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Fixed Point Approach To The Hyers-Ulam-Rassias Approximation Of Homomorphisms And Derivations On Non-Archimedean Random Lie C^*-Algebras

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Abstract. In this paper, using a fixed point method, we prove the generalized Hyers-Ulam stability of random homomorphisms in random C^*-algebras and random Lie C^*-algebras and of derivations on Non-Archimedean random C*-algebras and Non-Archimedean random Lie C*-algebras for the following m-variable additive functional equation:

$$\sum_{i=1}^{m} f(x_i) = \frac{1}{2m}\left[\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1\ ,i\neq j}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right)\right].$$

Keywords: Additive functional equation, fixed point, Non-Archimedean random space, homomorphism in C^*-algebras and Lie C^*-algebras, generalized Hyers-Ulam stability, derivation on C^*-algebras and Lie C^*-algebras

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations was originated from a question of Ulam [34] concerning about the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group (a metric which is defined on a set with group property) with the metric

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$d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x*y) = H(x)\diamond H(y)$ is stable (see also [17, 22]).
The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem ([1]–[28]). By a *non-Archimedean field* we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r|\,|s|$, and $|r + s| \le \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. Let X be a vector space over a field K with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$; (ii) for any $r \in K, x \in X$, $\|rx\| = |r|\|x\|$; (iii) the strong triangle inequality (ultrametric) holds; namely,

$$\|x + y\| \le \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space. From the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n-1\} \qquad (n > m),$$

holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

A non-Archimedean Banach algebra is a complete non-Archimedaen algebra $\mathcal{A}$ which satisfies $\|ab\| \le \|a\|.\|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions of non-Archimedean Banach algebras, we provide the following studied for further reading [11, 33].

If $\mathcal{U}$ is a non-Archimedean Banach algebra, then an involution on $\mathcal{U}$ is a mapping $t \to t^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

(i) $t^{**} = t$ for $t \in \mathcal{U}$; (ii) $(\alpha s + \beta t)^* = \overline{\alpha} s^* + \overline{\beta} t^*$; (iii) $(st)^* = t^* s^*$ for $s, t \in \mathcal{U}$.

If, in addition $\|t^*t\| = \|t\|^2$ for $t \in \mathcal{U}$, then $\mathcal{U}$ is a non-Archimedean C^*–algebra.

We recall a fundamental result in a fixed point theory. Let Ω be a set. A function $d : \Omega \times \Omega \to [0, \infty]$ is called a *generalized metric* on Ω if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$; (2) $d(x, y) = d(y, x)$ for all $x, y \in \Omega$; (3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in \Omega$.

Theorem 1.1. [10] *Let (Ω, d) be a complete generalized metric space and let $J : \Omega \to \Omega$ be a contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in \Omega$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \ge n_0$; (2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J; (3) y^* is the unique fixed point of J in the set $\Gamma = \{y \in \Omega \mid d(J^{n_0}x, y) < \infty\}$; (4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in \Gamma$.

In this paper, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean random C^*-algebras and non-Archimedean random Lie C^*-algebras for the following additive functional equation (see [**?**])

$$\sum_{i=1}^{m} f(x_i) = \frac{1}{2m}\left[\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1\ ,i\neq j}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right)\right] \tag{1.1}$$

2. RANDOM SPACES

In the section, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [9, 31, 32]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.1. ([31]) A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the

following conditions:
(a) T is commutative and associative; (b) T is continuous;
(c) $T(a,1)=a$ for all $a\in[0,1]$; (d) $T(a,b)\leq T(c,d)$ whenever $a\leq c$ and $b\leq d$ for all $a,b,c,d\in[0,1]$.

Typical examples of continuous t-norms are $T_P(a,b)=ab$, $T_M(a,b)=\min(a,b)$ and $T_L(a,b)=\max(a+b-1,0)$ (the Lukasiewicz t-norm).

