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On The Existence Of Nonnegative Solutions For Class Of Fractional Boundary Value Problems

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Abstract. In this paper, sufficient conditions for the existence of nonnegative solutions of a boundary value problem for a fractional order differential equation are provided. By applying Kranoselskii's fixed–point theorem in a cone, the existence of solutions of an auxiliary BVP formulated by truncating the response function is first proved. Then the Arzela–Ascoli theorem is used to take C^1 limits off sequences of such solutions.
Keywords: Boundary value problem; Nonnegative solutions; Caputo fractional derivative; Equicontinuous sets.

1. INTRODUCTION

Fractional differential equations have gained a considerable importance due to their varied applications in the field of visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of Physics, Chemistry and Biological sciences. So far there have been several fundamental works on the fractional derivative and fractional differential equations, written by Miller and Ross [4], Podlubny [5] and others [6–8]. Mathematical aspects of fractional order differential equations have been discussed in details by many authors [9–17].
Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. For boundary value

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problems with integral boundary conditions and comments on their importance, we refer the reader to the papers [1–4, 8, 11–15] and the references therein. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors, see [5, 6, 9, 10, 12]. The goal of this paper is to give existence and uniqueness results for the problem (1.1)-(1.3). Our approach here is based on the Krasnoselskii's fixed–point theorem, the Arzela–Ascoli theorem and the Banach contraction principle [20, 21, 27].
I established sufficient conditions ensuring that a fractional order differential equation admits a nonnegative solution, whose slope at the end of times depends on its values on the whole time interval. To be more precise, consider the following fractional differential equation:

$$\frac{d}{dt}\{{}^cD^{\alpha}_{0+}y(t)\} + q(t)f(y(t),\, y'(t)) = 0\,, \quad \text{a.a. } t \in [0,\ 1], \tag{1.1}$$

associated with the boundary conditions

$$y(0) = 0, \quad y'(0) = \upsilon > 0, \tag{1.2}$$

and

$${}^cD^{\alpha}_{0+}y(1) = \int_0^1 {}^cD^{\alpha}_{0+}y(s)\, dg(s), \tag{1.3}$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative of order $\alpha \in (1,\ 2)$, $f : [0,\ 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, $g : [0,\ 1] \longrightarrow [0,\ \infty)$ are given functions and in (1.3) the integral is meant in the Riemann–Stieljes sense.

2. THE BASIC TOOLS

We begin in this section with recall and introduce some notations, definitions and preliminary facts that will be used in the remainder of this paper [4, 5, 7, 20, 21].
We shall denote by $\mathbb{R}$ the real line, by $\mathbb{R}^+$ the interval $[0,\infty)$, and by I the interval $[0,\ 1]$. Let also $C^1_0(I)$ be the space of all functions $y : I \longrightarrow \mathbb{R}$, whose first derivative y' is absolutely continuous on I and $y(0) = 0$. The set $C^1_0(I)$ is a Banach space when it is furnished with the norm $\|\ \|$ defind by $\|y\| = \sup\{|y'(t)| \,:\, t \in I\}$. We denote by $L^1(I)$ the space of all functions $y : I \longrightarrow \mathbb{R}$ which are Lebesgue integrable on I, endowed with the usual norm $\|y\|_1 = \int_0^1 |y(t)|\, dt$. Finally $L^1_+(I)$ the space of all functions $y : I \longrightarrow \mathbb{R}^+$ which are Lebesgue integrable on I, endowed with the norm $\|y\|_1$.

A very usual technique to get such results is based on fixed-point theorems in cones and especially on the following well–known fixed–point theorem due to Krasnoselskii [21].

Theorem 2.1. *Let $\mathcal{B}$ be a Banach space and let $\mathbb{K}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and let*

$$F : \mathbb{K} \cap \left(\Omega_2 \backslash \overline{\Omega}_1\right) \longrightarrow \mathbb{K}$$

be a completely continuous operator such that either

$$\|Fu\| \leq \|u\|, \quad u \in \mathbb{K} \cap \partial\Omega_1, \quad \|Fu\| \geq \|u\|, \quad u \in \mathbb{K} \cap \partial\Omega_2$$

or

$$\|Fu\| \geq \|u\|, \quad u \in \mathbb{K} \cap \partial\Omega_1, \quad \|Fu\| \leq \|u\|, \quad u \in \mathbb{K} \cap \partial\Omega_2.$$

Then F has a fixed point in $\mathbb{K} \cap \Omega_2 \backslash \overline{\Omega}_1$.

