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Operator Valued Series and Vector Valued Multiplier Spaces

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Abstract. Let X, Y be normed spaces with $L(X,Y)$ the space of continuous linear operators from X into Y. If $\{T_j\}$ is a sequence in $L(X,Y)$, the (bounded) multiplier space for the series $\sum T_j$ is defined to be

$$M^\infty(\sum T_j) = \{\{x_j\} \in l^\infty(X) : \sum_{j=1}^{\infty} T_j x_j \textit{ converges}\}$$

and the summing operator $S : M^\infty(\sum T_j) \to Y$ associated with the series is defined to be $S(\{x_j\}) = \sum_{j=1}^{\infty} T_j x_j$. In the scalar case the summing operator has been used to characterize completeness, weakly unconditionall Cauchy series, subseries and absolutely convergent series. In this paper some of these results are generalized to the case of operator valued series The corresponding space of weak multipliers is also considered.

Keywords: multiplier convergent series, multipliers, compact operators, absolutely summing operators, summing operator

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1. Introduction

In a series of papers Aizpuru, Benitez-Trujillo and Perez-Fernandez ([PBA], [AP1] ,[AP2]) used the multiplier space

$$M^{\infty}(\sum x_j) = \{\{t_j\} \in l^{\infty} : \sum_{j=1}^{\infty} t_j x_j \text{ } converges\}$$

and the summing operator $S(\{t_j\}) = \sum_{j=1}^{\infty} t_j x_j$ of an arbitrary sequence $\{x_j\}$ in a normed space X to characterize weakly unconditionally Cauchy (wuc) series $\sum x_j$ and completeness and barrelledness of the normed space X. In an additional paper Swartz ([Sw3]) used the multiplier space and the summing operator to give characterizations of subseries and absolutely convergent series in terms of the summing operator. In this paper we will extend some of these results to the case of operator valued series and vector valued multipliers. The first question which arises, is what is the analogue of wuc series in the operator valued setting? There are several known characterizations of wuc series given in Proposition 3.8 of [Sw2]. The one which seems to be appropriate is the fact that a series $\sum x_j$ is wuc iff it is c_0 multiplier Cauchy (that is, the series $\sum_{j=1}^{\infty} t_j x_j$ is Cauchy for every $\{t_j\} \in c_0$); this condition is easily adapted to the operator valued case and gives results which are analogous to those obtained by Aizpuru, Benitez-Trujillo and Perez-Fernandez.

Main Results

Throughout let X, Y be normed spaces, $L(X, Y)$ the space of continuous linear operators from X into Y and $T_j \in L(X, Y)$ for $j \in \mathbb{N}$. If E is an X valued sequence space, the series $\sum T_j$ is E *multiplier convergent* (E multiplier Cauchy) if the series $\sum_{j=1}^{\infty} T_j x_j$ converges in Y (is Cauchy in Y) for every sequence $\{x_j\} \in E$. $c_0(X)$ ($l^{\infty}(X), c_{00}(X)$) is the space of all X valued sequences which converge to 0 (are bounded, are eventually 0). The multiplier space for the series $\sum T_j$ is defined to be

$$M^{\infty}(\sum T_j) = \{\{x_j\} \in l^{\infty}(X) : \sum_{j=1}^{\infty} T_j x_j \text{ } converges\};$$

the multiplier space for $\sum T_j$ is assumed to be equipped with the sup-norm $\|\cdot\|_{\infty}$. The summing operator associated with the series is defined to be

$$S : M^{\infty}(\sum T_j) \to Y, \ S(\{x_j\}) = \sum_{j=1}^{\infty} T_j x_j.$$

Similarly, the weak multiplier space is defined to be

$$M_w^\infty(\sum T_j) = \{\{x_j\} \in l^\infty(X) : \sum_{j=1}^\infty T_j x_j \text{ converges weakly}\}$$

and the weak summing operator is defined to be

$$S : M_w^\infty(\sum T_j) \to Y,\ S(\{x_j\}) = \sum_{j=1}^\infty T_j x_j \text{ (weak sum)}.$$

For a sufficient condition for equality between $M_w^\infty(\sum T_j)$ and $M^\infty(\sum T_j)$, we have

