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C_4-free zero-divisor graphs

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Abstract. In this paper we give a characterization for all commutative rings with 1 whose zero-divisor graphs are C_4-free.

Keywords: Zero-divisor graph; Bipartite graph.

1. INTRODUCTION

The graph theory terminology in general is followed in this study. [9]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. We denote the degree of a vertex v in G by $d_G(v)$, which is the number of edges incident to v. A graph G is *complete* if there is an edge between every pair of the vertices. A subset X of the vertices of a graph G is called *independent* if there is no edge with two endpoints in X. A graph G is called *bipartite* if its vertex set can be partitioned into two subsets X and Y such that every edge of G has one endpoint in X and other endpoint in Y. A graph G is said to be *star* if G contains one vertex in which all other vertices are joined to this vertex and G has no other edges. A *path* of length n is an ordered list of distinct vertices $v_0, v_1, ..., v_n$ such that v_i is adjacent to v_{i+1} for $i = 1, 2, ..., n-1$. We use $v_0 - v_1 - ... - v_n$ to refer such path. A (u, v)-path is a path with endpoints u and v. A *cycle* is a path $v_0, v_1, ..., v_n$ with an extra edge v_0v_n. A graph G is *connected* if it has a (u, v)-path for each pair $u, v \in V(G)$.

By the zero-divisor graph $\Gamma(R)$ of a ring R we mean the graph with vertices $Z(R) \setminus \{0\}$ such that there is an (undirected) edge between

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vertices a and b if and only if $a \neq b$ and $ab = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain. The concept of zero-divisor graphs has been studied extensively by many authors. For a list of references and the history of this topic the reader is referred to [1, 2, 3, 4, 6, 7].

Bipartie zero-divisor graphs are studied by Akbari et al. [2], Dancheng et al. [6], Demeyer et al. [7], and Jafari Rad et al. [8]. Dancheng et al. in [6] posed the following open question.

Question. How can one characterize the zero-divisor graphs which contain no rectangles?

In this paper I will characterize all commutative rings with 1 whose zero-divisor graphs are C_4-free.

We denote by K_n and C_n the complete graph and the cycle on n vertices. Also we denote by $K_{m,n}$ the complete bipartite graph.

Throughout, R will always be a commutative ring with $1 \neq 0$, unless we state R does not have 1. We also note that by $G \leq H$ for two graphs we mean that G is a subgraph of H, while by $R \leq S$ for two rings we mean that R is a subring of S.

Consider the following rings.
$T_4 = \{0, x, x+1, 1\}$, where $x^2 = 2x = 2 = 0$,
$T_8 = \{0, x, x^2, x+x^2, 1, 1+x, 1+x^2, 1+x+x^2\}$, where $x^3 = 2x = 2 = 0$,
$T_8' = \{0, x, x^2, x+x^2, 1, 1+x, 1+x^2, 1+x+x^2\}$, where $x^3 = 2x = 4 = 0$,
$T_9 = \{0, 1, -1, x, -x, 1+x, 1-x, x-1, -1-x\}$, where $x^2 = 3x = 3 = 0$.

We make use of the following.

Theorem 1.1. *([8]) Let R be a commutative ring with identity, and R is not an integral domain. Then $\Gamma(R)$ contains no triangle, if and only if R satisfies one of the following.*
(1) $Z(R) = I \cup J$, where I, J are commutative domains as rings, and $I \cap J = 0$.
(2) $R \cong \mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_9$, T_4, T_8, T_8' or T_9.

2. THE MAIN RESULT

We shall prove the following.

Theorem 2.1. *Let R be a commutative ring with identity, R is not an integral domain, and $|R| \notin \{8, 16, 32, 64\}$. Then $\Gamma(R)$ is C_4-free if and only if R satisfies one of the following.*
(1) $|R| = 9$ and $Nil(R) = \{0, x, -x\}$,
(2) $R \cong \mathbb{Z}_2 \times F$, where F is a field.