Definition 2.2. A map $f : I \longrightarrow \mathbb{R}$ is said to be L^1-Caratheodory if

(i) $t \longrightarrow f(t,\, u)$ is measurable for each $u \in \mathbb{R}$;
(ii) $t \longrightarrow f(t,\, u)$ is continuous for almost each $t \in I$;
(iii) for every $r > 0$ there exists $h_r \in L^1(I)$ such that $|f(t,\, u)| \leq h_r(t)$ for a.e. $t \in I$ and all $|u| \leq r$.

Definitions of Caputo and Remann$-$Liouville fractional derivative/integral and their relation are given below.

Definition 2.3. For a function u defined on an interval $[a,\ b]$, the Remann$-$Liouville fractional integral of f of order $\alpha > 0$ is defined by

$$I^{\alpha}_{a^+} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\, ds, \quad t > a,$$

and Remann$-$Liouville fractional derivative of u of order $\alpha > 0$ defined by

$$D^{\alpha}_{a^+} y(t) = \frac{d^n}{dt^n} \left\{ I^{n-\alpha}_{a^+} y(t) \right\},$$

where $n - 1 < \alpha \leq n$ while Caputo fractional derivative of y of order $\alpha > 0$ defined by is defined by

$${}^{c}D^{\alpha}_{a^+} y(t) = I^{n-\alpha}_{a^+} \left\{ y^{(n)}(t) \right\}.$$

An important of relation among of Caputo fractional derivative and Riemanna$-$Liouville fractional derivative is the following expression

$$D^{\alpha}_{a^+} y(t) = {}^{c}D^{\alpha}_{a^+} y(t) + \sum_{j=1}^{n-1} \frac{y^{(j)}(a)}{\Gamma(j-\alpha+1)} (t-a)^{j-\alpha} \tag{2.1}$$

We denote ${}^{c}D^{\alpha}_{a^+} y(t)$ as ${}^{c}D^{\alpha}_{a} y(t)$ and $I^{\alpha}_{a^+} y(t)$ as $I^{\alpha}_{a} y(t)$. Further ${}^{c}D^{\alpha}_{0^+} y(t)$ and $I^{\alpha}_{0^+} y(t)$ are referred as ${}^{c}D^{\alpha} y(t)$ and $I^{\alpha} y(t)$, respectively.

Theorem 2.4. *Let* $y \in C^m([0, b], \mathbb{R})$ *and* $\alpha, \beta \in (m-1, m)$, $m \in \mathbb{N}$. *Then*

(1) ${}^cD^\alpha I^\alpha y(t) = y(t)$.
(2) $I^\alpha I^\beta y(t) = I^{\alpha+\beta} y(t)$.
(3) $\lim_{t\to 0^+}\{{}^cD^\alpha y(t)\} = \lim_{t\to 0^+}\{I^\alpha y(t)\}$.
(4) ${}^cD^\alpha \lambda = 0$, *where* λ *is constant.*
(5) $I^\alpha\{{}^cD^\alpha y(t)\} = y(t) - \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{k!} t^k$.
(6) ${}^cD^\alpha\{D^m y(t)\} = {}^cD^{\alpha+m} y(t)$, $m = 0, 1, 2, \cdots$.
(7) *The interchange of the differentiation operators in formula (6) is allowed under conditions:*

$$ {}^cD^\alpha\{{}^cD^n y(t)\} = {}^cD^n\{{}^cD^\alpha y(t)\} = {}^cD^{\alpha+n} y(t), $$

as $y^{(j)}(0) = 0$ *for* $j = m, m+1, \cdots, n$.

Lemma 2.5. *(Lemma 2.22 [7]). Let* $\alpha > 0$. *Then* $I^\alpha({}^cD^\alpha y(t)) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{r-1} t^{r-1}$ *for some* $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, r-1$, $r = [\alpha] + 1$.