Proposition 1.1. *If $\sum T_j$ is $l^\infty(X)$ multiplier Cauchy, then*

$$M^\infty(\sum T_j) = M_w^\infty(\sum T_j).$$

Proof. Let $x \in M_w^\infty(\sum T_j)$ and let $y = \sum_{j=1}^\infty T_j x_j$ (weak sum). The partial sums of the series $\sum_{j=1}^\infty T_j x_j$ form a norm Cauchy sequence since $\sum T_j$ is l^∞ multiplier Cauchy and the series $\sum_{j=1}^\infty T_j x_j$ is weakly convergent so the series $\sum_{j=1}^\infty T_j x_j$ is norm convergent since the norm and weak topologies are linked ([Wi]6.1.9,6.1.11, [Sw2]A.4) so $x \in M^\infty(\sum T_j)$.

See [PBA], Lemma 3.3 and [Sw2], Lemma 8.26 for the scalar analogues. □

Continuity

We show that the continuity of the summing operator S can be characterized by the $c_0(X)$ multiplier Cauchy property of the series $\sum T_j$. For one of the implications below, we require the following lemma.

Lemma 1.2. *If $\sum T_j$ is $c_0(X)$ multiplier Cauchy, then* $\sup_j \|T_j\| < \infty$.

Proof. Let $t_j \to 0$ and $\|x_j\| \le 1$. Then $\{t_j x_j\} \in c_0(X)$ so $\sum_{j=1}^\infty t_j T_j x_j$ is Cauchy and $t_j T_j x_j \to 0$. Hence, $\{T_j x_j\}$ is bounded. Pick $x_j \in X$ such that $\|x_j\| = 1$ and $\|T_j x_j\| + 1 > \|T_j\|$. Then $\{\|T_j\|\}$ is bounded. □

Theorem 1.3. *The following are equivalent: (1) $S : M^\infty(\sum T_j) \to Y$ is continuous , (2) $\sum T_j$ is $c_0(X)$ multiplier Cauchy, (3)*

$$G = \{\sum_{j=1}^n T_j x_j : n \in \mathbb{N}, \|x_j\| \le 1\}$$

is bounded, (4) $S \mid_{c_{00}(X)} \to Y$ is continuous, (5) $S : M_w^\infty(\sum T_j) \to Y$ is continuous.

Proof. (1)⇒(2): Let $x = \{x_j\} \in c_0(X)$. If $j \in \mathbb{N}$ and $z \in X$, $e^j \otimes z$ will denote the sequence with z in the jth coordinate and 0 in the other coordinates. Set $x^k = \sum_{j=1}^k e^j \otimes x_j$ so $x^k \in M^\infty(\sum T_j)$ and $x^k \to x$ in $\|\cdot\|_\infty$. Since $S : M^\infty(\sum T_j) \to Y$ is continuous, $\{Sx^k\} = \{\sum_{j=1}^k T_j x_j^k\}$ is Cauchy so the series $\sum_j T_j x_j$ is Cauchy.

(2)⇒(3): If the conclusion fails, there exist $t_k \to 0$ and $\sum_{j=1}^{n_k} T_j x_j^k = y_k \in G$ and $t_k y_k \nrightarrow 0$. Hence, there exists $\delta > 0$ such that

$$(\#)\ \textit{for every } k \textit{ there exists } m_k > k \textit{ with } \|t_{m_k} y_{m_k}\| > \delta.$$

For $k = 1$ (#) implies $\|t_{m_1} y_{m_1}\| = \left\| t_{m_1} \sum_{j=1}^{m_1} T_j x_j^{m_1} \right\| > \delta$. By Lemma 2 there exists $l_1 > m_1$ such that $l \geq l_1$ implies $|t_l| \sum_{j=1}^{m_1} \|T_j\| < \delta/2$. For $k = l_1$ condition (#) implies there exists $m_2 > l_1$ such that

$$\left\| t_{m_2} \sum_{j=1}^{m_2} T_j x_j^{m_2} \right\| > \delta$$

[this is a slight abuse of the notation in (#) but it avoids multiple subscripts]. Note

$$\left\| t_l \sum_{j=1}^{k} T_j x_j \right\| \leq |t_l| \sum_{j=1}^{k} \|T_j\| \leq |t_l| \sum_{j=1}^{m_1} \|T_j\| < \delta/2$$

