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On Rad-H-supplemented Modules

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Abstract. Let M be a right R-module. We call M Rad-H-supplemented if for each $Y \leq M$ there exists a direct summand D of M such that $(Y+D)/D \subseteq (Rad(M)+D)/D$ and $(Y+D)/Y \subseteq (Rad(M)+Y)/Y$. It is shown that:

(1) Let $M = M_1 \oplus M_2$, where M_1 is a fully invariant submodule of M. If M is Rad-H-supplemented, then M_1 and M_2 are Rad-H-supplemented. (2) Let $M = M_1 \oplus M_2$ be a duo module and Rad-$\oplus$-supplemented. If M_1 is radical M_2-sejective (or M_2 is radical M_1-sejective), then M is Rad-H-supplemented. (3) Let $M = \oplus_{i=1}^{n} M_i$ be a finite direct sum of modules. If M_i is generalized radical M_j-projective for all $j > i$ and each M_i is Rad-H-supplemented, then M is Rad-H-supplemented.

Keywords: Rad-H-supplemented module, $FI - P^* - module$, Rad-H-cofinitely supplemented module.

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1. Introduction

In this paper, R denotes an associative ring with unity and all modules are unitary right R-modules. A submodule N of M is called *small* in M (denoted by $N \ll M$) if for every proper submodule L of M, $N+L \neq M$.

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Let N and L be submodules of M. Following [13], the module N is called a *supplement* of L in M if it is minimal with respect to the property $N + L = M$, equivalently, $N + L = M$ and $N \cap L \ll L$. The radical of an R-module M, denoted by Rad(M) is defined as the intersection of all maximal submodules of M. N is called a *Rad-supplement* of L in M, if $N + L = M$ and $N \cap L \subseteq Rad(L)$. M is called *supplemented* (*Rad-supplemented*) if for each submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll B$ $(A \cap B \subseteq Rad(B))$. M is called *weakly Rad-supplemented* if for each submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A \cap B \subseteq Rad(M)$. M is called $\oplus-supplemented$ if each submodule of M has a supplement that is a direct summand of M. M is called $Rad-\oplus-supplemented$ if each submodule of M has a Rad-supplement that is a direct summand of M. Recall that M is *lifting* if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$. A module M is called *H-supplemented* if for every submodule A of M there exists a direct summand D of M such that $A + X = M$ if and only if $D + X = M$ for every submodule X of M (see[9]). A module M is called *H-cofinitely supplemented* if for every cofinite submodule A of M (i.e. M/A finitely generated) there exists a direct summand D of M such that $A + X = M$ if and only if $D + X = M$ for every submodule X of M (see[6]).

M is called *Rad-H-supplemented* if for each $Y \leq M$ there exists a direct summand D of M such that $(Y + D)/D \subseteq (Rad(M) + D)/D$ and $(Y + D)/Y \subseteq (Rad(M) + Y)/Y$.

A submodule A of a module M is called *projection invariant* in M if $f(A) \leq A$ for any idempotent $f \in End(M)$. If for any $f \in End(M)$, $f(A) \leq A$, then A is called a *fully invariant* submodule of M. The module M is called a *duo* module, if every submodule of M is fully invariant. Recall that a module M has the *summand intersection property*, (SIP) if the intersection of any two direct summands of M is again a direct summand. Recall from [1] that a module M is said to have P^* property if for any submodule $N \leq M$ there exists a direct summand D of M such that $D \subseteq N$ and $N/D \subseteq Rad(M/D)$. We call M $FI - P^*$-module if for every fully invariant submodule A of M, there exists a direct summand D of M such that $D \subseteq A$ and $A/D \subseteq Rad(M/D)$. Clearly every module with property P^* is Rad-H-supplemented and every $Rad-\oplus-supplemented$ module has $FI - P^*$ property.

A module M is called $\oplus$-*cofinitely radical supplemented* (according to [5], *generalized* $\oplus$-*cofinitely supplemented*) if every cofinite submodule of M has a Rad-supplement that is a direct summand of M. Instead of a $\oplus$-cofinitely radical supplemented, we will use a $cgs^{\oplus}$-module.

