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Solving a System of Linear Equations by Homotopy Analysis Method

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Abstract. In this paper, an efficient algorithm for solving a system of linear equations based on the homotopy analysis method is presented. The proposed method is compared with the classical Jacobi iterative method, and the convergence analysis is discussed. Finally, two numerical examples are presented to show the effectiveness of the proposed method.
Keywords: Homotopy analysis method, System of Linear Equations, Jacobi iterative method.

1. INTRODUCTION

The homotopy analysis method (HAM) was proposed by Liao in His works [1-3]. In recent years, this method has been successfully employed to solve many types of linear or non-linear, homogeneous or nonhomogeneous, equations and systems of equations as well as problems in science and engineering [3-9]. The homotopy analysis method contains the auxiliary parameter $\hbar_i$ which provides us, with a simple, way adjusting and controling the convergence region and rate of convergence of the resulted series solution.

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In most cases, the homotopy analysis method leads to a very rapidly convergent series solution, and usually only a few iterations are sufficient to arrive at a very accurate solution, specially when the auxiliary parameter $\hbar_i$ and the base functions are properly chosen.
In this paper, the homotopy analysis method is used to find the solution of the linear system $Ax = b$. The convergence of the proposed method is also studied.

2. REVIEW OF THE HOMOTOPY ANALYSIS METHOD

Consider the following system of equations

$$\mathcal{N}_i[u_1(r,t), \cdots, u_n(r,t)] = 0, \qquad (i = 1, 2, \cdots, n) \tag{2.1}$$

subject to the initial conditions:

$$u_k(r,t) = c_k \qquad (k = 1, 2, \cdots, n),$$

where $\mathcal{N}_i$ are linear or nonlinear operators, r and t denote the independent variables and $u_i(r,t)$ are unknown functions. By means of generalizing the traditional homotopy analysis method, Liao [1-4] constructed the so called zero-order deformation equations for $i = 1, 2, \cdots, n$

$$(1-p)\mathcal{L}_i[\varphi_i(r,t;p) - u_{i0}(r,t)] = p\hbar_i H_i(r,t)\mathcal{N}_i[\varphi_1(r,t;p), \cdots, \varphi_n(r,t;p)], \tag{2.2}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar_i \neq 0$ are non-zero auxiliary parameters for $i = 1, 2, \cdots, n$, $H_i(r,t) \neq 0$ non-zero auxiliary functions, $\mathcal{L}_i$'s are auxiliary linear operator with the following property for $i = 1, 2, \cdots, n$

$$\mathcal{L}_i[\varphi_i(r,t)] = 0, \qquad \text{when} \qquad \varphi_i(r,t) = 0. \tag{2.3}$$

Also $u_{i0}(r,t)$ are initial guesses of $u_i(r,t)$ and $\varphi_i(r,t;p)$ are unknown functions, respectively. It is important, that one has a great freedom to choose auxiliary things in the homotopy analysis method. Obviously, when $p = 0$ and $p = 1$ we have

$$\varphi_i(r,t;0) = u_{i0}(r,t), \quad \varphi_i(r,t;1) = u_i(r,t), \quad (i = 1, 2, \cdots, n). \tag{2.4}$$

Thus, as p increases from 0 to 1, the solutions $\varphi_i(r,t;p)$ vary from the initial guesses $u_{i0}(r,t)$ to the solutions $u_i(r,t)$. Expanding $\varphi_i(r,t;p)$ in Taylor series with respect to p, we have

$$\varphi_i(r,t;p) = u_{i0}(r,t) + \sum_{m=1}^{+\infty} u_{im}(r,t)p^m, \quad (i = 1, 2, \cdots, n), \tag{2.5}$$

where

$$u_{im}(r,t) = \frac{1}{m!}\frac{\partial^m \varphi_i(r,t;p)}{\partial p^m}\Big|_{p=0}, \quad (i = 1, 2, \cdots, n). \tag{2.6}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar_i$, and the auxiliary function are so properly chosen, the series (5) converges at $p = 1$, and so we have

$$u_{im}(r,t) = u_{i0}(r,t) + \sum_{m=1}^{+\infty} u_{im}(r,t) \quad (i = 1, 2, \cdots, n). \tag{2.7}$$