for any $k \leq m_1, \|x_j\| \leq 1, l \geq l_1$ so $m_2 > m_1$ and, in particular,

$$\left\| t_{m_2} \sum_{j=1}^{m_1} T_j x_j^{m_1} \right\| < \delta/2.$$

Set $I_2 = [m_{1+1}, m_2]$. Then

$$\left\| t_{m_2} \sum_{j \in I_2} T_j x_j^{m_2} \right\| \geq \left\| t_{m_2} \sum_{j=1}^{m_2} T_j x_j^{m_2} \right\| - \left\| t_{m_2} \sum_{j=1}^{m_1} T_j x_j^{m_2} \right\| > \delta - \delta/2 = \delta/2.$$

Continuing this construction produces an increasing sequence $\{m_k\}$ and an increasing sequence of intervals $\{I_k\}$ such that

$$(*)\quad \left\| t_{m_k} \sum_{j \in I_k} T_j x_j^{m_k} \right\| > \delta/2.$$

Define $x = \{x_j\} \in c_0(X)$ by $x = \sum_{k=1}^\infty t_{m_k} \chi_{I_k} x^{m_k}$ [coordinate sum]. Then (*) implies $\sum_{j=1}^\infty T_j x_j$ fails the Cauchy criterion so $\sum T_j$ is not $c_0(X)$ multiplier Cauchy giving the desired contradiction.

(3)⇒(4): This follows immediately from (3).

(4)⇒(5): By (4) there exits $M > 0$ such that $\left\| \sum_{j=1}^{n} T_j x_j \right\| \leq M$ for $n \in \mathbb{N}, \|x_j\| \leq 1$. Then

$$\left\| S(\sum_{j=1}^{n} e^j \otimes x_j) \right\| = \left\| \sum_{j=1}^{n} T_j x_j \right\| \leq M$$

when $n \in \mathbb{N}, \|x_j\| \leq 1$. Thus, if $y' \in Y'$, then

$$\left| y'(\sum_{j=1}^{n} T_j x_j) \right| = \left| \sum_{j=1}^{n} y'(T_j x_j) \right| \leq M \left\| y' \right\|$$

for $n \in \mathbb{N}, \|x_j\| \leq 1$. Let $x \in M_w^\infty(\sum T_j)$ with $\|x\|_\infty \leq 1$. If $y' \in Y'$ and if $\sum_{j=1}^{\infty} T_j x_j$ denotes the weak sum of the series,

$$\left| y'(\sum_{j=1}^{\infty} T_j x_j) \right| = \left| \sum_{j=1}^{\infty} y'(T_j x_j) \right| = \left| \lim_n \sum_{j=1}^{n} y'(T_j x_j) \right| \leq M \left\| y' \right\|.$$

Hence, S is continuous.

Clearly (5) implies (1). □

See [AP1] for the analogue of these equivalences in the scalar case. The equivalence of (1) and (2) is the analogue of Proposition 2.1 of [AP2] (8.22 of [Sw2]); the other equivalences should be compared with those in 3.8 of [Sw2].

Completeness

We address the completeness of $M^\infty(\sum T_j)$.

Theorem 1.4. *If $M^\infty(\sum T_j)$ or $M_w^\infty(\sum T_j)$is complete, then $\sum T_j$ is $c_0(X)$ multiplier convergent.*

Proof. Let $x = \{x_j\} \in c_0(X)$. Consider the case of $M^\infty(\sum T_j)$ first. Then $x^k = \sum_{j=1}^{k} e^j \otimes x_j \in M^\infty(\sum T_j)$ and $x^k \to x$ in $\|\cdot\|_\infty$ so $x \in M^\infty(\sum T_j)$ and $\sum T_j$ is $c_0(X)$ multiplier convergent.

Next consider the case of $M_w^\infty(\sum T_j)$. Then by the argument above $x \in M_w^\infty(\sum T_j)$ so $\sum_{j=1}^{\infty} T_j x_j$ is weakly convergent. Since $c_0(X)$ is monotone and $\sum_{j=1}^{\infty} T_j x_j$ is weakly convergent for every $x \in c_0(X)$ this means the series $\sum_{j=1}^{\infty} T_j x_j$ is weakly subseries convergent and, therefore, norm subseries convergent by the Orlicz-Pettis Theorem ([Sw2]4.11). Hence, $\sum T_j$ is $c_0(X)$ multiplier convergent.