For the proof of this theorem we consider three cases, $\Gamma(R)$ has no triangle, $nil(R) = 0$ or $nil(R) \neq 0$.

We begin with the following lemma.

Lemma 2.2. *(a) If x is nilpotent, then$1 + x$ is invertable.*

(b) If $x \in R$, then $|\frac{R}{ann(x)}| = |Rx|$.

(c) Let $R = R_1 \times R_2$. If $\min\{|R_1|, |R_2|\} \geq 3$, Then $\Gamma(R)$ contains a C_4.

(d) $\Gamma(R_1 \times R_2 \times R_3)$ is C_4-free if and only if $R_i \cong \mathbb{Z}_2$ for $i = 1, 2, 3$.

(e) Let $R = R_1 \times R_2$, and $R_1 \cong \mathbb{Z}_2$. Then $\Gamma(R)$ contains a C_4 if and only if $\Delta(\Gamma(R_2)) \geq 2$.

Proof. Is elementary. □

Theorem 2.3. *Let R be a commutative ring with identity, and R is not an integral domain. If $\Gamma(R)$ has no triangle, then $\Gamma(R)$ is C_4-free if and only if $R = \mathbb{Z}_2 \times F$, where F is a field.*

Proof. By Theorem 1.1, $\Gamma(R)$ is C_3-free if and only if (1) or (2) holds in Theorem 1.1. If R satisfies (2), then $|Z(R)| \leq 4$. So $\Gamma(R)$ is C_4-free. Let R satisfies (1). By Lemma 2.2(c), $|I| = 2$ or $|J| = 2$. Let $|I| = 2$, and $I = \{0, x\}$. Hence by Lemma 2.2(b), $|\frac{R}{I}| = 2$, and so I is a maximal ideal of R. We deduce that $I + J = R$. Thus $R \cong I \times J$. Since $1 \in R$, I and J are fields. For the converse notice that $\Gamma(R)$ is a star. □

Proposition 2.4. *(a) Let a, b be two distinct elements of R. If $|(ann(a) \cap ann(b)) \setminus \{a, b, 0\}| \geq 2$, then $\Gamma(R)$ contains a C_4.*

(b) Let $Nil(R) = 0$. If $\Gamma(R)$ contains a triangle, then $\Gamma(R)$ is C_4-free if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. (b) Let $x - y - z - x$ be a triangle in $\Gamma(R)$. Let $I = ann(x)$ and $J = ann(y)$. Since $y, z \in I$, $|I| \geq 3$. Also $I \cap Rx = 0$ since $Nil(R) = 0$. By Lemma 2.2(c), $|Rx| = 2$, and by Lemma 2.2(b), $|\frac{R}{I}| = 2$. Similarly, $|\frac{R}{J}| = 2$. On the other hand by case(a), $|I \cap J| = 2$. Now $|\frac{R}{I \cap J}| \leq |\frac{R}{I}||\frac{R}{J}|$. This implies that $|R| \leq 8$, and $I = \{0, y, z, y + z\}$. Note that $|I| = |J| = |ann(z)| = 4$. Consequently, $ann(x^2) = I$, $ann(y^2) = J$ and $ann(z^2) = ann(z)$. Since $I \neq J$, we have $I + J = R$ and $|R| = 8$. We deduce that $R \cong I \times Rx$. We next show that $y^2 = y$. Suppose that $y^2 \neq y$. Hence $y^2 = z$ or $y^2 = y + z$. If $y^2 = z$, then $y^3 = yz = 0$, a contradiction. So $y^2 = y + z$. Therefore $y^2(y - 1) = yz = 0$, and $y - 1 \in J = \{0, x, z, x + z\}$. In each possiblity for $y - 1$ we get a contradiction, since $y(y - 1) = 0$. Thus $y^2 = y$. Similarly, $z^2 = z$, $y(y + z) = y$, and $z(y + z) = z$. We conclude that $I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It proves the nontrivial side, and the proof is complete. □