Define the vectors

$$\overrightarrow{u_{in}} = \{u_{i0}(r,t), u_{i1}(r,t), \cdots, u_{in}(r,t)\} \quad (i = 1, 2, \cdots, n). \tag{2.8}$$

Differentiating (7) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we obtain the mth-order deformation equation for $i = 1, 2, \cdots, n$:

$$\mathcal{L}_i[u_{im}(r,t) - \chi_m u_{i,m-1}(r,t)] = \hbar_i H_i(r,t) \mathcal{R}_{im}(\vec{u}_{1,m-1}, \cdots, \vec{u}_{n,m-1}, r, t), \tag{2.9}$$

where

$$\mathcal{R}_{im}(\vec{u}_{1,m-1}, \cdots, \vec{u}_{n,m-1}, r, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}_i[\varphi_1(r,t;p), \cdots, \varphi_n(r,t;p)]}{\partial p^{m-1}} \bigg|_{p=0} \tag{2.10}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases} \tag{2.11}$$

3. SYSTEM OF LINEAR EQUATIONS

Consider the linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{b}$ is a known vector and $\mathbf{x}$ is the unknown vector of variables. It is also assumed that the system has a unique solution. The i-th equation of this system can be expressed as follows:

$$a_{i,1}x_1 + \cdots + a_{i,i-1}x_{i-1} + a_{i,i}x_i + a_{i,i+1}x_{i+1} + \cdots + a_{i,n}x_n = b_i. \tag{3.1}$$

Without loss of generality, suppose that $a_{i,i} \neq 0$. It is now possible to obtain x_i from (12), which concludes that

$$x_i = \frac{b_i}{a_{i,i}} - \frac{1}{a_{i,i}} \Big(a_{i,1}x_1 + \cdots + a_{i,i-1}x_{i-1} + a_{i,i+1}x_{i+1} + \cdots a_{i,n}x_n \Big). \tag{3.2}$$

In the following, we will use HAM to find the solution of (13). It is concluded from (1) and (13) that

$$\mathcal{N}_i[x_1, \cdots, x_n] = x_i - \frac{b_i}{a_{ii}} + \frac{1}{a_{ii}} \Big(a_{i1}x_1 + \cdots + a_{i,i-1}x_{i-1} + a_{i,i+1}x_{i+1} + \cdots a_{in}x_n \Big) = 0. \tag{3.3}$$

In (8) and (9) for all i we define $H_i(r,t) = 1$, $u_{im}(r,t) = x_{im}$, $\hbar_i = \hbar$, and $\mathcal{L}_i$ the identity linear operator. Also, we consider the function $\varphi_i(p)$ as follows:

$$\varphi_i(p) = x_{i0} + \sum_{m=1}^{+\infty} x_{im} p^m \qquad (i = 1, 2, \cdots, n),$$
$$x_{im} = \frac{1}{m!} \frac{\partial^m \varphi_i(p)}{\partial p^m}\bigg|_{p=0}. \tag{3.4}$$

The mth-order deformation equation is obtained as

$$x_{im} - \chi_m x_{i,m-1} = \hbar \mathcal{R}_{im}(\overrightarrow{x}_{1,m-1}, \cdots, \overrightarrow{x}_{n,m-1}), \tag{3.5}$$

where

$$\mathcal{R}_{im}(\overrightarrow{x}_{1,m-1}, \cdots, \overrightarrow{x}_{n,m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}_i[\varphi_1(p), \cdots, \varphi_n(p)]}{\partial p^{m-1}}\bigg|_{p=0}$$
$$= \frac{1}{(m-1)!} \frac{\partial^{m-1}\Big(\varphi_i(p) - \frac{b_i}{a_{ii}} + \frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} \varphi_j(p)\Big)}{\partial p^{m-1}}\Bigg|_{p=0}$$
$$= \frac{1}{(m-1)!} \frac{\partial^{m-1}\Big(x_{i0} + \sum_{m=1}^{+\infty} x_{im} p^m - \frac{b_i}{a_{ii}} + \frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}\Big(x_{j0} + \sum_{m=1}^{+\infty} x_{jm} p^m\Big)\Big)}{\partial p^{m-1}}\Bigg|_{p=0}$$
$$= \begin{cases} x_{i0} - \frac{b_i}{a_{ii}} + \frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_{j0} & m = 1, \\ x_{i,m-1} + \frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_{j,m-1} & m > 1. \end{cases}$$