Definition 2.2. ([32]) A *non- Archemidean random normed space* (briefly, NA-RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\mu_x(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=0$; (RN2) $\mu_{\alpha x}(t)=\mu_x(\frac{t}{|\alpha|})$ for all $x\in X$, $\alpha\neq 0$; (RN3) $\mu_{x+y}(t)\geq T(\mu_x(t),\mu_y(t))$ for all $x,y\in X$ and all $t\geq 0$.

Every normed space $(X,\|.\|)$ defines a non- Archemidean random normed space (X,μ,T_M), where $\mu_x(t)=\frac{t}{t+\|x\|}$ for all $t>0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 2.3. [32] A *non- Archemidean random normed algebra* (X,μ,T,T') is a non- Archemidean random normed space (X,μ,T) with algebraic structure such that

(RN-4) $\mu_{xy}(t)\geq T'(\mu_x(t),\mu_y(t))$ for all $x,y\in X$ and all $t>0$. in which T' is a continuous t-norm.

Every non- Archemidean normed algebra $(X,\|\cdot\|)$ defines a non- Archemidean random normed algebra (X,μ,T_M), where $\mu_x(t)=\frac{t}{t+\|x\|}$ for all $t>0$ iff

$$\|xy\|\leq\|x\|\|y\|+t\|y\|+t\|x\|\ (x,y\in X;\ t>0).$$

This space is called the induced non- Archemidean random normed algebra.

Definition 2.4. (1) Let (X,μ,T_M) and (Y,μ,T_M) be non- Archemidean random normed algebras. An $\mathbb{R}$-linear mapping $f:X\to Y$ is called a *homomorphism* if

$$f(xy)=f(x)f(y)$$

for all $x,y\in X$.

(2) An $\mathbb{R}$-linear mapping $f:X\to X$ is called a *derivation* if

$$f(xy)=f(x)y+xf(y)$$

for all $x,y\in X$.

Definition 2.5. Let $(\mathcal{U}, \mu, T, T')$ be a non-Archimedean random Banach algebra, then an involution on $\mathcal{U}$ is a mapping $u \to u^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

(i) $u^{**} = u$ for $u \in \mathcal{U}$; (ii) $(\alpha u + \beta v)^* = \overline{\alpha} u^* + \overline{\beta} v^*$; (iii) $(uv)^* = v^* u^*$ for $u, v \in \mathcal{U}$.

If, in addition $\mu_{u^*u}(t) = T'(\mu_u(t), \mu_u(t))$ for $u \in \mathcal{U}$ and $t > 0$, then $\mathcal{U}$ is a non-Archimedean random C^*–algebra.

Definition 2.6. Let (X, μ, T) be an NA-RN-space.
(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.
(3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

3. Stability of homomorphisms and derivations in non-Archimedean random C^*-algebras

Throughout this section, assume that $\mathcal{A}$ is a non-Archimedean random C^*-algebra with norm $\mu^{\mathcal{A}}_{\cdot}$ and that $\mathcal{B}$ is a non-Archimedean random C^*-algebra with norm $\mu^{\mathcal{B}}_{\cdot}$.

Theorem 3.1. [35] *Let V and W be real vector spaces. A mapping $f : V \to W$ with $f(0) = 0$ satisfies in the following functional equation*

$$\sum_{i=1}^{m} f(x_i) = \frac{1}{2m} \left[\sum_{i=1}^{m} f\left(m x_i + \sum_{j=1\ ,i \neq j}^{m} x_j \right) + f\left(\sum_{i=1}^{m} x_i \right) \right]$$

if and only f f is additive.

For a given mapping $f : \mathcal{A} \to \mathcal{B}$, we define

$$D_\mu f(x_1, ..., x_m) := \sum_{i=1}^{m} \mu f(x_i) - \frac{1}{2m} [\sum_{i=1}^{m} f\left(\mu m x_i + \sum_{j=1\ ,i \neq j}^{m} x_j \right) + f\left(\sum_{i=1}^{m} \mu x_i \right)]$$

for a fixed positive integer m with $m \geq 2$ and for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}; |\lambda| = 1\}$ and all $x_1, ..., x_m \in \mathcal{A}$.