Lemma 2.5. *If $\Gamma(R)$ is C_4-free and $x^n = 0$, then $n \leq 4$.*

Proof. If $n \geq 5$, then $x^{n-1} - x^{n-2} - x^{n-3} - (x^{n-1} + x^{n-2}) - x^{n-1}$ is a C_4, a contradiction. □

Let

$$A = \{x : x^2 = 0, x \neq 0\},$$
$$B = \{x : x^3 = 0, x^2 \neq 0\},$$

and

$$C = \{x : x^4 = 0, x^3 \neq 0\}.$$

Proposition 2.6. *(a) If $C \neq \emptyset$, then $\Gamma(R)$ is C_4-free if and only if $|R| = 16$, and $Nil(R) = \{0, x, x^2, x^3, x+x^2, x+x^3, x^2+x^3, x+x^2+x^3\}$, where $x \in C$.*

(b) Let $Nil(R) = \{x : x^3 = 0\}$ and $B \neq \emptyset$. Then $\Gamma(R)$ is C_4-free if and only if $R = \{0, x, x^2, x+x^2, 1, 1+x, 1+x^2, 1+x+x^2\}$, where $x \in B$ and $char(R) \in \{2, 4, 8\}$.

Proof. (a) Let $x \in C$, and let $C_1 = \{0, x, x^2, x^3, x + x^2, x + x^3, x^2 + x^3, x + x^2 + x^3\}$. We show that $ann(x^2) \subseteq C_1$. If $r \in ann(x^2) \setminus C_1$, then $rx^2 = 0$ and so $r - x^2 - x^3 - (x^2 + x^3) - r$ is a C_4, a contradiction. Now we obtain $ann(x^2) = \{0, x^2, x^3, x^2 + x^3\}$. For any $r \in R$, $rx^2 \in ann(x^2)$. This implies that $rx^2 = 0$, $rx^2 = x^2$, $rx^2 = x^3$, or $rx^2 = x^2 + x^3$. We deduce that $r \in ann(x^2)$, $r - 1 \in ann(x^2)$, $r - 1 - x \in ann(x^2)$ or $r - x \in ann(x^2)$. We obtain $|\frac{R}{ann(x^2)}| = 4$. Since $|ann(x^2)| = 4$, we obtain $|R| = 16$. But $ann(x^3) \neq R$. So $|ann(x^3)| = 8$. Therefore $Nil(R) = ann(x^3)$ is the unique maximal ideal of R. We have
$R = \{0, x, x^2, x^3, x + x^2, x + x^3, x^2 + x^3, x + x^2 + x^3, 1, 1 + x, 1 + x^2, 1 + x^3, 1 + x + x^2, 1 + x + x^3, 1 + x^2 + x^3, 1 + x + x^2 + x^3\}$.

By Lemma 2.2, $Z(R) = Nil)R)$. Thus $\Gamma(R) = \Gamma(Nil(R))$, and so is C_4-free. The converse is trivial.

(b) Let $x \in B$. First we show that $ann(x) = \{0, x^2\}$. Let $r \in ann(x) \setminus \{0, x^2\}$. We have $r - x - x^2 - (x + x^2) - r$ is a C_4, a contradiction. So $ann(x) = \{0, x^2\}$. Hence $Rx^2 = \{0, x^2\}$ and $|\frac{R}{ann(x^2)}| = 2$. Next we show that $ann(x^2) = \{0, x, x^2, x+x^2\}$. Let $r \in ann(x^2) \setminus \{0, x, x^2, x+x^2\}$. Therefore $rx^2 = 0$ and $rx \in ann(x)$. Thus $rx = 0$ or $rx = x^2$. Since $r \notin ann(x)$, we have $rx = x^2$, and so $(r - x)x = 0$. This implies that $r - x \in ann(x) = \{0, x^2\}$. Therefore $r \in \{0, x, x^2, x + x^2\}$, a contradiction. Hence $|R| = 8$, and $Nil(R) = \{0, x, x^2, x + x^2\}$. For the converse, notice that $|Z(R)| = 3$. □