for $i = 1, 2, \cdots, n$. Assuming $x_{i0} = \frac{b_i}{a_{ii}}$ for $i = 1, \cdots, n$, we have

$$\begin{cases} x_{i1} = \frac{\hbar}{a_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_{j0} \\ x_{i,m} = (1+\hbar) x_{i,m-1} + \frac{\hbar}{a_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_{j,m-1} \qquad (m \geq 2). \end{cases} \tag{3.6}$$

In order to make a better comparison between this method and the Jacobi iterative method [10] we rewrite the HAM scheme in matrix form. Toward this end, let $\mathbf{D}$ stands for the diagonal matrix whose diagonal elements are the same as $\mathbf{A}$. It is also assumed that $-\mathbf{L}$ is the strictly lower-triangular part of $\mathbf{A}$, and $-\mathbf{U}$ is the strictly upper-triangular part of $\mathbf{A}$. Using this notations, $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$ and the equation $\mathbf{Ax} = \mathbf{b}$, or equivalently $(\mathbf{D} - \mathbf{L} - \mathbf{U})\mathbf{x} = \mathbf{b}$, yields

$$\mathbf{Dx} = (\mathbf{L} + \mathbf{U})\mathbf{x} + \mathbf{b}, \tag{3.7}$$

which can further be written as

$$\mathbf{x} = \mathbf{D^{-1}}(\mathbf{L} + \mathbf{U})\mathbf{x} + \mathbf{D^{-1}b}. \tag{3.8}$$

This results the following matrix form for the homotopy analysis method:

$$\begin{cases} \mathbf{x}^{(0)} = \mathbf{D}^{-1}\mathbf{b} \\ \mathbf{x}^{(1)} = \hbar\mathbf{D}^{-1}(\mathbf{L}+\mathbf{U})\mathbf{x}^{(0)} \qquad (m = 2, 3, \cdots) \\ \mathbf{x}^{(m)} = \mathbf{T}_{\mathbf{A}}(\hbar)\mathbf{x}^{(m-1)}, \end{cases} \tag{3.9}$$

where $\mathbf{T}_{\mathbf{A}}(\hbar) = (1+\hbar)\mathbf{I} + \hbar\mathbf{D}^{-1}(\mathbf{L}+\mathbf{U})$.
Note that the iteration matrix of the homotopy analysis method is similar to the Jacobi iterative method. But, the initial value of the homotopy analysis method is $\mathbf{D}^{-1}b$, while the initial value of the Jacobi iterative method is usually assumed to be the zero vector.

4. CONVERGENCE ANALYSIS OF THE PROPOSED METHOD

According to (20), it can be easily shown that

$$\mathbf{x}^{(m)} = \Big(\mathbf{T}_{\mathbf{A}}(\hbar)\Big)^{m-1}\mathbf{x}^{(1)}, \qquad (m = 2, 3, \cdots). \tag{4.1}$$

Hence, the solution can be written as

$$\mathbf{x} = \mathbf{x}^{(0)} + \mathbf{x}^{(1)} + \mathbf{x}^{(2)} + \mathbf{x}^{(3)} + \cdots, \tag{4.2}$$

or equivalently,

$$\mathbf{x} = \mathbf{x}^{(0)} + \Big[\mathbf{I} + \mathbf{T}_{\mathbf{A}}(\hbar) + (\mathbf{T}_{\mathbf{A}}(\hbar))^2 + (\mathbf{T}_{\mathbf{A}}(\hbar))^3 + \cdots\Big]\mathbf{x}^{(1)}. \tag{4.3}$$

Now, the following theorem can be proved.

