of three positive solutions for the problem (1).

The basic space used in this paper is a real Banach space $E = \prod_{i=1}^{n}(C[0,1],\mathbb{R})$, with the norm $\|(u_1, u_2, \ldots, u_n)\| := \sum_{i=1}^{n}\|u_i\|$, where $\|u_i\| = \max_{t\in[0,1]}|u_i(t)|$. For convenience, we make the following assumptions:

(A) $\alpha_i \geq 0, \beta_i \geq 0, \gamma_i \geq 0, \delta_i \geq 0, \rho_i = \alpha_i\gamma_i + \beta_i\gamma_i + \alpha_i\delta_i > 0, \rho_i - \mu_{i2}\psi(\eta_i) > 0, \rho_i - \mu_{i1}\varphi(\xi_i) > 0, \mu_{i1}, \mu_{i2} > 0, \Delta_i < 0$, for $i = 1, 2, \ldots, n$, and $\sigma \in (0, \frac{1}{2})$,

$$\Delta_i = \begin{vmatrix} -\mu_{i1}\psi(\xi_i) & \rho_i - \mu_{i1}\varphi(\xi_i) \\ \rho_i - \mu_{i2}\psi(\eta_i) & -\mu_{i2}\varphi(\eta_i) \end{vmatrix}, \qquad i = 1, 2, \ldots, n,$$

where

$$\psi_i(t) = \beta_i + \alpha_i t, \quad \varphi_i(t) = \gamma_i + \delta_i - \gamma_i t, \quad t \in [0,1], \qquad i = 1, 2, \ldots, n, \tag{1.2}$$

are linearly independent solutions of the equation $x''(t) = 0, t \in [0,1]$. Obviously, ψ_i is non-decreasing on $[0,1]$ and φ_i is non-increasing on $[0,1]$.

(B) $f_i \in C([0,1] \times \prod_{i=1}^{n}[0,+\infty), [0,+\infty))$, are continuous and $a_i : (0,1) \to [0,+\infty)$ is continuous and $a_i(t) \neq 0$, for $i = 1, 2, \ldots, n$ on any subinterval of $(0,1)$, and

$$0 < \int_0^1 a_i(s)ds < +\infty.$$

For the convenience of the reader, we present here the Leggett-Williams fixed point theorem [6].
Given a cone K in a real Banach space E, a map α is said to be a nonnegative continuous concave (resp. convex) functional on K provided

that $\alpha : K \to [0. + \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),$$
$$(resp.\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y)),$$

for all $x, y \in K$ and $t \in [0, 1]$.
Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha, a, b)$ by

$$P_r = \{x \in K | \|x\| < r\},$$

and

$$P(\alpha, a, b) = \{x \in K | a \leq \alpha(x), \|x\| \leq b\}.$$

Theorem 1.1. *(Leggett-Williams fixed point theorem). Let* $A : \overline{P_c} \to \overline{P_c}$ *be a completely continuous operator and let* α *be a nonnegative continuous concave functional on* K *such that* $\alpha(x) \leq \|x\|$ *for all* $x \in \overline{P_c}$. *Suppose there exist* $0 < a < b < d \leq c$ *such that*
(A1) $\{x \in P(\alpha, b, d) | \alpha(x) > b\} \neq \emptyset$*, and* $\alpha(Ax) > b$ *for* $x \in P(\alpha, b, d)$*,*
(A2) $\|Ax\| < a$ *for* $\|x\| \leq a$*, and*
(A3) $\alpha(Ax) > b$ *for* $x \in P(\alpha, b, c)$ *with* $\|Ax\| > d$.
Then A *has at least three fixed points* x_1*,* x_2*, and* x_3 *and such that* $\|x_1\| < a, b < \alpha(x_2)$ *and* $\|x_3\| > a$*, with* $\alpha(x_3) < b$.

Inspired and motivated by the works mentioned above, in this work we will consider the existence of positive solutions to BVP (1). we shall first give a new form of the solution, and then determine the properties of the Green's function for associated linear boundary value problems; finally, by employing the Leggett-Williams fixed point theorem, some sufficient conditions guaranteeing the existence of three positive solutions. The rest of the article is organized as follows: in Section 2, we present some preliminaries that will be used in Section 3. The main results and proofs will be given in Section 3. Finally, in Section 4, an example are given to demonstrate the application of our main result.