We give some new characterizations of Rad-H-supplemented modules. We investigate radical sejective modules. The direct sum of two Rad-H-supplemented modules need not be Rad-H-supplemented. We investigate finite direct sums of Rad-H-supplemented modules.

2. Rad-H-supplemented modules

Proposition 2.1. *Let M be a module. If M is Rad-H-supplemented, then for each $Y \leq M$, there exists $X \leq M$ and a direct summand D of M with $Y \subseteq X$ and $D \subseteq X$ such that $X/Y \subseteq (Rad(M) + Y)/Y$ and $X/D \subseteq (Rad(M) + D)/D$.*

Proof. It follows from the definition Rad-H-supplemented module. □

Theorem 2.2. *The following are equivalent for a module M:*
(1) M is $FI - P^*$.
(2) Every fully invariant submodule of M has a Rad-supplement which is a direct summand.

Proof. (1) ⇒ (2) Suppose that M is $FI - P^*$. Then for every fully invariant submodule A of M, there exists a direct summand D of M such that $A/D \subseteq Rad(M/D)$. Let $M = D \oplus D'$ for some submodule D' of M. Since $D \subseteq A$, then $A + D' = M$ and from $A/D \subseteq Rad(M/D)$, we have $A \subseteq Rad(M) + D$. Hence $A \cap D' \subseteq RadD'$. So A has a Rad-supplement which is a direct summand.

(2) ⇒ (1) Let A be a fully invariant submodule in M. Then $M = M_1 \oplus M_2$ such that $A + M_2 = M$ and $A \cap M_2 \subseteq Rad(M_2)$. Since A is a fully invariant submodule in M, $A = (A + M_1) \cap (A + M_2) = A + M_1$. Hence $M_1 \leq A$, $A = (A \cap M_2) \oplus M_1 \subseteq Rad(M_2) \oplus M_1 \subseteq Rad(M) + M_1$. Hence $A/M_1 \subseteq Rad(M/M_1)$. So M is $FI - P^*$. □

Let M be a right R-module. We call M *Rad-H-cofinitely supplemented* if for every cofinite submodule A of M (i.e. the factor module M/A is finitely generated), there exists a direct summand D of M such that $(A + D)/D \subseteq (Rad(M) + D)/D$ and $(A + D)/A \subseteq (Rad(M) + A)/A$.

A module M is called *local* if the sum of all proper submodules of M is a proper submodule of M. Recall from [2] that a module M is called *w-local* if it has a unique maximal submodule. Clearly, local modules are w-local.

Proposition 2.3. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2. If M_2 is a $cgs^{\oplus}$-module and every cofinite submodule of M is fully invariant and contains M_1, then M is Rad-H-cofinitely supplemented.*

Proof. Suppose that M_2 is a $cgs^{\oplus}$-module, then by [10, Theorem 2.3], there exists a submodule K of M_2 such that K is a direct summand of M,

$M = K + N$ and $N \cap K \subseteq Rad(K)$ for every cofinite submodule N/M_1 of M/M_1. Hence N be a cofinite submodule of M. Thus $M = K \oplus K'$ for some submodule K' of M. Since $N = (N + K) \cap (N + K') = N + K'$, we have $K' \leq N$. So $N = (N \cap K) \oplus K' \subseteq Rad(K) \oplus K' \subseteq Rad(M) + K'$. Hence $(N+K')/K' \subseteq (Rad(M)+K')/K'$ and $(N+K')/N \subseteq (Rad(M)+N)/N$. Therefore M is Rad-H-cofinitely supplemented. □

Proposition 2.4. *Let M be an R-module. Assume that for every maximal submodule A of M there exists a direct summand D of M such that $(A+D)/D \subseteq Rad(M/D)$ and every cofinite submodule contains D. Then:*

(1) M is $cgs^{\oplus}$-module.

(2) M is a w-local module if $Rad(M) \neq M$.