Theorem 4.1. *The sequence*

$$\mathbf{x}^{[m]} = \mathbf{x}^{(0)} + \sum_{k=0}^{m}(\mathbf{T}_{\mathbf{A}}(\hbar))^k\mathbf{x}^{(1)}, \qquad (m = 0, 1, 2, \cdots), \tag{4.4}$$

is a Chauchy sequence if

$$\|\mathbf{T}_{\mathbf{A}}(\hbar)\| < 1. \tag{4.5}$$

Proof. We must show that

$$\lim_{m\to\infty}\left\|\mathbf{x}^{[m+p]} - \mathbf{x}^{[m]}\right\| = 0.$$

In order to show this, first we write

$$\mathbf{x}^{[m+p]} - \mathbf{x}^{[m]} = \left[\sum_{k=1}^{p}(\mathbf{T}_{\mathbf{A}}(\hbar))^{m+k}\right]\mathbf{x}^{(1)},$$

which yields

$$\left\|\mathbf{x}^{[m+p]} - \mathbf{x}^{[m]}\right\| \le \|\mathbf{x}^{(1)}\| \sum_{k=1}^{p}\|\mathbf{T}_{\mathbf{A}}(\hbar)\|^{m+k}.$$

Let $\gamma = \left\|\mathbf{T_A}(\hbar)\right\|$, then

$$\left\|\mathbf{x}^{[m+p]} - \mathbf{x}^{[m]}\right\| \leq \|\mathbf{x}^{(1)}\|\gamma^m \sum_{k=1}^{p} \gamma^k \leq \|\mathbf{x}^{(1)}\| \Big(\frac{\gamma^{p+1}-1}{\gamma-1}\Big)\gamma^m.$$

Now if $\gamma < 1$, then we have

$$\lim_{m\to\infty}\left\|\mathbf{x}^{[m+p]} - \mathbf{x}^{[m]}\right\| \leq \|\mathbf{x}^{(1)}\| \Big(\frac{\gamma^{p+1}-1}{\gamma-1}\Big)\lim_{m\to\infty}\gamma^m.$$

Hence, we obtain

$$\lim_{m\to\infty}\left\|\mathbf{x}^{[m+p]} - \mathbf{x}^{[m]}\right\| = 0,$$

which completes the proof. □

5. NUMERICAL RESULTS

In this section, two examples are presented. In the first example a system of linear equations is solved by the homotopy analysis method and the Jacobi iterative method and then the results are compared with each other.

Example 5.1. In this example, we approximate the solution of the system

$$\begin{cases} 10x_1 + x_2 + x_3 = 24 \\ -x_1 + 20x_2 + x_3 = 21 \\ x_1 - 2x_2 + 100x_3 = 300 \end{cases}$$

where the exact solution of which is $\mathbf{x} = (x_1, x_2, x_3)^t = (2, 1, 3)^t$. The comparison will be made by applying the two methods separately.

(i) *The Homotopy Analysis Method*
From (20), we have $x_{1,0} = \frac{24}{10}$, $x_{2,0} = \frac{21}{20}$, and $x_{3,0} = \frac{300}{100}$,

$$\begin{cases} x_{1,m+1} = (1+\hbar) + \frac{\hbar}{10}\big(x_{2,m} + x_{3,m}\big) \\ x_{2,m+1} = (1+\hbar) + \frac{\hbar}{20}\big(-x_{1,m} + x_{3,m}\big) \\ x_{3,m+1} = (1+\hbar) + \frac{\hbar}{100}\big(x_{1,m} - 2x_{2,m}\big) \end{cases} \qquad (m = 0, 1, 2, \cdots)$$

For $\hbar = -1$ and $\hbar = -0.99$, the solutions in the first five iterations are listed in Tables 1 and 2 respectively .
(ii) *The Jacobi iterative method*
We convert $Ax = b$ into the form $x = Tx + c$. To do this, solve i-th equation for $x_i, i = 1, 2, 3$ to obtain:

$$\begin{cases} x_1 = \frac{1}{10}(-x_2 - x_3) + \frac{24}{10}, \\ x_2 = \frac{1}{20}(x_1 - x_3) + \frac{21}{20}, \\ x_3 = \frac{1}{100}(-x_1 + 2x_3) + \frac{300}{100}. \end{cases} ,$$

For an initial approximation, we let $x^{(0)} = (0, 0, 0)$. Approximate solutions are presented in Table 3.