Lemma 2.7. *(a) Let $Nil(R) = \{x : x^2 = 0\}$, $A \neq \emptyset$, and $char(Nil(R)) \neq 2$. Then $\Gamma(R)$ is C_4-free if and only if $|R| = 9$ and $Nil(R) = \{0, x, -x\}$.*

(b) Let $Nil(R) = \{x : x^2 = 0\}$, $A \neq \emptyset$, $char(Nil(R)) = 2$, and $Nil(R)$ has at least two nontrivial distinct elements x, y such that $xy \neq 0$. Then $\Gamma(R)$ is C_4-free if and only if $|R| = 16$ and $Nil(R) = \{0, x, xy, x + xy, y, y + x, y + xy, y + x + xy\}$.

Proof. (a) Let $x \in B$, and $x \neq -x$. We show that $ann(x) = \{0, x, -x\}$. Let r, s be two distinct elements of $ann(x) \setminus \{0, x, -x\}$. Therefore $r - x - s - (-x) - r$ is a C_4, a contradiction. So $|ann(x) \setminus \{0, x, -x\}| \leq 1$. On the other hand $\{0, x, -x\} \leq ann(x)$, and so $3 \mid |ann(x)|$. We deduce that $ann(x) = \{0, x, -x\}$, and $\{0, x, -x\} = Rx$. By Lemma 2.2(b), $|\frac{R}{ann(x)}| = 3$, and so $|R| = 9$. Hence $R = \{0, x, -x, 1, 1+x, 1-x, -1, -1+x, -1-x\}$. By Lemma 2.2(a), $Z(R) = \{0, x, -x\} = Nil(R)$. For the converse by Lemma 2.2(a), $Z(R) = \{0, x, -x\}$ and $\Gamma(R) = K_2$.

(b) We first show that $ann(x) = \{0, x, xy, x + xy\}$. Let $r \in ann(x) \setminus \{0, x, xy, x+xy\}$. Hence $r - x - (x+xy) - xy - r$ is a C_4, a contradiction. So $ann(x) = \{0, x, xy, x + xy\}$. On the other hand $Rx \leq ann(x)$, which implies that $Rx = ann(x)$. By lemma 2.2(b), $|\frac{R}{ann(x)}| = 4$. Thus $|R| = 16$, and $Nil(R) = \{0, x, xy, x + xy, y, y + x, y + xy, y + x + xy\}$. For the converse by Lemma 2.2, $Z(R) = Nil(R)$, and $\Gamma(R)$ is C_4-free. □

Lemma 2.8. *(a) Let $Nil(R) = \{x : x^2 = 0\}$, $A \neq \emptyset$, $char(Nil(R)) = 2$, and for any pair of elements x, y in $Nil(R)$, $xy = 0$. If $\Gamma(R)$ is C_4-free then $|Nil(R)| \leq 4$.*

(b) Let $Nil(R) = \{0, x\}$, and $\Gamma(R)$ contains a triangle. Then $\Gamma(R)$ contains a C_4.

(c) Let $Nil(R) = \{0, x, y, x + y\}$, where $2x = 2y = xy = 0$. If $\Gamma(R)$ is C_4-free, then $|R| \in \{8, 16, 32, 64\}$.

Proof. (a) Is trivial.

(b) Assume to the contrary that $\Gamma(R)$ is C_4-free. Let $a - b - c - a$ is a triangle in $\Gamma(R)$. Let $r \in R$. If $ra \notin \{0, a, b, c\}$, then $a - b - ra - c - a$ is a C_4, a contradiction. So $ra \in \{0, a, b, c\}$. We consider two cases.