Proof. (1) Suppose that N is a cofinite submodule of M. Then M/N is finitely generated. Hence M/N has a maximal submodule Q/N. So Q is a maximal submodule of M. By hypothesis, there exists a direct summand P of M such that $(Q + P)/P \subseteq Rad(M/P)$. Let $M = P \oplus P'$ for some submodule P' of M. Hence $M = N + P'$ and $N \cap P' \subseteq Rad(P')$. This shows that every cofinite submodule of M has a Rad-supplement that is a direct summand of M. So M is $cgs^{\oplus}$-module.

(2) Let M be a module satisfying the assumptions of Proposition and $Rad(M) \neq M$. Let A be a maximal submodule of M. Then there exists a direct summand D of M such that $(A + D)/D \subseteq Rad(M/D)$ and every cofinite submodule contains D. In particular, every maximal submodule of M contains D. So $D \subseteq Rad(M)$. Since D is a direct summand of M, we have $Rad(M/D) = (Rad(M)+D)/D = Rad(M)/D$. Thus $A \subseteq A + D \subseteq Rad(M)$. But $Rad(M) \subseteq A$. Then Rad(M)=A. So M contains only one maximal submodule. Hence M is a w-local module. Consequently, every module which satisfies the assumption of Proposition is either radical (i.e. having no maximal submodules) or w-local. □

Proposition 2.5. *Let M be a Rad-H-cofinitely supplemented module. Then for each maximal submodule Y of M, there exists a Rad-supplement L of Y and a Rad-supplement K of L such that $(Y+K)/K \subseteq Rad(M/K)$ and every homomorphism $f : M \to M/(K \cap L)$ can be lifted to the homomorphism $\bar{f} : M \to M$.*

Proof. Suppose that Y is a maximal submodule of M. Then there exists $D, D' \leq M$ such that $M = D \oplus D'$, $(Y + D)/D \subseteq (Rad(M) + D)/D$. It is easy to show that D' is a Rad-supplement of Y and D is a Rad-supplement of D'. So it follows by taking $D = K$ and $D' = L$. □

Proposition 2.6. *Let M be Rad-H-supplemented and N a fully invariant submodule of M. Then M/N is Rad-H-supplemented.*

Proof. Let $L/N \leq M/N$. Since M is Rad-H-supplemented, by Proposition 2.1, there exists $X \leq M$ and a direct summand D of M such that $X/D \subseteq (Rad(M) + D)/D$ and $X/L \subseteq (Rad(M) + L)/L$. Let $M = D \oplus D'$, where $D' \leq M$. Since N is a fully invariant submodule of M, $N = (D \cap N) + (D' \cap N) = (D + N) \cap (D' + N)$. So $(D + N)/N \oplus (D' + N)/N = M/N$. It is easy to see that $\frac{X/N}{(D+N)/N} \subseteq \frac{Rad(M/N)+(D+N)/N}{(D+N)/N}$ and $\frac{X/N}{L/N} \subseteq \frac{Rad(M/N)+L/N}{L/N}$. Therefore M/N is Rad-H-supplemented. □

Theorem 2.7. *Let $M = M_1 \oplus M_2$, where M_1 is a fully invariant submodule of M. If M is Rad-H-supplemented, then M_1 and M_2 are Rad-H-supplemented.*

Proof. By Proposition 2.6, M_2 is Rad-H-supplemented. Next, we show that M_1 is Rad-H-supplemented. Let K be a submodule of M_1. Since M is Rad-H-supplemented, there exists a direct summand D of M such that $(K + D)/K \subseteq (Rad(M) + K)/K$ and $(K + D)/D \subseteq (Rad(M) + D)/D$. Write $M = D \oplus D'$, where $D' \leq M$. Since M_1 is a fully invariant submodule of M, $M_1 = (M_1 \cap D) \oplus (M_1 \cap D')$. Hence $(M_1 \cap D)$ is a direct summand of M_1. We know that $K + D \subseteq Rad(M) + K$ and $K + D \subseteq Rad(M) + D$. It is easy to see that $K + (D \cap M_1) \subseteq Rad(M_1) + K$ and $K + (D \cap M_1) \subseteq Rad(M_1) + (D \cap M_1)$. So $\frac{K+(D\cap M_1)}{K} \subseteq \frac{Rad(M_1)+K}{K}$ and $\frac{K+(D\cap M_1)}{(D\cap M_1)} \subseteq \frac{Rad(M_1)+(D\cap M_1)}{(D\cap M_1)}$. Hence M_1 is Rad-H-supplemented. □