3. THE MAIN RESULTS

Consider equation (1.1) associated with the conditions (1.2), (1.3). It is clear that, without loss of generality, we can assume that $g(0) = 0$. By a solution of this boundary value problem for we mean a function $y \in C_0^1(I)$ satisfying condition (1.3), as well as equation (1.1) for almost all $t \in I$. Searching for the existence of solutions, we shall first reformulate the problem to an operator equation of the form $y = Fy$, where F is a suitable operator. To find F, consider an equation of the form

$$ \frac{dz(t)}{dt} = -h(t), \quad \text{a.e. on } I, \tag{3.1} $$

subject to condition (1.1), (1.3), where

$$ {}^cD^\alpha y(t) = z(t). \tag{3.2} $$

By integration in Eq. (3.1) we get

$$ z(t) = z(1) + \int_t^1 h(r)dr, \quad t \in I. \tag{3.3} $$

Using Eq. (3.2) into Eq. (3.3) yields

$$ {}^cD^\alpha y(t) = {}^cD^\alpha y(1) + \int_t^1 h(r)dr, \quad t \in I. \tag{3.4} $$

Now, multiply equation (3.4) by $dg(t)$ and integrate over [0, 1], to get

$$\begin{aligned}\int_0^1 {}^cD^\alpha y(t)dg(t) &= {}^cD^\alpha y(1)\int_0^1 dg(t) + \int_0^1\int_t^1 h(r)drdg(t)\\ &= {}^cD^\alpha y(1)(g(1)-g(0)) + \int_0^1\int_t^1 h(r)drdg(t)\end{aligned}$$

Therefore from (1c) it follows that

$$ {}^cD^\alpha y(1) = \gamma\int_0^1\int_t^1 h(r)\,dr\,dg(t), \qquad (3.5)$$

where

$$\gamma = \frac{1}{1-g(1)}$$

Substituting Eq. (3.5) into Eq. (3.4) we obtain

$$ {}^cD^\alpha y(t) = \gamma\int_0^1\int_t^1 h(r)\,dr\,dg(t) + \int_t^1 h(r)dr.$$

Using Lemma 2.5 we get

$$y(t) = -c_0 - c_1 t + \frac{\gamma\, t^\alpha}{\Gamma(\alpha+1)}\int_0^1\int_t^1 h(r)\,dr\,dg(t) + \frac{1}{\Gamma(\alpha)}\int_0^t\int_s^1 (t-s)^{\alpha-1}h(r)drds,$$

and

$$y'(t) = -c_1 + \frac{\gamma\, t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1\int_t^1 h(r)\,dr\,dg(t) + \frac{\alpha-1}{\Gamma(\alpha)} + \int_0^t\int_s^1 (t-s)^{\alpha-2}h(r)drds.$$

Applying the condition (1.2), we find that

$$y(t) = \upsilon t + \frac{\gamma\, t^\alpha}{\Gamma(\alpha+1)}\int_0^1\int_t^1 h(r)\,dr\,dg(t) + \frac{1}{\Gamma(\alpha)}\int_0^t\int_s^1 (t-s)^{\alpha-1}h(r)drds,$$

where $\upsilon = y'(0) > 0$. This process shows that solving the boundary value problem (1.1)–(1c) is equivalent to solving the operator equation $y = Fy$ in $C_0^1(I)$ where F is the operator defined by

$$\begin{aligned}(Fy)(t) &= \upsilon t + \frac{\gamma\, t^\alpha}{\Gamma(\alpha+1)}\int_0^1\int_t^1 q(u)f(y(u,\, y'(u))\,du\,dg(t)\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t\int_s^1 (t-s)^{\alpha-1}q(u)f(y(u,\, y'(u))\,duds. \qquad (3.6)\end{aligned}$$

Before presenting the following results, we give the notation and the list of our assumptions, used in this paper. Let

$$\sigma := \|q\|_1\{\gamma g(1) + \upsilon + 1\}$$

and $\mathbb{K}_+ := \{y \in C_0^1(I) : y \geq 0,\, y \text{ is nondecreasing and } y' \text{ is nonincreasing}\}$ which is a cone in $C_0^1(I)$.

(H1) The function f is a real–valued continuous function defined at least on $I \times \mathbb{R}^2$, satisfying the inequality $f(u, v) \geq 0$ when $u \geq 0, v \geq 0$. Also, $q \in L^1_+(I)$ and $g : I \to \mathbb{R}$ is nondecreasing function.
(H2) There exists a nondecreasing function $\psi : \mathbb{R}^+ \to (0, +\infty)$ such that $f(u, v) \leq \psi(v)$ for all $(u, v) \in \mathbb{R}^+$.
(H3) The function $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is an L^1-Caratheodory,
(H4) There exists a positive real value μ such that $|f(u(t), u'(t)) - f(v(t), v'(t))| \leq \mu|u(t) - v(t)|$ for all $u, v \in C^1_0(I)$ and $t \in [0, 1]$. Moreover, $\mu\|q\|_1 < \Gamma(\alpha + 1)$.