Note that a $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a ${}^*-homomorphism$ in non-Archimedean random C^*-algebras if H satisfies

$$H(xy) = H(x)H(y) \text{ and } H(x^*) = H(x)^*$$

for all $x, y \in \mathcal{A}$.

We prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean random C^*-algebras for the functional equation $D_\lambda f(x_1, ..., x_m) = 0$.

Theorem 3.2. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ for which there are functions $\varphi : \mathcal{A}^m \to D^+$, $\psi : \mathcal{A}^2 \to D^+$ and $\eta : \mathcal{A} \to D^+$ such that $|m| < 1$ is far from zero and*

$$\mu^{\mathcal{B}}_{D_\lambda f(x_1,\ldots,x_m)}(t) \geq \varphi_{x_1,\ldots,x_m}(t), \tag{3.1}$$

$$\mu^{\mathcal{B}}_{f(xy)-f(x)f(y)}(t) \geq \psi_{x,y}(t), \tag{3.2}$$

$$\mu^{\mathcal{B}}_{f(x^*)-f(x)^*}(t) \geq \eta_x(t), \tag{3.3}$$

for all $\lambda \in \mathbb{T}^1$, all $x_1, ..., x_m, x, y \in A$ and $t > 0$. If there exists an $L < 1$ such that

$$\varphi_{mx_1,\ldots,mx_m}(|m|Lt) \geq \varphi_{x_1,\ldots,x_m}(t), \tag{3.4}$$

$$\psi_{mx,my}(|m|^2Lt) \geq \psi_{x,y}(t), \tag{3.5}$$

$$\eta_{mx}(|m|Lt) \geq \eta_x(t), \tag{3.6}$$

for all $x, y, x_1, ..., x_m \in \mathcal{A}$ and $t > 0$, then there exists a unique random ${}^*-homomorphism$ $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x)-H(x)}(t) \geq \varphi_{x,0,\ldots,0}((|m| - |m|L)t) \tag{3.7}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. It follows from (3.4), (3.5), (3.6) and $L < 1$ that

$$\lim_{n\to\infty} \varphi_{m^nx_1,\ldots,m^nx_m}(|m|^n t) = 1, \tag{3.8}$$

$$\lim_{n\to\infty} \psi_{m^nx,m^ny}(|m|^{2n} t) = 1, \tag{3.9}$$

$$\lim_{n\to\infty} \eta_{m^nx}(|m|^n t) = 1, \tag{3.10}$$

for all $x, y, x_1, ..., x_m \in \mathcal{A}$ and $t > 0$. Let us define Ω to be the set of all mappings $g : \mathcal{A} \longrightarrow \mathcal{B}$ and introduce a generalized metric on Ω as follows:

$$d(g, h) = \inf\{k \in (0, \infty) \ : \ \mu^{\mathcal{B}}_{g(x)-h(x)}(kt) > \phi_{x,0,\ldots,0}(t), \ \forall x \in \mathcal{A}, t > 0\}.$$

It is easy to show that (Ω, d) is a generalized complete metric space (see [8]). Now we consider the function $J : \Omega \longrightarrow \Omega$ defined by $Jg(x) = \frac{1}{m}g(mx)$ for all $x \in \mathcal{A}$ and $g \in \Omega$. Note that for all $g, h \in \Omega$ we have

$$d(g,h) < k \Longrightarrow \mu^{\mathcal{B}}_{g(x)-h(x)}(kt) > \phi_{x,0,\ldots,0}(t) \Longrightarrow \mu^{\mathcal{B}}_{\frac{1}{m}g(mx)-\frac{1}{m}h(mx)}(kt)$$
$$> \phi_{mx,0,\ldots,0}(|m|t) \Longrightarrow \mu^{\mathcal{B}}_{\frac{1}{m}g(mx)-\frac{1}{m}h(mx)}(kLt)$$
$$> \phi_{x,0,\ldots,0}(t) \Longrightarrow d(Jg, Jh) < kL.$$