2. PRELIMINARIES

In this section, we present some notations and preliminary lemmas that will be used in the proof of the main result.

Definition 2.1. Let X be a real Banach space. A non-empty closed convex set $P \subset X$ is called a cone of X if it satisfies the following conditions:
(1) $x \in P, \mu \geq 0$ implies $\mu x \in P$,
(2) $x \in P, -x \in P$ implies $x = 0$.

Let $y_i(t) = -\phi_{p_i}(u_i''(t))$, for $i = 1, 2, \ldots, n$, then the following boundary value problems

$$\begin{cases} (\phi_{p_i}(u_i''(t)))' + a_i(t) f_i(t, u_1(t), \ldots, u_n(t)) = 0, & 0 \le t \le 1, \\ u_i''(0) = 0, \end{cases}$$

is turned into the following boundary value problems

$$\begin{cases} y_i'(t) - a_i(t) f_i(t, u_1(t), \ldots, u_n(t)) = 0, \ 0 \le t \le 1, \\ y_i(0) = 0, \end{cases} \quad i = 1, 2, \ldots, n. \tag{2.1}$$

Lemma 2.2. *The BVP (3) has a unique solution*

$$y_i(t) = \int_0^t a_i(s) f_i(s, u_1(s), \ldots, u_n(s)) ds, \qquad i = 1, 2, \ldots, n. \tag{2.2}$$

Lemma 2.3. *If (A) holds, then for $y_i(t) \in C([0,1]), i = 1, 2, \ldots, n$, the following boundary value problems*

$$\begin{cases} u_i''(t) + \phi_{p_i}^{-1}(y_i(t)) = 0, & 0 \le t \le 1, \\ \alpha_i u_i(0) = \beta_i u_i'(0) = \mu_{i1} u_i(\xi_i), \\ \gamma_i u_i(1) + \delta_i u_i'(1) = \mu_{i2} u_i(\eta_i), \end{cases} \tag{2.3}$$

have a unique solution

$$u_i(t) = \int_0^1 G_i(t,s) \phi_{p_i}^{-1}(y_i(s)) ds + A_i(\phi_{p_i}^{-1}(y_i)) \psi_i(t) + B_i(\phi_{p_i}^{-1}(y_i)) \varphi_i(t), \tag{2.4}$$

where

$$G_i(t,s) = \frac{1}{\rho_i} \begin{cases} \varphi_i(t)\psi_i(s), & s \le t, \\ \varphi_i(s)\psi_i(t), & t \le s, \end{cases} \tag{2.5}$$

$$A_i(\phi_{p_i}^{-1}(y_i)) = \frac{1}{\Delta_i} \begin{vmatrix} \mu_{i1} \int_0^1 G_i(\xi_i, s) \phi_{p_i}^{-1}(y_i) ds & \rho_i - \mu_{i1}\varphi_i(\xi_i) \\ \mu_{i2} \int_0^1 G_i(\eta_i, s) \phi_{p_i}^{-1}(y_i) ds & -\mu_{i2}\varphi_i(\eta_i) \end{vmatrix}, \tag{2.6}$$

$$B_i(\phi_{p_i}^{-1}(y_i)) = \frac{1}{\Delta_i} \begin{vmatrix} -\mu_{i1}\psi_i(\xi_i) & \mu_{i1} \int_0^1 G_i(\xi_i, s) \phi_{p_i}^{-1}(y_i) ds \\ \rho_i - \mu_{i2}\psi_i(\eta_i) & \mu_{i2} \int_0^1 G_i(\eta_i, s) \phi_{p_i}^{-1}(y_i) ds \end{vmatrix}, \tag{2.7}$$

Proof. The proof follows by routine calculations. □

Remark 2.4. For fixed integrable function y, it is obvious that $A_i(\phi_{p_i}^{-1}(y)), B_i(\phi_{p_i}^{-1}(y))$, for $i = 1, \ldots, n$, are constants.