Lemma 3.1. *Consider the functions f, q and g satisfying assumptions (H1)-(H4). Then the operator $F : C^1_0(I) \longrightarrow C^1_0(I)$ is completely continuous and the operator F maps the cone $\mathbb{K}_+$ into itself.*

Proof. The proof will be given in three steps.
Step 1. F is continuous.
Let y_m be a sequence such that $y_m \to y$ in $C^1_0(I)$. Then

$$\begin{aligned} |F(y_m)(t) - F(y)(t)| &\leq \int_0^t \int_s^1 \frac{|q(u)|\,|f(y_m(u), y'_m(u)) - f(y(u), y'(u))|}{\Gamma(\alpha)(t-s)^{1-\alpha}}\, duds, \\ &\leq \frac{\mu\|q\|_1\|y_m - y\|}{\Gamma(\alpha)} \int_0^t \int_s^1 (t-s)^{\alpha-1} du\, ds \\ &\leq \frac{\mu\|y_m - y\|}{\Gamma(\alpha+1)} t^\alpha. \end{aligned}$$

Since f is continuous and y_m, y belong to $C^1_0(I)$, then $\|F(y_m)(t) - F(y)(t)\| \to 0$ as $m \to \infty$.
Step 2. F maps bounded sets into bounded sets in $C^1_0(I)$.
Indeed, it is enough to show that there exists a positive constant ρ such that for each $y \in B_r = \{y \in C^1_0(I) : \|y\| \leq r\}$ one has $\|F(y)\| \leq \rho$. Let $y \in B_r$. Then by $(\mathbf{H}_2)$, for each $t \in [0, 1]$ we have

$$\begin{aligned} \|F(y)(t)\| &\leq \|q\|_1 \psi(y'(0))\{\gamma g(1) + 1\} + \upsilon \\ &\leq \|q\|_1 \psi(\|y\|)\{\gamma g(1) + 1\} \leq \|q\|_1 \psi(r)\{\gamma g(1) + 1\} + \upsilon := \rho \end{aligned}$$

Step 3. F maps bounded sets into equicontinuous sets of $C^1_0(I)$.
Let $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ and B_r be a bounded set of $C^1_0(I)$ as in step 2. Let $y \in B_r$ and $t \in [0, 1]$ we have

$$\begin{aligned} |F(y)(t_2) - F(y)(t_1)| &\leq |F(y)(t_2) - F(y)(t_1)| \\ &\leq \upsilon|t_2 - t_1| + \gamma(g(1) + 1)\|q\|\|\psi\||t_2^\alpha - t_1^\alpha|. \end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero. Then $F(B_r)$ is equicontinuous. As a consequence of Step 1 to 3 together with Arzela-Ascoli theorem we can conclude that $F : C^1_0(I) \longrightarrow C^1_0(I)$

is completely continuous.
It is easy to see that, under condition (H1), the operator F maps the cone $\mathbb{K}_+$ into itself. $\square$

Lemma 3.2. *Consider the functions f, q and g satisfying assumption (H1). Then there exists $m > 0$ such that for any $y \in \mathbb{K}_+$ with $\|y\| = m$, we have $\|Fy\| \geq \|y\|$.*

Proof. We assum the contrary. Then for every positive integer n, there exists a function $y_n \in \mathbb{K}_+$, with $\|y_n\| = n^{-1}$ and $\|Fy_n\| < \|y_n\|$. Let $z_n = y_n'$. Then for all n and every $s \in [0, 1]$ we have

$$0 \leq z_n(s) \leq z_n(0) = \|y_n\|,$$

which implies that $z_n \to 0$ in $AC(I)$. So, we must have

$$0 \geq \lim_{n\to\infty} z_n(0) \geq \lim_{n\to\infty} (Fy_n)'(0) = \upsilon$$

which is a contracdiction.

$\square$

Now we are ready to give our first main result.

Theorem 3.3. *Consider the functions f, q and g satisfying assumptions (H1) and (H3). Then the boundary value problem (1.1)–(1.3) has at least one solution.*

Proof. For each positive integer n, define the function

$$f_n(u, v) = \min\{f(u, v),\ n\}$$

and consider the problem (3.7), (1.2), (1.3) where Eq. (3.7) stands for the equation

$$^cD^\alpha y(t) = q(t) f_n(y(t),\, y'(t)), \quad \text{a.a } t \in [0, 1]. \tag{3.7}$$