From this it is easy to see that $d(Jg, Jk) \leq Ld(g, h)$ for all $g, h \in \Omega$, that is, J is a self-function of Ω with the Lipschitz constant L. Putting $\mu = 1$, $x = x_1$ and $x_2 = x_3 = \ldots = x_m = 0$ in (3.1) we have $\mu^{\mathcal{B}}_{f(mx)-mf(x)}(t) \geq \phi_{x,0,\ldots,0}(t)$ for all $x \in \mathcal{A}$ and $t > 0$. Then

$$\mu^{\mathcal{B}}_{f(x)-\frac{1}{m}f(mx)}(t) \geq \phi_{x,0,\ldots,0}(|m|t)$$

for all $x \in \mathcal{A}$ and $t > 0$, that is, $d(Jf, f) \leq \frac{1}{|m|} < \infty$. Now, from the fixed point alternative, it follows that there exists a fixed point H of J in Ω such that

$$H(x) = \lim_{n\to\infty} |m|^{-n} f(m^n x) \tag{3.11}$$

for all $x \in \mathcal{A}$, since $\lim_{n\to\infty} d(J^n f, H) = 0$. On the other hand, it follows from (3.1), (3.8), and (3.11) that

$$\mu^{\mathcal{B}}_{D_\lambda H(x_1,\ldots,x_m)}(t) = \lim_{n\to\infty} \mu^{\mathcal{B}}_{\frac{1}{m^n}Df(m^n x_1,\ldots,m^n x_m)}(t)$$
$$\geq \lim_{n\to\infty} \phi_{m^n x_1,\ldots,m^n x_m}(|m|^n t) = 1$$

By a similar method noted above, we get $\lambda H(mx) = H(m\lambda x)$ for all $\lambda \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Thus one can show that the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear. It follows from (3.2), (3.9) and (3.11) that

$$\mu^{\mathcal{B}}_{H(xy)-H(x)H(y)}(t) = \lim_{n\to\infty} \mu^{\mathcal{B}}_{f(m^{2n}xy)-f(m^n x)f(m^n y)}(|m|^{2n}t)$$
$$\geq \lim_{n\to\infty} \psi_{m^n x, m^n y}(|m|^{2n}t) = 1$$

for all $x, y \in \mathcal{A}$. So $H(xy) = H(x)H(y)$ for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is a homomorphism, satisfying (3.7),intended. Also by (3.3), (3.10), (3.11) and by a similar method, we have $H(x^*) = H(x)^*$.

□

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ such that*

$$\mu^{\mathcal{B}}_{D_\lambda f(x_1,\ldots,x_m)}(t) \geq \frac{t}{t + \theta \cdot (\|x_1\|^r_{\mathcal{A}} + \|x_2\|^r_{\mathcal{A}} + \cdots + \|x_m\|^r_{\mathcal{A}})},$$
$$\mu^{\mathcal{B}}_{f(xy)-f(x)f(y)}(t) \geq \frac{t}{t + \theta \cdot (\|x\|^r_{\mathcal{A}} \cdot \|y\|^r_{\mathcal{A}})}, \ \mu^{\mathcal{B}}_{f(x^*)-f(x)^*}(t) \geq \frac{t}{t + \theta \cdot \|x\|^r_{\mathcal{A}}}$$

for all $\lambda \in \mathbb{T}^1$, *all* $x_1, ..., x_m, x, y \in \mathcal{A}$ *and* $t > 0$. *Then there exists a unique* $^*-$*homomorphism* $H : \mathcal{A} \to \mathcal{B}$ *such that*

$$\mu^{\mathcal{B}}_{f(x)-H(x)}(t) \geq \frac{(|m| - |m|^r)t}{(|m| - |m|^r)t + \theta|m| - |m|^r\|x\|^r_{\mathcal{A}}}$$