Theorem 2.8. *Let $M = M_1 \oplus M_2$. Assume that for every submodule N of M_1 there exists a direct summand K of M such that $M_2 \leq K$, $(N + K)/K \subseteq (Rad(M) + K)/K$ and $(N + K)/N \subseteq (Rad(M) + N)/N$. Then M_1 is Rad-H-supplemented.*

Proof. Let L be a submodule of M_1. By hypothesis, there exists a direct summand K of M such that $M_2 \leq K$, $(L + K)/K \subseteq (Rad(M) + K)/K$ and $(L + K)/L \subseteq (Rad(M) + L)/L$. Now $K = (K \cap M_1) \oplus M_2$. Hence $K \cap M_1$ is a direct summand of M_1. Now $L + K \subseteq Rad(M) + L$ and $L + K \subseteq Rad(M) + K$. It is easy to see that $L + (K \cap M_1) \subseteq Rad(M_1) + L$ and $L + (K \cap M_1) \subseteq Rad(M_1) + (K \cap M_1)$. So $\frac{L+(K\cap M_1)}{(K\cap M_1)} \subseteq \frac{Rad(M_1)+(K\cap M_1)}{(K\cap M_1)}$ and $\frac{L+(K\cap M_1)}{L} \subseteq \frac{Rad(M_1)+L}{L}$. Therefore M_1 is Rad-H-supplemented. □

3. Radical Sejectivity

Let M_1 and M_2 be modules such that $M = M_1 \oplus M_2$. We say M_1 is *radical M_2-sejective* if for every $A \leq M$ such that $M = A + M_2$, there exists $K \leq M$ such that $M = K \oplus M_2$ and $(A + K)/A \subseteq (Rad(M) +$

$A)/A$. M_1 and M_2 are called *relatively radical sejective* if M_1 is radical M_2-sejective and M_2 is radical M_1-sejective.

Theorem 3.1. *Let $M = M_1 \oplus M_2$ be a duo module and Rad-$\oplus$-supplemented. If M_1 is radical M_2-sejective (or M_2 is radical M_1-sejective), then M is a Rad-H-supplemented module.*

Proof. Let N be a submodule of M. Since M is Rad-$\oplus$-supplemented, there exists decomposition $M = M_1 \oplus M_2$ such that $M = N + M_2$ and $N \cap M_2 \subseteq Rad(M_2)$ for some submodules M_1 and M_2. Since M_1 is radical M_2-sejective there exists $K \leq M$ such that $M = K \oplus M_2$ and $(N + K)/N \subseteq (Rad(M) + N)/N$. Now we show $(N + K)/K \subseteq (Rad(M) + K)/K$. Since M is a duo module, N is fully invariant and $N = (N + K) \cap (N + M_2) = N + K$. Hence $K \leq N$, $N + K = N = (N \cap M_2) \oplus K \subseteq Rad(M) + K$. So $(N + K)/K \subseteq (Rad(M) + K)/K$. Therefore M is Rad-H-supplemented. □

Proposition 3.2. *Let M be an $FI - P^*$-module and X be a fully invariant submodule of M which is a direct summand in M. Then X is $FI - P^*$.*

Proof. Let A be a fully invariant submodule in X. Then A is a fully invariant submodule in M. Since M is $FI - P^*$, A contains a direct summand B of M such that $A/B \subseteq Rad(M/B)$. Let $M = B \oplus B'$ for some submodule B' of M. Since $A \subseteq Rad(B') + B$, we have $A \subseteq Rad(X) + B$. So $A/B \subseteq Rad(X/B)$. Also B is a direct summand of X. Therefore X is $FI - P^*$. □

Let M and N be modules. Then N is called *generalized radical M-projective* if for any $K \leq M$ and any homomorphism $f : N \to M/K$, there exists a homomorphism $h : N \to M$ such that $Im(f - \pi h) \subseteq (Rad(M) + K)/K$, where $\pi : M \to M/K$ is a natural epimorphism.