Case 1. $x \notin \{a, b, c\}$. We have $ra \in \{0, a, b, c\}$. If $ra = b$, then $ra^2 = ba = 0$ and so $(ra)^2 = 0$. Hence $ra = 0$, a contradiction. Thus $ra \neq b$. Similarly $ra \neq c$. We deduce that $Ra = \{0, a\}$ and by Lemma 2.2(b), $|\frac{R}{ann(a)}| = 2$. Similarly, $|\frac{R}{ann(b)}| = 2$. Since $a, b \notin ann(a) \cap ann(b)$, by Proposition 2.4(a) $|ann(a) \cap ann(b)| = 2$. On the other hand $ann(a) + ann(b) = R$, and so $|R| = 8$. Also $ann(a) = \{0, b, c, b + c\}$ and $R = \{0, b, c, b + c, a, a + b, a + c, a + b + c\}$. But $x \in R$. So $x = b + c$, $a + b$, $a + c$ or $a + b + c$. If $x = b + c$, then $x^2 = b^2 + c^2 = 0$ which implies that $b^2 = -c^2$ and so $b^3 = -c^2 b = 0$. Hence $b = x$, a contradiction. So $x \neq b + c$. By a similar discussion we obtain $x \notin R$, a contradiction.

Case 2. $x \in \{a, b, c\}$. Without loss of generality assume that $x = c$. If $a + x \neq b$, then $a - x - (a + x) - b - a$ is a C_4, a contradiction. So $a + x = b$, and hence $(a + x)a = 0$, that is $a^2 = 0$. By assumption $a = x$, a contradiction.

(c) Let $I = ann(x)$ and $J = ann(y)$. By Proposition 2.4(a), $|I \cap J| = 4$. On the other hand $Rx \subseteq Nil(R)$. By lemma 2.2, $|\frac{R}{I}| \in \{2, 4\}$. Similarly, $|\frac{R}{J}| \in \{2, 4\}$. We conclude that $|R| \in \{8, 16, 32, 64\}$. □

As a consequence of Lemmas 2.7 and 2.8 we obtain the following.

Proposition 2.9. *Assume that R contains a triangle, $A \neq \emptyset$, and $|R| \notin \{8, 16, 32, 64\}$. Then $\Gamma(R)$ is C_4-free if and only if $|R| = 9$ and $|Nil(R)| = 3$.*

Now the result follows from Theorem 2.3, and Propositions 2.4, 2.6 and 2.9.

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References

[1] S. Akbari, and A. Mohammadian, Zero-divisor graphs of non-commutative rings, *Journal of Algebra,* **296** *(2006), 462-479.*

[2] S. Akbari, H.R. Maimani, and S. Yassemi, When a zero-divisor graph is planar or a complete r-partite graph, *Journal of Algebra,* **270** *(2003), 169180.*

[3] D.F. Anderson, A. Frazier, A. Lauve, and P.S. Livingston, The zero-divisor graph of a commutative ring II, *in: Ideal Theoretic Methods in Commutative Algebra (Columbia, MO, 1999), Dekker, New York,(2001), 6172.*

[4] D.F. Anderson, and P.S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra,* **217** *(1999), 434447.*

[5] M.F. Atiyah, Ian G. Macdonald, Introduction to Commutative Algebra, *Addison-Wesley Publishing Co, Reading, Mass.-London-Don Mills, Ont, (1969).*

[6] L. Dancheng, and W. Tongsuo, On bipartite zero-divisor graphs, *Discrete Mathematics,* **309** *(2009), 755762.*

[7] F. Demeyer, and K. Schneider, Automorphisms and zero divisor graphs of commutative rings, *International J. Commutative Rings,* **1** *Issue 3 (2002), 93106.*

[8] N. Jafari Rad, and S.H. Jafari, A characterization of bipartite zero-divisor graphs, accepted for publication.

[9] D.B. West, "Introduction To Graph Theory", *Prentice-Hall of India Pvt. Ltd.* (2003).