We next address the converse of this result. □

Theorem 1.5. *If $\sum T_j$ is $c_0(X)$ multiplier convergent and X, Y are complete, then $M^\infty(\sum T_j)$ is complete.*

Proof. Suppose $\{x^k\}$ is Cauchy in $M^\infty(\sum T_j)$. There exists $x \in l^\infty(X)$ such that $x^k \to x$ in $\|\cdot\|_\infty$ [X is complete so $l^\infty(X)$ is complete]. Theorem 2 implies there exists $M > 0$ such that $\left\|\sum_{j=1}^n T_j x_j\right\| \le M$ for $n \in \mathbb{N}, \|x_j\| \le 1$. Let $\epsilon > 0$. Fix n such that $\|x^n - x\| < \epsilon$. Since $\sum_{j=1}^\infty T_j x_j^n$ converges, there exists N such that $q > p \ge N$ implies $\left\|\sum_{j=p}^q T_j x_j^n\right\| < \epsilon$. For each j , $\left\|x_j^n - x_j\right\| / \epsilon \le 1$ so $\left\|\sum_{j=p}^q T_j(x_j^n - x_j)/\epsilon\right\| \le 2M$ or $\left\|\sum_{j=p}^q T_j(x_j^n - x_j)\right\| \le 2M\epsilon$ for $q > p \ge N$. Hence, if $q > p \ge N$,

$$\left\|\sum_{j=p}^q T_j x_j\right\| \le \left\|\sum_{j=p}^q T_j x_j^n\right\| + 2M\epsilon \le \epsilon + 2M\epsilon.$$

Thus, the series $\sum_{j=1}^\infty T_j x_j$ satisfies the Cauchy criterion and is convergent since Y is complete.

A similar result holds for $M_w^\infty(\sum T_j)$.

Theorem 1.6. *If $\sum T_j$ is $c_0(X)$ multiplier convergent and X, Y are complete, then $M_w^\infty(\sum T_j)$ is complete.*

□

Proof. Suppose $\{x^k\} \subset M_w^\infty(\sum T_j)$ is Cauchy and let $x \in l^\infty$ be such that $x^k \to x$ in $\|\cdot\|_\infty$ [X is complete so $l^\infty(X)$ is complete]. Theorem 3 implies there exists $M > 0$ such that $\left\|\sum_{j=1}^n T_j x_j\right\| \le M$ for $n \in \mathbb{N}, \|x_j\| \le 1$. Let $\epsilon > 0$. There exists n such that $\left\|x^k - x\right\|_\infty < \epsilon/3M$ for $k \ge n$. Then

$$(*) \quad \left\|\sum_{j=1}^m T_j(x_j^k - x_j)\right\| \le \epsilon/3$$

for $m \in \mathbb{N}, k \ge n$. Thus, $\left\|\sum_{j=1}^m T_j(x_j^k - x_j^l)\right\| \le 2\epsilon/3$ for $k, l \ge n, m \in \mathbb{N}$. Let $z_k = \sum_{j=1}^\infty T_j x_j^k$ [weak sum] so we have $\|z_k - z_l\| \le 2\epsilon/3$ for $k, l \ge n$. Since Y is complete, there exists $z \in Y$ such that $z_k \to z$. We claim that the series $\sum_{j=1}^\infty T_j x_j$ converges weakly to z. There exists $N > n$ such that $\|z_k - z\| < \epsilon/3$ for $k \ge N$. If $y' \in Y', \|y'\| \le 1$, then from (*)

$$\left|y'(\sum_{j=1}^m T_j(x_j^k - x_j)\right| \le \epsilon$$

for $m \in \mathbb{N}, k \geq n$. Fix N. There exists N_1 such that

$$\left| y'(\sum_{j=1}^{m} T_j(x_j^N) - z_N) \right| < \epsilon/3$$

for $m \geq N_1$. Hence, if $m \geq N_1$,

$$\left| y'(\sum_{j=1}^{m} T_j(x_j) - z) \right| \leq \left| y'(\sum_{j=1}^{m} T_j(x_j - x_j^N) \right| + \|z_k - z\| + \left| y'(\sum_{j=1}^{m} T_j(x_j^N) - z_N) \right| < \epsilon.$$

This establishes the claim.