for all $x \in \mathcal{A}$ *and* $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi_{x_1,\ldots,x_m}(t) = \frac{t}{t + \theta \cdot (\|x_1\|^r_{\mathcal{A}} + \|x_2\|^r_{\mathcal{A}} + \cdots + \|x_m\|^r_{\mathcal{A}})},$$

$$\psi_{x,y}(t) := \frac{t}{t + \theta \cdot (\|x\|^r_{\mathcal{A}} \cdot \|y\|^r_{\mathcal{A}})}, \ \eta_x(t) = \frac{t}{t + \theta \cdot \|x\|^r_{\mathcal{A}}}$$

for all $x_1, ..., x_m, x, y \in A$, $L = |m|^{r-1}$ and $t > 0$, we get the desired result. □

We prove the generalized Hyers-Ulam stability of derivations on non-Archimedean random C^*-algebras for the functional equation $D_\lambda f(x_1, ..., x_m) = 0$.

Theorem 3.4. *Let* $f : \mathcal{A} \to \mathcal{A}$ *be a mapping with* $f(0) = 0$ *for which there are functions* $\varphi : \mathcal{A}^m \to D^+$, $\psi : \mathcal{A}^2 \to D^+$ *and* $\eta : \mathcal{A} \to D^+$ *such that* $|m| < 1$ *is far from zero and*

$$\mu^{\mathcal{A}}_{D_\lambda f(x_1,\ldots,x_m)}(t) \geq \varphi_{x_1,\ldots,x_m}(t), \mu^{\mathcal{A}}_{f(xy)-f(x)y-xf(y)}(t) \geq \psi_{x,y}(t),$$
$$\mu^{\mathcal{A}}_{f(x^*)-f(x)^*}(t) \geq \eta_x(t),$$

for all $\lambda \in \mathbb{T}^1$ *and all* $x_1, ..., x_m, x, y \in A$ *and* $t > 0$. *If there exists an* $L < 1$ *such that* (3.4), (3.5) *and* (3.6) *hold, then there exists a unique derivation* $\delta : \mathcal{A} \to \mathcal{A}$ *such that*

$$\mu^{\mathcal{A}}_{f(x)-\delta(x)}(t) \geq \varphi_{x,0,\ldots,0}((|m| - |m|L)t)$$

for all $x \in \mathcal{A}$ *and* $t > 0$.

4. STABILITY OF HOMOMORPHISMS AND DERIVATIONS IN NON-ARCHIMEDEAN LIE C^*-ALGEBRAS

A non-Archimedean random C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy-yx}{2}$ on $\mathcal{C}$, is called a *Lie non-Archimedean random* C^**-algebra*.

Definition 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be random Lie C^*-algebras. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a *non-Archimedean Lie* C^**-algebra homomorphism* if $H([x, y]) = [H(x), H(y)]$ for all $x, y \in \mathcal{A}$.

Throughout this section, assume that $\mathcal{A}$ is a non-Archimedean random Lie C^*-algebra with norm $\mu^{\mathcal{A}}$ and that $\mathcal{B}$ is a non-Archimedean random Lie C^*-algebra with norm $\mu^{\mathcal{B}}$.

We prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean random Lie C^*-algebras for the functional equation $D_\lambda f(x_1, ..., x_m) = 0$.

Theorem 4.2. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ for which there are functions $\varphi : \mathcal{A}^m \to D^+$ and $\psi : \mathcal{A}^2 \to D^+$ such that* (3.1) *and* (3.3) *hold and*

$$\mu^{\mathcal{B}}_{f([x,y])-[f(x),f(y)]}(t) \geq \psi_{x,y}(t) \tag{4.1}$$

for all $\lambda \in \mathbb{T}^1$, all $x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ and (3.4)*,* (3.5) *and* (3.6) *hold, then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that* (3.7) *hold.*