Proposition 3.3. *Let $M = M_1 \oplus M_2$. If M_1 is generalized radical M_2-projective, then M_1 is radical M_2-sejective.*

Proof. Let $K \leq M$ and $M = K + M_2$. Consider epimorphism $\pi : M_2 \to M/K$ given by $m_2 \to m_2 + K$ and the homomorphism $h : M_1 \to M/K$ given by $m_1 \to m_1 + K$. Since M_1 is generalized radical M_2-projective, there exists a homomorphism $\bar{h} : M_1 \to M_2$ and a submodule X of M with $K \subseteq X$ such that $Im(h - \pi\bar{h}) = X/K \subseteq (Rad(M) + K)/K$. Let M_3=$\{a - (a)\bar{h}|\ a \epsilon M_1\}$. Clearly $M = M_2 \oplus M_3$. Since $K + M_3 \subseteq X$, $(K + M_3)/K \subseteq X/K$. Hence, $(K + M_3)/K \subseteq (Rad(M) + K)/K$. So M_1 is radical M_2-sejective.

□

Proposition 3.4. *Let M be a Rad-H-supplemented module. Then $M/Rad(M)$ is semisimple.*

Proof. Let $N/Rad(M) \leq M/Rad(M)$. Since M is Rad-H-supplemented, there exists a direct summand D of M such that $(N+D)/D \subseteq (Rad(M)+D)/D$ and $(N+D)/N \subseteq (Rad(M)+N)/N$. Let $M = D \oplus D'$ for some submodule D' of M. Then $M = D' + N$. It follows that $M/Rad(M) = N/Rad(M) + (D' + Rad(M))/Rad(M)$. Since $N \cap D' \subseteq Rad(D')$, $M/Rad(M) = N/Rad(M) \oplus (D' + Rad(M))/Rad(M)$. Hence $M/Rad(M)$ is semisimple. □

Recall that a module M is *semilocal* provided that $M/Rad(M)$ is semisimple.

Remark 3.5. Any Rad-H-supplemented is weakly Rad-supplemented.

Proof. Let M Rad-H-supplemented module. By proposition 3.4, M semilocal. C. Lomp [8] proved that a module M is semilocal iff M is weakly Rad-supplemented. Thus M weakly Rad-supplemented. □

Rad-H-supplemented$\Longrightarrow$weakly Rad-supplemented.

Theorem 3.6. *Let $M = M_1 \oplus M_2$. Then:*

(1) If M_1 is radical M_2-sejective (or M_2 is radical M_1-sejective) and M_1, M_2 are Rad-H-supplemented, then M is Rad-H-supplemented.

(2) If M_1 is generalized radical M_2-projective (or M_2 is generalized radical M_1-projective) and M_1, M_2 are Rad-H-supplemented, then M is Rad-H-supplemented.

Proof. (1) Let $Y \leq M$.

Case 1: $M = Y + M_2$. Since M_1 is radical M_2-sejective, there exists $M_3 \leq M$ such that $M = M_3 \oplus M_2$ and $(Y + M_3)/Y \subseteq (Rad(M)+Y)/Y$. Since $M/M_3 \cong M_2$, M/M_3 Rad-H-supplemented. Now consider the submodule $(Y + M_3)/M_3$ of M/M_3. By Proposition 2.1, there exists $X/M_3 \leq M/M_3$ and a direct summand D/M_3 of M/M_3 such that $\frac{X/M_3}{(Y+M_3)/M_3} \cong \frac{X}{(Y+M_3)} \subseteq \frac{Rad(M)+(Y+M_3)}{(Y+M_3)}$ and $\frac{X/M_3}{D/M_3} \cong \frac{X}{D} \subseteq \frac{Rad(M)+D}{D}$. Clearly, $M = D \oplus (M_2 \cap D')$, so D is a direct summand of M. It is easy to see that $X/Y \subseteq (Rad(M) + Y)/Y$. Therefore, M is Rad-H-supplemented.