Theorems 4 and 5 give analogues of Theorems 2.1 and 3.2 of [PBA] (see also 8.20 and 8.24 of [Sw2]).

Theorem 1.7. *If Y is not complete, there exists a $c_0(X)$ multiplier Cauchy, absolutely convergent series $\sum T_j$ such that $M^\infty(\sum T_j)$ ($M_w^\infty(\sum T_j)$) is not complete.*

□

Proof. Since Y is not complete, there exists a series $\sum y_j$ in Y such that $\sum_{j=1}^\infty y_j$ doesn't converge but $\sum_{j=1}^\infty j \|y_j\| < \infty$. Let $x_0 \in X, \|x_0\| = 1$. Pick $x_0' \in X'$ such that $x_0'(x_0) = \|x_0\| = 1$. Define $T_j \in L(X, Y)$ by $T_j x = x_0'(x) j y_j$. Note that $\|T_j\| = j \|y_j\|$ so the series $\sum T_j$ is absolutely convergent. We claim $\sum T_j$ is $c_0(X)$ multiplier Cauchy. For this let $x = \{x_j\} \in c_0(X)$. Then for $q > p$,

$$\left\| \sum_{j=p}^{q} T_j x_j \right\| \leq \sum_{j=p}^{q} \|T_j\| \|x_j\| \leq \max_{p \leq j \leq q} \|x_j\| \sum_{j=1}^{\infty} j \|y_j\| \to 0$$

as $p \to \infty$ establishing the claim. Now $x = \{x_0/j\} \in c_0(X)$ and

$$\sum_{j=1}^{\infty} T_j x_j = \sum_{j=1}^{\infty} \frac{1}{j} x_0'(x_0) j y_j = \sum_{j=1}^{\infty} y_j$$

doesn't converge so $\{x_0/j\} \notin M^\infty(\sum T_j)$. But, $\sum_{j=1}^n e^j \otimes x_0/j \in M^\infty(\sum T_j)$ and $\sum_{j=1}^n e^j \otimes x_0/j \to \{x_0/j\}$ in $\|\cdot\|_\infty$ so $M^\infty(\sum T_j)$ is not complete.

For the case of $M_w^\infty(\sum T_j)$ the series $\sum y_j$ above is also not weakly convergent since the partial sums are $\|\cdot\|$ Cauchy and the norm and weak topologies are linked ([Wi]6.1.6,6.1.11, [Sw2]A.4) so the proof above also works in this case.

From Theorems 4,5 and 6, we have the analogue of Theorems 2.2 and 3.4 of [PBA]; see also 8.21 of [Sw2].

Corollary 1.8. *Suppose X is complete. Then Y is complete iff for every $c_0(X)$ multiplier Cauchy series $\sum T_j$ the space $M^\infty(\sum T_j)$ ($M_w^\infty(\sum T_j)$) is complete.*

□

Recall that Y is complete iff $L(X,Y)$ is complete so the corollary could be restated in these terms ([Sw1]5.8,8.1.18).

Compactness

We next consider compactness for the summing operator. The scalar case was addressed in [Sw3].

Proposition 1.9. *If $S : M^\infty(\sum T_j) \to Y$ is compact (precompact, weakly compact), then each T_j is compact (precompact, weakly compact).*

Proof. Fix j. Then $\{e^j \otimes x : \|x\| \le 1\} \subset M^\infty(\sum T_j) \subset l^\infty(X)$ is bounded so $\{S(e^j \otimes x) : \|x\| \le 1\} = \{T_j x : \|x\| \le 1\}$ is relatively compact (precompact, relatively weakly compact).

Thus, if we want to consider compactness (precompactness, weak compactness) for the summing operator, we must consider the appropriate space of operators. The space of precompact (compact, weakly compact) operators will be denoted by $PC(X,Y)$ $(K(X,Y), W(X,Y))$.