Table 1. The homotopy analysis method for $\hbar = -1$

Iteration	x_1	x_2	x_3
0	2.400000	1.050000	3.000000
1	1.995000	1.020000	2.997000
2	1.998300	0.999900	3.000450
3	1.999965	0.999893	3.000015
4	2.000009	0.999998	2.999998
5	2.000000	1.000001	3.000000

Table 2. The homotopy analysis method for $\hbar = -0.99$

Iteration	x_1	x_2	x_3
0	2.400000	1.050000	3.000000
1	2.009050	1.030300	3.007030
2	2.020304	1.020600	3.020510
3	2.029930	1.030490	3.030207
4	2.037991	1.040490	3.040310
5	2.046001	1.050390	3.050425

Table 3. Jacobi method

Iteration	x_1	x_2	x_3
0	0.000000	0.0000000	0.000000
1	2.400000	1.0500000	3.000000
2	1.995000	1.0200000	2.997000
3	1.998300	0.9999000	3.000450
4	1.999965	0.9998925	3.000015
5	2.000009	0.9998975	2.999998
6	2.000000	1.0000010	3.000000

Considering the Tables 1 and 2, we see that the two Tables are the same for the first rows. In other words, if the initial values in the Jacobi iterative method are selected $x^{(0)} = (\frac{24}{10}, \frac{21}{20}, \frac{300}{100})$, then the Adomian decomposition method and the Jacobi iterative method are exactly the same [4].

Example 5.2. The aim of this example is to approximate the solution of the system

$$\begin{cases} 4x_1 + x_2 - x_3 = 7 \\ -x_1 + 6x_2 + 2x_3 = 9 \\ x_2 - 3x_3 = 5 \end{cases},$$

where the exact solution of which is $\mathbf{x} = (x_1, x_2, x_3)^t = (1, 2, -1)^t$. From (20), we have $x_{1,0} = \frac{7}{4}$, $x_{2,0} = \frac{9}{6}$, and $x_{3,0} = -\frac{5}{3}$. The six term approximation of the solution vector $\mathbf{x}$ is

$$\mathbf{x} \simeq \mathbf{x}_0 + \mathbf{x}_1 + \cdots + \mathbf{x}_5,$$

which yields

$$\hbar = -1.1 \Rightarrow \mathbf{x}^t \simeq [0.92092, 2.06033, -0.93491],$$
$$\hbar = -0.9 \Rightarrow \mathbf{x}^t \simeq [1.07051, 1.95926, -1.06693],$$
$$\hbar = -1.0 \Rightarrow \mathbf{x}^t \simeq [0.99565, 2.00984, -1.00087].$$

References

[1] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, PhD thesis, Shanghai Jiao Tong University, 1992.

[2] S.J. Liao, An approximate solution technique which does not depend upon small parameters: a special example, *Int J Nonlinear Mech,* **30** *(1995), 371-380.*

[3] S.J. Liao, Homotopy analysis method: a new analytical technique for nonlinear problems, *Commun Nonlinear Sci Numer Simulat,* **2** *(1997), 95-100.*

[4] S.J. Liao, Beyond perturbation: introduction to the homotopy analysis method, *Boca Raton: CRC Press, Chapman & Hall; 2003.*

[5] H. Jafari, S. Seifi, A. Alipoor and M. Zabihi, An iterative method for solving linear and nonlinear fractional diffusion-wave equation, *J. Nonlinear Fract Phenom Sci. Eng., in press.*

[6] H. Jafari and S. Seifi, Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation, *Commun Nonlinear Sci. Numer Simulat,* **14** *(2009), 2006-2012.*

[7] A.S. Bataineh, M.S.M. Noorani and I. Hashim, Approximate analytical solutions of systems of PDEs by homotopy analysis method, *J Comput Math Appl,* **55** *(2008), 2913-2923.*

[8] H. Xu, S.J. Liao and X.C. You, Analysis of nonlinear fractional partial differential equations with the homotopy analysis method, *Commun Nonlinear Sci Numer Simulat,* **14** *(2009), 1152-1156.*

[9] A.M. Wazwaz, The variational iteration method for solving linear and nonlinear systems of PDEs, *Comput Math Appl,* **54** *(2007), 895-902.*