For convenience, let

$\Lambda_0 = \min\left\{\frac{\varphi_1(1-\sigma)}{\varphi_1(\sigma)}, \frac{\psi_1(\sigma)}{\psi_1(1)}, \ldots, \frac{\varphi_n(1-\sigma)}{\varphi_n(\sigma)}, \frac{\psi_n(\sigma)}{\psi_n(1)}\right\},$
$\Lambda_1 = \max\left\{1, \|\psi_1\|, \|\varphi_1\|, \ldots, \|\psi_n\|, \|\varphi_n\|\right\},$
$\Lambda_2 = \min\left\{ \min_{t\in[\sigma,1-\sigma]} \varphi_1(t), \min_{t\in[\sigma,1-\sigma]} \psi_1(t), \ldots, \right.$

$\min_{t\in[\sigma,1-\sigma]} \varphi_n(t), \min_{t\in[\sigma,1-\sigma]} \psi_n(t), 1\Big\}$,
$\lambda = \min\left\{\Lambda_0, \frac{\Lambda_2}{\Lambda_1}\right\}$.

If (A) and (B) hold, then from Lemmas 1 and 2, we know that

$$\mathbf{u} = (u_i(t), \ldots, u_n(t)$$

is a solution of the BVP (1) if and only if

$$u_i(t) = \int_0^1 G_i(t,s)\phi_{p_i}^{-1}(W_i(s))ds + A_i(\phi_{p_i}^{-1}(W_i(s)))\psi_i(t) + \qquad (2.8)$$
$$B_i(\phi_{p_i}^{-1}(W_i(s)))\varphi_i(t), \quad 0 \le t \le 1,$$

where $W_i(s) = \int_0^s a_i(\tau)f_i(\tau, u_1(\tau), \ldots, u_n(\tau))d\tau$, for $i = 1, \ldots, n$.

We need some properties of the functions $G_i, i = 1, \ldots, n$, in order to discuss the existence of positive solutions.
For the Green's functions $G_i(t,s) i = 1, \ldots, n,$, we have the following two Lemmas [7].

Lemma 2.5. *If (A) and (B) hold, then*

$$0 \le G_i(t,s) \le G_i(s,s), \qquad t,s \in [0,1], \qquad (2.9)$$

and

$$G_i(t,s) \ge \Lambda_0 G_i(s,s), \qquad t \in [\sigma, 1-\sigma], s \in [0,1], \qquad (2.10)$$

for $i = 1, \ldots, n$.

Denote

$$K = \{\mathbf{u} = (u_1, \ldots, u_n) \in E | u_i(t) \ge 0, \min_{t\in[\sigma,1-\sigma]}(\sum_{i=1}^n u_i(t)) \ge \lambda\|\mathbf{u}\|, i = 1, \ldots, n\}.$$

It is obvious that K is cone.

Define the operator $T : E \to E$ by

$$T(\mathbf{u})(t) = (T_1(\mathbf{u})(t), \ldots, T_n(\mathbf{u})(t)), \qquad \forall t \in (0,1), \qquad (2.11)$$

where

$$\begin{cases} T_1(\mathbf{u})(t) = \int_0^1 G_1(t,s)\phi_{p_1}^{-1}(W_1(s))ds + A_1(\phi_{p_1}^{-1}(W_1(s)))\psi_1(t) \\ \qquad\qquad\qquad\qquad +B_1(\phi_{p_1}^{-1}(W_1(s)))\varphi_1(t),\ 0 \leq t \leq 1, \\ \vdots \\ T_n(\mathbf{u})(t) = \int_0^1 G_n(t,s)\phi_{p_n}^{-1}(W_n(s))ds + A_n(\phi_{p_n}^{-1}(W_n(s)))\psi_n(t) \\ \qquad\qquad\qquad\qquad +B_n(\phi_{p_n}^{-1}(W_n(s)))\varphi_n(t),\ 0 \leq t \leq 1, \end{cases} \tag{2.12}$$

and $W_i(s) = \int_0^s a_i(\tau)f_i(\tau,\mathbf{u}(\tau))d\tau$, and $\mathbf{u} = (u_1,\ldots,u_n)$, for $i = 1,\ldots,n$. Evidently, $(u_1(t),\ldots,u_n(t))$ is a solution of the BVP (1) if and only if $(u_1(t),\ldots,u_n(t))$ is a fixed point of operator T.