Theorem 1.10. *Let $T_j \in PC(X,Y)$. If $\sum T_j$ is $l^\infty(X)$ multiplier convergent, then $S : M^\infty(\sum T_j) = l^\infty(X) \to Y$ is precompact.*

□

Proof. Let $\epsilon > 0$. The series $\sum_{j=1}^\infty T_j x_j$ are uniformly convergent for $\|x_j\| \le 1$ ([Sw2] 11.11) so there exists N such that

$$(*)\quad \left\| \sum_{j=n}^\infty T_j x_j \right\| < \epsilon \ for\ n \ge N\ and\ \|x_j\| \le 1.$$

Define $S_n : M^\infty(\sum T_j) = l^\infty(X) \to Y$ by $S_n(\{x_j\}) = \sum_{j=1}^n T_j x_j$. Then each S_n is precompact since each T_j is precompact. By (*) $\|S_n - S\| \to 0$ so S is precompact ([Sw1] 28.2]). □

Theorem 1.11. *Let Y be complete and $T_j \in K(X,Y)$ $[W(X,Y)]$. If $\sum T_j$ is $l^\infty(X)$ multiplier convergent, then $S : M^\infty(\sum T_j) = l^\infty(X) \to Y$ is compact [weakly compact].*

Proof. Using the notation in the proof above, the operators S_n are compact [weakly compact] and converge to S in norm so S is compact {weakly compact} by [DS]VI.5.5, [Ta],7.1, [Sw1],28.3 {[DS]VI.4.4, [Sw1]29.3}. □

We consider the converse.

Proposition 1.12. *If $S : M^\infty(\sum T_j) \to Y$ is compact (weakly compact), then*

$$F = \{\sum_{j\in\sigma} T_j x_j : \sigma finite, \|x_j\| \leq 1\}$$

is relatively compact (relatively weakly compact).

Proof. The set $E = \{\sum_{j\in\sigma} e^j \otimes x_j : \sigma \text{ finite}, \|x_j\| \leq 1\} \subset M^\infty(\sum T_j)$ is bounded and $SE = F$.

Theorem 1.13. *If $S : M^\infty(\sum T_j) \to Y$ is compact (weakly compact), then the series $\sum T_j$ is $l^\infty(X)$ multiplier convergent.*

□

Proof. If $x \in l^\infty(X)$, the relative compactness of the set F in Proposition 12 implies that the series $\sum_{j=1}^\infty T_j x_j$ is subseries convergent in the norm topology of Y ([Sw2] 2.48]). This proves the first statement when S is compact. If S is weakly compact, the relative weak compactness of the set F in Proposition 12 implies the series $\sum_{j=1}^\infty T_j x_j$ is subseries convergent in the weak topology of Y ([Sw2] 2.48]). But then the Orlicz-Pettis Theorem gives that the series is subseries convergent in the norm topology.

The conclusion in the theorem above implies that the series $\sum_{j=1}^\infty T_j x_j$ actually converge uniformly for $\|x_j\| \leq 1$ ([Sw2]11.11).

Continuity and bounded multiplier series □

Consider the duality between $l^\infty(X)$ and $l^1(X')$: if $x = \{x_j\} \in l^\infty(X)$ and $y = \{y_j\} \in l^1(X')$, then $\langle y, x\rangle = \sum_{j=1}^\infty \langle y_j, x_j\rangle$ defines a duality between $l^\infty(X)$ and $l^1(X')$. Then also $M^\infty(\sum T_j) \subset l^\infty(X)$ and $l^1(X')$ ($M_w^\infty(\sum T_j)$ and $l^1(X')$) similarly form a dual pair.

If E, F are a pair of vector spaces in duality, we denote the weak topology on E from F by $\sigma(E, F)$. We consider sequential continuity with respect to $\sigma(M^\infty(\sum T_j), l^1(X'))$ and $\sigma(M_w^\infty(\sum T_j), l^1(X'))$. Note that for this we must consider the T_j to be completely continuous operators [recall a linear operator is completely continuous if it carries weakly convergent sequences to norm convergent sequences].

Proposition 1.14. *If the summing operator $S : M^\infty(\sum T_j) \to Y$ is sequentially*

$$\sigma(M^\infty(\sum T_j), l^1(X')) - \|\cdot\|$$

continuous, then each T_k is completely continuous. A similar statement holds for $M_w^\infty(\sum T_j)$.