From (H_3), we have $f_n(u, v) \leq \psi(v)$ for all $u, v \in \mathbb{R}^+$, $n = 1, 2, \cdots$.
Since the function f_n satisfies Assumption (H1), by Lemma 3.2, there exists a positive real number m_n such that for every $y \in \mathbb{K}_+$ with $\|y\| = m_n$, it holds that $\|F_n y\| \geq \|y\|$, where

$$\begin{aligned} (F_n y)(t) &= \upsilon t + \frac{\gamma\, t^\alpha}{\Gamma(\alpha+1)} \int_0^1 \int_t^1 q(u) f_n(y(u,\, y'(u))\, du\, dg(t) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_s^1 (t-s)^{\alpha-1} q(u) f_n(y(u,\, y'(u))\, duds. \end{aligned} \tag{3.8}$$

Moreover, if $y \in \mathbb{K}_+$ is such that $\|y\| = n\sigma =: M_n$, then

$$|F_n y(t)| \leq \upsilon + \gamma n \int_0^1 \int_t^1 |q(s)| dudg(t) + n \int_0^t |q(u)| du.$$

Then,

$$\|F_n y\| \leq n\|q\|_1\{\gamma g(1) + \upsilon + 1\} = n\sigma = M_n = \|y\|.$$

Hence, by Theorem 2.1, there exists a solution $y_n \in C_0^1(I)$ of the problem (3.2), (1.2), (1.3), such that $m_n \leq \|y_n\| \leq M_n$.
Now, prove that $\Omega := \{y \in C_0^1(I):$ y is solution of the (1.1)–(1.3)$\}$ is compact. Let $\{y_m\}_{m=1}^{\infty}$ be a sequence in Ω, then

$$\begin{aligned} y_m(t) &= \upsilon t + \frac{\gamma\, t^{\alpha}}{\Gamma(\alpha+1)} \int_0^1 \int_t^1 q(u) f(y_m(u),\, y_m'(u))\, du\, dg(t) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_s^1 (t-s)^{\alpha-1} q(u) f(y_m(u),\, y_m'(u))\, duds. \end{aligned}$$

As in Steps 2 and 3 we can easily prove that there exists $L > 0$ such that $\|y_m\| < L$, for all $m \geq 1$, and the set $\{y_m\}_{m=1}^{\infty}$ is equicontinuous in $C_0^1(I)$. Hence by Arzela-Ascoli theorem we can conclude that there exists a subsequence of $\{y_m\}_{m=1}^{\infty}$ converging to y in $C_0^1(I)$. Using f is an L^1-Carathedory we can prove that

$$\begin{aligned} y(t) &= \upsilon t + \frac{\gamma\, t^{\alpha}}{\Gamma(\alpha+1)} \int_0^1 \int_t^1 q(u) f(y(u),\, y'(u))\, du\, dg(t) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_s^1 (t-s)^{\alpha-1} q(u) f(y(u),\, y_m(u))\, duds. \end{aligned}$$

Therefore Ω is compact. □

Theorem 3.4. *Consider the functions f, q and g satisfying assumptions (H1) and (H4). Then the boundary value problem (1.1)–(1.3) has a unique solution in* $C_0^1(I)$.

Proof. It is clear to show that, F is a contraction and hence the Banach contraction principle yields that F has a unique fixed point which is a solution to (1.1)-(1.2). □

Example 3.5. Consider the following boundary value problem:

$$\frac{d}{dt}\left\{{}^cD^{\frac{3}{2}} y(t)\right\} = e^{-t}\left\{\frac{1}{2}\cos^2 y(t) + y'(t)\right\}, \quad t \in [0,\, 1], \tag{3.9}$$

$$y(0) = 0, \quad y'(0) = \upsilon > 0 \tag{3.10}$$

$${}^cD^{\frac{3}{2}} y(1) = \int_0^1 {}^cD^{\frac{3}{2}} y(s) dg(s). \tag{3.11}$$

We observe that

$$g(t) = \begin{cases} 0, & \text{if } 0 \leq t < \frac{1}{2}, \\ \frac{1}{2}, & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

the boundary condition (1.3) reduces to the boundary condition (11c). Moreover, if we set

$$f(u, v) = \frac{1}{2}\cos^2 u + v \ \text{ and } \ \psi(v) := \frac{1}{2} + v,$$

we can see that $f(u, v) \leq \psi(v)$. Then by Theorem 3.3 the boundary value problem (3.1)–(3.3) has atleast one solution.

4. CONCLUSIONS

Nonnegative solutions for nonlinear fractional differential equations comprising of standard Caputo fractional derivative have been discussed. The conditions on coefficients when the solutions are unique and further unique as well as positive, have been worked out. The present work provides insights in the equations encountered in fractional order dynamical systems and controllers which further may be explored.