Lemma 2.6. *If (A) and (B) holds, then the operator defined in (13) satisfies $T(K) \subseteq K$.*

Proof. For any $\mathbf{u} = (u_1,\ldots,u_n) \in K$, then from properties of $G_i(t,s)$, we have $T_i(\mathbf{u})(t) \geq 0, t \in [0,1]$, and it follows form (18) that

$$T_i(\mathbf{u})(t) = \int_0^1 G_i(t,s)\phi_{p_i}^{-1}(W_i(s))ds + A_i(\phi_{p_i}^{-1}(W_i(s)))\psi_i(t) + B_i(\phi_{p_i}^{-1}(W_i(s)))\varphi_i(t)$$
$$\leq \int_0^1 G_i(s,s)\phi_{p_i}^{-1}(W_i(s))ds + \Lambda_1[A_i(\phi_{p_i}^{-1}(W_i(s))) + B_i(\phi_{p_i}^{-1}(W_i(s)))].$$

Thus,

$$\|T_i(\mathbf{u})\| \leq \int_0^1 G_i(s,s)\phi_{p_i}^{-1}(W_i(s))ds + \Lambda_1[A_i(\phi_{p_i}^{-1}(W_i(s))) + B_i(\phi_{p_i}^{-1}(W_i(s)))].$$

On the other hand, for $t \in [\sigma, 1-\sigma]$, we have

$$\min_{t\in[\sigma,1-\sigma]} T_i(\mathbf{u})(t) = \min_{t\in[\sigma,1-\sigma]} \Big[\int_0^1 G_i(t,s)\phi_{p_i}^{-1}(W_i(s))ds +$$
$$A_i(\phi_{p_i}^{-1}(W_i))\psi_i(t) + B_i(\phi_{p_i}^{-1}(W_i))\varphi_i(t)\Big]$$
$$\geq \Lambda_0 \int_0^1 G_i(s,s)\phi_{p_i}^{-1}(W_i(s))ds + A_i(\phi_{p_i}^{-1}(W_i))\psi_i(t) + B_i(\phi_{p_i}^{-1}(W_i))\varphi_i(t)$$
$$\geq \Lambda_0 \int_0^1 G_i(s,s)\phi_{p_i}^{-1}(W_i(s))ds + \tfrac{\Lambda_2}{\Lambda_1}.\Lambda_1.[A_i(\phi_{p_i}^{-1}(W_i)) + B_i(\phi_{p_i}^{-1}(W_i))]$$
$$\geq \lambda\big[\int_0^1 G_i(s,s)\phi_{p_i}^{-1}(W_i(s))ds + \Lambda_1.[A_i(\phi_{p_i}^{-1}(W_i)) + B_i(\phi_{p_i}^{-1}(W_i))]\big]$$
$$\geq \lambda\|T_i(\mathbf{u})\|.$$

Therefore

$$\min_{t\in[\sigma,1-\sigma]} \big(T_1(\mathbf{u})(t),\ldots,T_n(\mathbf{u})(t)\big) \geq \lambda\|T_1(\mathbf{u})\| + \ldots + \lambda\|T_n(\mathbf{u})\|$$
$$= \lambda\|(T_1(\mathbf{u}),\ldots,T_n(\mathbf{u}))\|.$$

From the above, we get $T(K) \subseteq K$. This completes the proof of Theorem 4. $\square$

3. MAIN RESULTS

In this section, we discuss the existence of positive solutions of BVP (1). We define the nonnegative continuous concave functional on K by

$$\alpha(u_1, \ldots, u_n) = \min_{\sigma \le t \le 1-\sigma} (u_1(t) + \cdots + u_n(t)).$$

It is obvious that, for each $\mathbf{u} = (u_1, \ldots, u_n) \in K, \alpha(\mathbf{u}) \le \|\mathbf{u}\|$.