Proof. Fix k. Let $x_j \to 0$ in $\sigma(X, X')$. We claim

$$\sigma(M^\infty(\sum T_j), l^1(X')) - \lim_j e^k \otimes x_j = 0.$$

Let $y = \{y_j\} \in l^1(X')$. Then $\langle y, e^k \otimes x_j \rangle = \langle y_k, x_j \rangle \to 0$ as $j \to \infty$ justifying the claim. Then $\lim_j \|S(e^k \otimes x_j)\| = \lim_j \|T_k x_j\| = 0$. The same proof works for $M_w^\infty(\sum T_j)$.

Theorem 1.15. *If $S : M^\infty(\sum T_j) \to Y$ is sequentially $\sigma(M^\infty(\sum T_j), l^1(X'))-\|\cdot\|$ continuous, then $\sum T_j$ is $l^\infty(X)$ multiplier Cauchy [so if Y is complete, then $\sum T_j$ is $l^\infty(X)$ multiplier convergent and $l^\infty(X) = M^\infty(\sum T_j)$]. A similar statement holds for $M_w^\infty(\sum T_j)$.*

□

Proof. Let $x = \{x_j\} \in l^\infty(X)$ and set $x^k = \chi_{\{1,\dots,k\}} x$ so $x^k \in M^\infty(\sum T_j)$. We claim $\{x^k\}$ is $\sigma(M^\infty(\sum T_j), l^1(X'))$ Cauchy. Let $y = \{y_j\} \in l^1(X')$. Then $\langle y, x^k \rangle = \sum_{j=1}^k \langle y_j, x_j \rangle \to \sum_{j=1}^\infty \langle y_j, x_j \rangle$ justifying the claim. The continuity of S implies $\{Sx^k\} = \{\sum_{j=1}^k T_j x_j\}$ is norm Cauchy so $\sum T_j$ is $l^\infty(X)$ multiplier Cauchy. The same proof works for $M_w^\infty(\sum T_j)$.

We next consider the converse of this theorem.

Lemma 1.16. *If $B \subset l^\infty(X)$ is $\sigma(l^\infty(X), l^1(X'))$ bounded, then B is $\|\cdot\|_\infty$ bounded.*

□

Proof. Let $t = \{t_j\} \in l^1$ and $x' \in X'$. Then $tx' \in l^1(X')$ so

$$\sup\{|\langle tx', x \rangle| : x \in B\} = \sup\{\left|\sum_{j=1}^\infty t_j \langle x', x_j \rangle\right| : x \in B\} < \infty.$$

Since $t \in l^1$ is arbitrary, $\{\{\langle x', x_j \rangle\} : x \in B\} \subset l^\infty$ is $\|\cdot\|_\infty$ bounded. Therefore, $\sup\{|\langle x', x_j \rangle| : x \in B, j \in \mathbb{N}\} < \infty$ so by the Uniform Boundedness Principle, $\sup\{\|x_j\| : x \in B, j \in \mathbb{N}\} < \infty$ or $\sup\{\|x\|_\infty : x \in B\} < \infty$. □

Lemma 1.17. *If $\sigma(l^\infty(X), l^1(X')) - \lim x^j = 0$, then for every l , $\sigma(X, X') - \lim_j x_l^j = 0$.*

Proof. Let $x' \in X'$. Then $e^l \otimes x' \in l^1(X')$ so $\langle e^l \otimes x', x^j \rangle = \left\langle x', x_l^j \right\rangle \to 0$ as $j \to \infty$.

Theorem 1.18. *Suppose each T_j is completely continuous and $\sum T_j$ is $l^\infty(X)$ multiplier convergent. Then $S : M_w^\infty(\sum T_j) \to Y$ is sequentially $\sigma(M_w^\infty(\sum T_j), l^1(X')) - \|\cdot\|$ continuous.*