Proof. By the same reasoning as in the proof of Theorem 3.2, we can find the mapping $H : \mathcal{A} \to \mathcal{B}$ given by

$$H(x) = \lim_{n\to\infty} |m|^{-n} f(m^n x) \tag{4.2}$$

for all $x \in \mathcal{A}$. It follows from (3.5) and (4.2) that

$$\begin{aligned} \mu^{\mathcal{B}}_{H([x,y])-[H(x),H(y)]}(t) &= \lim_{n\to\infty} \mu^{\mathcal{B}}_{f(m^{2n}[x,y])-[f(m^n x),f(m^n y)]}(|m|^{2n} t) \\ &\geq \lim_{n\to\infty} \psi_{m^n x, m^n y}(|m|^{2n} t) = 1 \end{aligned}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then $H([x, y]) = [H(x), H(y)]$ for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is a Lie C^*-algebra homomorphism satisfying (3.7), as intended. □

Corollary 4.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ such that*

$$\mu^{\mathcal{B}}_{D_\lambda f(x_1,\dots,x_m)}(t) \geq \frac{t}{t + \theta(\|x_1\|^r_A + \|x_2\|^r_A + \cdots + \|x_m\|^r_A)},$$

$$\mu^{\mathcal{B}}_{f([x,y])-[f(x),f(y)]}(t) \geq \frac{t}{t + \theta \cdot \|x\|^r_A \cdot \|y\|^r_A}, \quad \mu^{\mathcal{B}}_{f(x^*)-f(x)^*}(t) \geq \frac{t}{t + \theta \cdot \|x\|^r_{\mathcal{A}}}$$

for all $\lambda \in \mathbb{T}^1$, all $x_1, ..., x_m, x, y \in \mathcal{A}$ and $t > 0$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x)-H(x)}(t) \geq \frac{(|m| - |m|^r)t}{(|m| - |m|^r)t + \theta\|x\|^r_{\mathcal{A}}}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. The proof follows from Theorem 4.2 and a method similar to Corollary 3.3. □

Definition 4.4. Let $\mathcal{A}$ be a non-Archimedean random Lie C^*-algebra. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a *Lie derivation* if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{A}$.

We prove the generalized Hyers-Ulam stability of derivations on non-Archimedean random Lie C^*-algebras for the functional equation $D_\lambda f(x_1, ..., x_m) = 0$.

Theorem 4.5. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$ for which there are functions $\varphi : A^m \to D^+$ and $\psi : A^2 \to D^+$ such that* (3.1) *and* (3.3) *hold and*

$$\mu^{\mathcal{A}}_{f([x,y])-[f(x),y]-[x,f(y)]}(t) \quad \geq \quad \psi_{x,y}(t), \tag{4.3}$$

for all $x, y \in \mathcal{A}$. If there exists an $L < 1$ and (3.4), (3.5) *and* (3.6) *hold, then there exists a unique Lie derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that such that* (3.7) *hold.*

Proof. By the same reasoning as the proof of Theorem 4.2, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying (3.7); the mapping $\delta : \mathcal{A} \to \mathcal{A}$ is given by

$$\delta(x) = \lim_{n\to\infty} |m|^{-n} f(m^n x) \tag{4.4}$$

for all $x \in \mathcal{A}$. It follows from (3.5) and (4.4) that

$$\begin{aligned}
&\mu^{\mathcal{A}}_{\delta([x,y])-[\delta(x),y]-[x,\delta(y)]}(t) \\
&\qquad = \lim_{n\to\infty} \mu^{\mathcal{A}}_{f(m^{2n}[x,y])-[f(m^n x),\cdot m^n y]-[m^n x, f(m^n y)]}(|m|^{2n} t) \\
&\qquad \geq \lim_{n\to\infty} \psi_{m^n x, m^n y}(|m|^{2n} t) = 1
\end{aligned}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{A}$. Thus $\delta : \mathcal{A} \to \mathcal{A}$ is a Lie derivation satisfying (3.7). ☐

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