Throughout this section, for convenience, let

$$\widetilde{A_i} = \frac{1}{\Delta_i} \begin{vmatrix} \mu_{i1} & \rho_i - \mu_{i1}\varphi_i(\xi_i) \\ \mu_{i2} & -\mu_{i2}\varphi_i(\eta_i) \end{vmatrix}, \ \widetilde{B_i} = \frac{1}{\Delta_i} \begin{vmatrix} -\mu_{i1}\psi_i(\xi_i) & \mu_{i1} \\ \rho_i - \mu_{i2}\psi_i(\eta_i) & \mu_{i2} \end{vmatrix},$$

$$M_i = \max_{0 \le t \le 1} \int_0^1 G_i(t,s)ds, \ m_i = \min_{\sigma \le t \le 1-\sigma} \int_\sigma^{1-\sigma} G_i(t,s)ds, \quad i = 1, \ldots, n.$$

Also, we assume that $p_i, i = 1, \ldots, n$, are positive numbers satisfying $\frac{1}{p_1} + \cdots + \frac{1}{p_n} \le 1$.

Assume that (A) and (B) hold. In addition, assume there exist nonnegative numbers a, b, c such that $0 < a < b \le \min\{\lambda, \frac{m_1}{p_1 M_1}, \ldots, \frac{m_n}{p_n M_n}\}c$, and $f_i(t, u_1, \ldots, u_n)$, satisfy the following conditions:

H1) $f_i(t, u_1, \ldots, u_n) < \frac{1}{\int_0^1 a_i(t)dt} \phi_{p_i}\big(\frac{c}{p_i M_i[1+\Lambda_1 \widetilde{A_i} + \Lambda_1 \widetilde{B_i}]}\big)$, for any $t \in [0,1], u_1 + \cdots + u_n \in [0, c]$,

H2) $f_i(t, u_1, \ldots, u_n) < \frac{1}{\int_0^1 a_i(t)dt} \phi_{p_i}\big(\frac{a}{p_i M_i[1+\Lambda_1 \widetilde{A_i} + \Lambda_1 \widetilde{B_i}]}\big)$, for any $t \in [0,1], u_1 + \cdots + u_n \in [0, a]$,

H3) $f_i(t, u_1, \ldots, u_n) > \frac{1}{\int_\sigma^{1-\sigma} a_i(t)dt} \phi_{p_i}\big(\frac{b}{m_i[1+\Lambda_2 \widetilde{A_i} + \Lambda_2 \widetilde{B_i}]}\big)$, for some $i = 1, 2, \ldots, n$, and for any $t \in [0,1], u_1 + \cdots + u_n \in [b, \frac{b}{\lambda}]$.

Then the system (1) has at least three positive solutions $\mathbf{u}_i = (u_1^{(i)}, \ldots, u_n^{(i)}), i = 1, 2, 3$, such that $\|\mathbf{u}_1\| < a, b < \alpha(\mathbf{u}_2)$, and $\|\mathbf{u}_3\| > a$, with $\alpha(\mathbf{u}_3) < b$.

Proof. First, we show that $T : \overline{P_c} \to \overline{P_c}$ is a completely continuous operator. If $(u_1, \ldots, u_n) \in \overline{P_c}$, by condition (H1), for $i = 1, \ldots, n$, we have

$$A_i(\phi_{p_i}^{-1}(W_i))$$
$$\le \frac{1}{\Delta_i} \begin{vmatrix} \mu_{i1} \int_0^1 G_i(\xi_i, s)\phi_{p_i}^{-1}(\int_0^s a_i(\tau) f_i(\tau, u_1(\tau), \ldots, u_n(\tau))d\tau)ds & \rho_i - \mu_{i1}\varphi_i(\xi_i) \\ \mu_{i2} \int_0^1 G_i(\xi_i, s)\phi_{p_i}^{-1}(\int_0^s a_i(\tau) f_i(\tau, u_1(\tau), \ldots, u_n(\tau))d\tau)ds & -\mu_{i2}\varphi_i(\eta_i) \end{vmatrix}$$
$$\le \frac{c}{p_i[1+\Lambda_1 \widetilde{A_i} + \Lambda_1 \widetilde{B_i}]} \widetilde{A_i},$$