Case 2: $M \neq Y + M_2$. Since M_1, M_2 are Rad-H-supplemented, then M_1, M_2 are weakly Rad-supplemented. From [11, Propositions 3.2, 3.7], M/Y is weakly Rad-supplemented. So there exists a submodule K/Y of M/Y such that $M/Y = K/Y + (Y + M_2)/Y$ and $(K \cap (Y + M_2))/Y \subseteq Rad(M/Y)$. Then $M = K + M_2$. Since M_1 is radical M_2-sejective, there exists $M_4 \leq M$ such that $M = M_2 \oplus M_4$ and $(K + M_4)/K \subseteq (Rad(M) + K)/K$. Now M/M_2 and M/M_4 are Rad-H-supplemented. Therefore, there exists submodules X_1/M_2 of M/M_2, X_2/M_4 of M/M_4, direct summands D_1/M_2 of M/M_2 and

D_2/M_4 of M/M_4 such that $\frac{X_1}{(Y+M_2)} \subseteq \frac{Rad(M)+(Y+M_2)}{(Y+M_2)}$, $\frac{X_1}{D_1} \subseteq \frac{Rad(M)+D_1}{D_1}$, $\frac{X_2}{(K+M_4)} \subseteq \frac{Rad(M)+(K+M_4)}{(K+M_4)}$ and $\frac{X_2}{D_2} \subseteq \frac{Rad(M)+D_2}{D_2}$. Clearly, $D_1 \cap D_2$ is a direct summand of M and $\frac{(X_1 \cap X_2)}{(D_1 \cap D_2)} \subseteq \frac{Rad(M)+(D_1 \cap D_2)}{(D_1 \cap D_2)}$. Since $X_2 \subseteq Rad(M) + (K + M_4) \subseteq Rad(M) + K$, $X_2 \cap M_2 \subseteq Rad(M_2) + (K \cap M_2)$ and $(X_2 \cap M_2) + Y \subseteq Rad(M) + Y$. As $X_1 \subseteq Rad(M) + Y + M_2$, $X_1 \cap X_2 \subseteq Rad(M) + Y$. Thus $\frac{(X_1 \cap X_2)}{Y} \subseteq \frac{Rad(M)+Y}{Y}$. So M is Rad-H-supplemented.

(2) By Proposition 3.3, M_1 is radical M_2-sejective. So the proof follows by (1). □

Lemma 3.7. *Let A, M_1, M_2,...,M_n be modules. If each M_i is generalized radical A-projective for $i = 1, 2, ..., n$, then $\oplus_{i=1}^{n} M_i$ is generalized radical A-projective.*

Proof. The proof is straightforward. □

Corollary 3.8. *Let $M = \oplus_{i=1}^{n} M_i$ be a finite direct sum of modules. If M_i is generalized radical M_j-projective for all $j > i$ and each M_i is Rad-H-supplemented, then M is Rad-H-supplemented.*

Proof. It follows from Theorem 3.6(2) and Lemma 3.7. □

Proposition 3.9. *Let $M = M_1 \oplus M_2$. Then M_2 is $FI - P^*$ if and only if for every fully invariant submodule N/M_1 of M/M_1, there exists a direct summand K of M such that $K \leq M_2$, $M = K + N$ and $N \cap K \subseteq Rad(K)$.*

Proof. Suppose that M_2 is $FI - P^*$. Let N/M_1 be a fully invariant submodule of M/M_1. It is easy to see that $N \cap M_2$ is fully invariant in M_2. Since M_2 is $FI - P^*$, there exists a decomposition $M_2 = K \oplus K'$ such that $M_2 = (N \cap M_2) + K$ and $N \cap K \subseteq Rad(K)$. Clearly, $M = K + N$.

Conversely, suppose that M/M_1 has the stated property. Let H be a fully invariant submodule of M_2. It is easy to see that $(H \oplus M_1)/M_1$ is fully invariant in M/M_1. By hypothesis, there exists a direct summand L of M such that $L \leq M_2$, $M = L + H + M_1$ and $L \cap (H + M_1) \subseteq Rad(L)$. By modularity, we have $M_2 = L + H$. It follows easily that L is a Rad-supplement of H in M_2. Therefore, M_2 is $FI - P^*$ by Theorem 2.2. □

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