□

Proof. Let $\epsilon > 0$ and $x^j \to 0$ in $\sigma(M_w^\infty(\sum T_j), l^1(X'))$. By Lemma 16, $\sup_j \|x^j\|_\infty < \infty$ and for convenience assume $\|x^j\|_\infty \le 1$ for all j. The series $\sum_{j=1}^{\infty} T_j x_j$ converge uniformly for $\|x_j\| \le 1$ ([Sw2]11.11]) so there exists N such that $\left\|\sum_{j=N}^{\infty} T_j x_j\right\| < \epsilon$ for $\|x_j\| \le 1$. By Lemma 17 and the complete continuity of each T_l, $\lim_j \left\|T_l x_l^j\right\| = 0$. Therefore, there exists J such that $\left\|\sum_{l=1}^{N-1} T_l x_l^j\right\| < \epsilon$ for $j \ge J$. Hence, if $j \ge J$, then

$$\left\|Sx^j\right\| = \left\|\sum_{l=1}^{\infty} T_l x_l^j\right\| \le \left\|\sum_{l=1}^{N-1} T_l x_l^j\right\| + \left\|\sum_{l=N}^{\infty} T_l x_l^j\right\| < 2\epsilon.$$

Remark 1.19. Note that since $l^1(X') \subset l^\infty(X)'$, the $\sigma(M_w^\infty(\sum T_j), l^1(X'))-\|\cdot\|$ sequential continuity of the summing operator S implies that S is completely continuous.

□

Absolutely summing operators:

We make a few remarks concerning absolutely convergent series and absolutely summing operators. We only have necessary conditions for the summing operator to be absolutely summing.

Theorem 1.20. *If $S : M_w^\infty(\sum T_j) \to Y$ is absolutely summing, then each T_k is absolutely summing and $\pi(T_k) \le 2\pi(S)$, where $\pi(S)$ is the absolutely summing norm of S ([DJT]).*

Proof. For any $\{x^1, ..., x^m\} \subset M_w^\infty(\sum T_j)$,

$$\sum_{j=1}^{m} \left\|Sx^j\right\| \le 2\pi(S) \sup\{\left\|\sum_{j\in\sigma} x^j\right\|_\infty : \sigma \subset \{1, ..., m\}\}$$

(see Theorem 15 of [Sw3] for this characterization of absolutely summing operators). Fix k. Let $\{x_1, ..., x_m\} \subset X$. Then

$$\begin{aligned}\textstyle\sum_{j=1}^{m} \left\|S(e^k \otimes x_j)\right\| = \sum_{j=1}^{m} \|T_k x_j\| &\le 2\pi(S) \sup\{\left\|\textstyle\sum_{j\in\sigma} e^k \otimes x_j\right\|_\infty : \sigma \subset \{1, ..., m\}\} \\ &= 2\pi(S) \sup\{\left\|\textstyle\sum_{j\in\sigma} x_j\right\| : \sigma \subset \{1, ..., m\}\}\end{aligned}$$

so T_k is absolutely summing and $\pi(T_k) \le 2\pi(S)$.

Theorem 1.21. *If $S : M_w^\infty(\sum T_j) \to Y$ is absolutely summing and if $x \in l^\infty(X)$, then $\sum_{j=1}^{\infty} \|T_j x_j\| \le 2\pi(S) \|x\|_\infty$.*

□

Proof. For every m ,

$$\begin{aligned}\textstyle\sum_{j=1}^{m} \left\|S(e^j \otimes x_j)\right\| = \sum_{j=1}^{m} \|T_j x_j\| &\le 2\pi(S) \sup\{\left\|\textstyle\sum_{j\in\sigma} e^j \otimes x_j\right\|_\infty : \sigma \subset \{1, ..., m\}\} \\ &= 2\pi(S) \sup\{\|x_j\| : 1 \le j \le m\} \le 2\pi(S) \|x\|_\infty\end{aligned}$$

and the result follows.

Corollary 1.22. *If $S : M_w^\infty(\sum T_j) \to Y$ is absolutely summing, then $\sum_{j=1}^\infty \|T_j\| < \infty$,i.e., $\sum T_j$ is absolutely convergent.*

□

Proof. For each j, pick $x_j \in X, \|x_j\| \leq 1$, such that $\|T_j x_j\| \geq \|T_j\| - 1/2^j$. Theorem 21 gives the result. □

In [Sw3] it was shown in the scalar case that the summing operator S is absolutely summing iff the series $\sum x_j$ is absolutely convergent. Obviously, the main problem here is to give conditions which characterize when the summing operator $S : M_w^\infty(\sum T_j) \to Y$ is absolutely summing. Corollary 22 gives necessary conditions but sufficient conditions are missing.

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