and

$$B_i(\phi_{p_i}^{-1}(W_i))$$

$\leq \frac{1}{\Delta_i}\begin{vmatrix} -\mu_{i1}\psi_i(\xi_i) & \mu_{i1}\int_0^1 G_i(\xi_i,s)\phi_{p_i}^{-1}(\int_0^s a_i(\tau)f_i(\tau,u_1(\tau),\dots,u_n(\tau))d\tau)ds \\ \rho_i-\mu_{i2}\psi_i(\eta_i) & \mu_{i2}\int_0^1 G_i(\xi_i,s)\phi_{p_i}^{-1}(\int_0^s a_i(\tau)f_i(\tau,u_1(\tau),\dots,u_n(\tau))d\tau)ds \end{vmatrix}$
$\leq \frac{c}{p_i[1+\Lambda_1\widetilde{A_i}+\Lambda_1\widetilde{B_i}]}\widetilde{B_i},$

Thus,

$\|T_i(u_1,\dots,u_n)\| = \max_{0\leq t\leq 1}|T_i(u_1,\dots,u_n)(t)|$
$=\max_{0\leq t\leq 1}\Big(\int_0^1 G_i(t,s)\phi_{p_i}^{-1}(W_i(s))ds + A_i(\phi_{p_i}^{-1}(W_i))\psi_i(t)+$
$B_i(\phi_{p_i}^{-1}(W_i))\varphi_1(t)\Big) \leq \frac{c}{p_i[1+\Lambda_1\widetilde{A_i}+\Lambda_1\widetilde{B_i}]}+\frac{c}{p_i[1+\Lambda_1\widetilde{A_i}+\Lambda_1\widetilde{B_i}]}\widetilde{A_i}\varphi_1(t)$
$+\frac{c}{p_i[1+\Lambda_1\widetilde{A_i}+\Lambda_1\widetilde{B_i}]}\widetilde{B_i}\varphi_i(t)$
$\leq \frac{c}{p_i[1+\Lambda_1\widetilde{A_i}+\Lambda_1\widetilde{B_i}]}[1+\Lambda_1\widetilde{A_i}+\Lambda_1\widetilde{B_i}]=\frac{c}{p_i}.$

thus

$\|T(u_1,\dots,u_n)\| = \sum_{i=1}^n\|T_i(u_1,\dots,u_n)\| \leq \sum_{i=1}^n \frac{c}{p_i}\leq c.$

Therefore, $\|T(u_1,\dots,u_n)\|\leq c$, that is, $T:\overline{P_c}\to\overline{P_c}$. The operator T is completely continuous by an application of the Ascoli-Arzela theorem.
In the same way, the condition (H2) implies that the condition (A2) of Theorem 1 is satisfied. We now show that condition (A1) of Theorem 1 is satisfied. Clearly, $\{(u_1,\dots,u_n)\in P(\alpha,b,\frac{b}{\lambda})|\alpha(u_1,\dots,u_n)>b\}\neq\emptyset$. If $(u_1,\dots,u_n)\in P(\alpha,b,\frac{b}{\lambda})$, then $b\leq u_1(s)+\cdots+u_n(s)\leq\frac{b}{\lambda}, s\in[\sigma,1-\sigma]$. By condition (H3), for $i=1,\dots,n$, we get

$A_i(\phi_{p_i}^{-1}(W_i))$
$\geq \frac{1}{\Delta_i}\begin{vmatrix} \mu_{i1}\int_\sigma^{1-\sigma} G_i(\xi_i,s)\phi_{p_i}^{-1}(\int_\sigma^{1-\sigma} a_i(\tau)f_i(\tau,u_1(\tau),\dots,u_n(\tau))d\tau)ds & \rho_i-\mu_{i1}\varphi_i(\xi_i) \\ \mu_{i2}\int_\sigma^{1-\sigma} G_i(\xi_i,s)\phi_{p_i}^{-1}(\int_\sigma^{1-\sigma} a_i(\tau)f_i(\tau,u_1(\tau),\dots,u_n(\tau))d\tau)ds & -\mu_{i2}\varphi_i(\eta_i) \end{vmatrix}$
$\geq \frac{b}{[1+\Lambda_2\widetilde{A_i}+\Lambda_2\widetilde{B_i}]}\widetilde{A_i},$