References

[1] W. G. Glockle, T. F. Nonnenmacher, A fractional calculus approach of self-similar protein dynamics, *Biophys. J.,* **68** *(1995) 46-53.*

[2] R. Hilfer, Applications of fractional calculus in phisics, *World Scientific, Singapore, 2000.*

[3] F. Metzler, W. Schick, H. G. Kilian, T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phy.,* **103** *(1995) 7180-7186.*

[4] K. B. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, *Wiely, New york, 1993.*

[5] I. Podlubny, Fractional differential equations, *Academic Press, New York, 1999.*

[6] D. Baleanu, A. K. Golmankhaneh, R. Nigmatullin, R., Fractional Newtonian mechanics, *Central European Journal of Physics,* **8** *(2010) 120-125.*

[7] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives: Theory and applications, *Gordon and Breach, Yverdon, 1993.*

[8] B. N. Lundstrom, M. H. Higgs, W. J. Spain and A. L. Fairhall, Fractional differentiation by neocortical pyramidal neurons, *Nature Neuroscience,* **11** *(11) (2008).*

[9] I. Poudlubny, Geometric and physical interpretation of fractional integration and fractional differentiation, *Fract. Calc. Appl. Anal.,* **5** *(2002) 367-386.*

[10] A. A. Kilbas, Hari M. Srivastava, Juan J. Trujillo, Theory and applications of fractional differential equations, in: *Nrth-Holland Mathematics Studies,* **204***, Elsevier Science B. V, Amesterdam, 2006.*

[11] V. Lashmikantham, S. Leela, J. Vasundhara, Theory of fractional dynamic systems, *Cambridge Academic Publishers, Cambridge, 2009.*

[12] R. P Agarwal, M. Benchohra, S. Hamani, Asurvey on existence result for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.,* **109** *(2010) 973-1033.*

[13] Varsha Daftardar- Gejji, A. Babakhani, Analysis of a System of Fractional Differential Equations, *Journal of Mathematical Analysis and Applications,* **293** *(2004) 511-522.*

[14] A. Babakhani, E. Enteghami, Existence of positive solutions for multiterm fractional differential equations of finite delay with polynomial coefficients, *Abstract and Applied Analysis, Vol. 2009, Article ID 768920, (2009).*

[15] A. Babakhani, Positive solutions for system of nonlinear fractional differential equations in two dimensions with delay, *Abstract and Applied Analysis,* **2010** *(2010), Article number 536317.*

[16] D. Baleanu, J. J. Trujillo, A new method of finding the fractional Euler-Lagrange and Hamilton equations within Caputo fractional derivatives, *Communications in Nonlinear Science and Numerical Simulation,* **15** *(5) (2010) 1111-1115.*

[17] D. Baleanu, J. J. Trujillo, New applications of fractional variational principles, *Rep. Math. Phys.,* **61** *(2008) 331-335.*

[18] Jinhua Wang, Honjun Xiang, and Zhigang Liu, Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations, *Hindawi Publishing Corporation, Inter. Jour. Diff. Equ.,* **2010** *, Article ID 186928, (2010).*

[19] A. Arara, M. Benchohra, N. Hamidi, J. J. Nieto, Fractional order differential equations on an unbounded domain, *Nonlinear Anal.,* **72** *(2010) 580-586.*

[20] R. R. Goldberg, Methods of Real Analysis, *Oxford and IFH Publishing Company, New Delhi, 1970.*

[21] A. Granas, J. Dugundji, Fixed point theory, *Springer Verlag, 2003.*

[22] C. Li, W. Deng, Remarks on fractional derivatives, *Appl. Math. Comput.,* **187** *(2007) 777-t84.*

[23] C. Li, W. Deng, Remarks on fractional derivatives, *Appl. Math. Comput.,* **187** *(2007) 777-784.*

[24] R. Hilfer, Applications of fractional calculus in physics, *Singapore, World Scientific, (2000).*

[25] R. Gorenflo and S. Vessella, Abel integral equations: Analysis and applications, *Lecture Notes in Mathematics, 1461 Springer-Verlag, Berlin, (1991).*

[26] Rabha W. Ibrahim and Shaher Momani, Upper and lower bounds of solutions for fractionla integral equations, *Surveys in Mathematics and its Applications,* **2** *(2007), 145-156.*

[27] M.C. Joshi, R.K. Bose, Some Topics in Nonlinear Functional Analysis, Wiley Eastern, New Delhi, (1985).