and

$B(\phi_{p_i}^{-1}(W_i))$
$\geq \frac{1}{\Delta_i}\begin{vmatrix} -\mu_{i1}\psi_i(\xi_i)\mu_{i1}\int_\sigma^{1-\sigma} G_i(\xi_i,s)\phi_{p_i}^{-1}(\int_\sigma^{1-\sigma} a_i(\tau)f_i(\tau,u_1(\tau),\dots,u_n(\tau))d\tau)ds \\ \rho_i-\mu_{i2}\psi_i(\eta_i) \end{vmatrix} \quad \mu_{i2}\int_\sigma^{1-\sigma} G_i(\xi_i,s)\phi_{p_i}^{-1}(\int_\sigma^{1-\sigma} a_i($
$\geq \frac{b}{[1+\Lambda_2\widetilde{A_i}+\Lambda_2\widetilde{B_i}]}\widetilde{B_i}.$

Thus,

$\alpha(T(u_1,\dots,u_n)(t)) = \min_{\sigma\le t\le 1-\sigma}(T_1(u_1,\dots,u_n)(t)+\dots+T_n(u_1,\dots,u_n)(t))$
$\ge \min_{\sigma\le t\le 1-\sigma}\Big(\int_0^1 G_1(t,s)\phi_{p_1}^{-1}(W_1(s))ds + A_1(\phi_{p_1}^{-1}(W_1))\psi_1(t)$
$+ B_1(\phi_{p_1}^{-1}(W_1))\varphi_1(t)\Big)$
$+$
$\vdots$
$+ \min_{\sigma\le t\le 1-\sigma}\Big(\int_0^1 G_n(t,s)\phi_{p_n}^{-1}(W_n(s))ds + A_n(\phi_{p_n}^{-1}(W_n))\psi_n(t)$
$+ B_n(\phi_{p_n}^{-1}(W_n))\varphi_n(t)\Big)$
$\ge \frac{b}{1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}} + \frac{b}{1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}}\widetilde{A_1}\psi_1(t) + \frac{b}{1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}}\widetilde{B_1}\varphi_1(t)$
$\ge \frac{b}{1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}}[1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}] = b$

Therefore, condition (A1) of Theorem 1 is satisfied.
Finally, we show that the condition (A3) of Theorem 1 is also satisfied.
If $(u_1,\dots,u_n)\in P(\alpha,b,c)$, and $\|T(u_1,\dots,u_n)\| > \frac{b}{\lambda}$, then
$\alpha(T(u_1,\dots,u_n)(t)) = \min_{\sigma\le t\le 1-\sigma} T(u_1,\dots,u_n)(t)$
$\ge \lambda\|T(u_1,\dots,u_n)\| > b.$
Therefore, the condition (A3) of Theorem 1 is also satisfied. By Theorem 1, there exist three positive solutions $\mathbf{u}_i = (u_1^{(i)},\dots,u_n^{(i)}), i=1,2,3,$ such that $\|\mathbf{u}_1\| < a, b < \alpha(\mathbf{u}_2)$, and $\|\mathbf{u}_3\| > a$, with $\alpha(\mathbf{u}_3) < b$. we have the conclusion. □

4. APPLICATION

Example 4.1. Consider the following boundary value problem system:

$$\begin{cases} (\phi_{p_1}(u_1''(t)))' + a_1(t)f_1(t,u_1(t),u_2(t)) = 0 & 0\le t\le 1, \\ (\phi_{p_2}(u_2''(t)))' + a_2(t)f_2(t,u_2(t),u_2(t)) = 0 & 0\le t\le 1, \\ u_1(0)-u_1'(0) = u_1(\frac{1}{4}), \quad u_1(1)+u_1'(1) = \frac{1}{2}u(\frac{1}{2}), & u_1''(0)=0, \\ u_2(0)-u_2'(0) = u_2(\frac{1}{4}), \quad u_2(1)+u_2'(1) = \frac{1}{2}u_2(\frac{1}{2}), & u_2''(0)=0, \end{cases} \tag{4.1}$$

where $n=2, a_i(t)=1, \alpha_i=\beta_i=\gamma_i=\delta_i=1$, and

$$f_i(t,u_1,u_2) = \begin{cases} \frac{t}{1000} + \frac{u_1+u_2}{1000}, & t\in[0,1], 0\le u_1+u_2\le 1, \\ \frac{t}{1000} + 2((u_1+u_2)^2-(u_1+u_2)) + \frac{1}{1000}, & t\in[0,1], 1<u_1+u_2<2, \\ \frac{t}{1000} + 2\big[\log_2(u_1+u_2) + \frac{u_1+u_2}{2}\big] + \frac{1}{1000}, & t\in[0,1], 2\le u_1+u_2\le 4, \\ \frac{t}{1000} + 4\sqrt{u_1+u_2} + \frac{1}{1000}, & t\in[0,1], 4<u_1+u_2<+\infty, \end{cases}$$

for $i=1,2$. Choose $\sigma=\frac{1}{4}, p=q=3, p_1=p_2=2$. Then by direct calculations, we can obtain that
$\rho_i = 3\ \psi_i(t) = 1+t,\ \varphi_i(t) = 2-t,\ t\in[0,1],$

$\Delta_i = \begin{vmatrix} -\mu_{i1}\psi_i(\xi_i)\rho_i - \mu_{i1}\varphi_1(\xi_1) \\ \rho_i - \mu_{i2}\psi_i(\eta_i) - \mu_{i2}\varphi_i(\eta_i) \end{vmatrix} = -\frac{15}{8},\ \widetilde{A_i} = \frac{11}{15},\ \widetilde{B_i} = \frac{23}{8},$
$M_i = \max_{0\leq t\leq 1}\int_0^1 G_i(t,s)ds = \frac{4}{3},$
$m_i = \min_{\sigma\leq t\leq 1-\sigma}\int_\sigma^{1-\sigma} G_i(t,s)ds = \frac{25}{96},$
$\int_0^1 a_i(t)dt = 1,\ \int_\sigma^{1-\sigma} a_i(t)dt = \frac{1}{2}\ i = 1,2,$
$\Lambda_0 = \min\left\{\frac{\varphi_1(1-\sigma)}{\varphi_1(\sigma)}, \frac{\psi_2(\sigma)}{\psi_2(1)}, \frac{\varphi_2(1-\sigma)}{\varphi_2(\sigma)}, \frac{\psi_2(\sigma)}{\psi_2(1)}\right\} = \frac{5}{7},$
$\Lambda_1 = \max\left\{1, \|\psi_1\|, \|\varphi_1\|, \|\psi_2\|, \|\varphi_2\|\right\} = 2,$
$\Lambda_2 = \min\Big\{\min_{t\in[\sigma,1-\sigma]}\varphi_1(t), \min_{t\in[\sigma,1-\sigma]}\psi_1(t), \min_{t\in[\sigma,1-\sigma]}\varphi_2(t),$
$\min_{t\in[\sigma,1-\sigma]}\psi_2(t), 1\Big\} = 1,$
$\lambda = \min\left\{\Lambda_0, \frac{\Lambda_2}{\Lambda_1}\right\} = \frac{1}{2},$

and

$$\frac{1}{p_i M_i[1+\Lambda_1\widetilde{A_i}+\Lambda_1\widetilde{B_i}]} = \frac{45}{986}, \qquad \frac{1}{m_i[1+\Lambda_2\widetilde{A_i}+\Lambda_2\widetilde{B_i}]} = \frac{2304}{2765}, \qquad i = 1,2.$$

So we choose $a = 1, b = 2, c = 200$, Then, by Theorem 2, system (15) has at least three positive solutions $\mathbf{u}_i = (u_1^{(i)}, u_2^{(i)}), i = 1,2,3$, such that $\|\mathbf{u}_1\| < a, b < \alpha(\mathbf{u}_2)$, and $\|\mathbf{u}_3\| > a$, with $\alpha(\mathbf{u}_3) < b$.

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