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Solving the inverse problem of determining an unknown control parameter in a semilinear parabolic equation

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Abstract. The inverse problem of identifying an unknown source control parameter in a semilinear parabolic equation under an integral overdetermination condition is considered. The series pattern solution of the proposed problem is obtained by using the weighted homotopy analysis method (WHAM). A description of the method for solving the problem and finding the unknown parameter is derived. Finally, two numerical examples are investigated to illustrate this method.

Keywords: Semilinear parabolic equation; Inverse problem; Unknown control parameter; Weighted homotopy analysis method; Series solution.

2000 Mathematics subject classification: xxxx, xxxx; Secondary xxxx.

1. Introduction

Consider the following semilinear parabolic partial differential equation

$$u_t = u_{xx} + p(t)u + F(x,t), \; (x,t) \in Q \equiv [0,L] \times [0,T], \tag{1.1}$$

where $F(x,t)$ is a known function and p(t) is an unknown function. The inverse problem of determining the unknown parameter $p(t)$ in the equation (1.1), subject to the suitable initial and boundary conditions and the overdetermination condition, occur in many physical phenomena,

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for example, in the study of heat conduction processes, thermoelasticity, chemical diffusion, vibration problems and control theory [1-4]. In these applications, the unknown parameter $p(t)$ is called control function. Thus these inverse problems has been considered by many researchers in many theoretical papers, in recent years, notably [2-15]. In these papers two types of the overdetermination conditions are considered, the condition

$$u(x_0, t) = E(t),\ 0 \le t \le T,$$

where $0 \le x_0 \le L$, and the condition

$$\int_0^{s(t)} u(x,t)dx = E(t),\ 0 \le t \le T,$$

where $s(t)$ and $E(t)$ are two known functions. Authors use different methods for solving these inverse problems, such as finite difference method [6, 7, 8], The Legendre-tau technique [9],the parareal-inverse problem algorithm [10], the Sinc-collocation method [11], the Adomian decomposition procedure [12], The radial basis functions method [13], the Bernstein-Galerkin method [14] and a finite difference-Runge-kutta method [15].
In this paper, we consider the equation (1.1) subject to the initial condition

$$u(x, 0) = u_0(x),\ 0 \le x \le L, \tag{1.2}$$

the boundary condition

$$u(0, t) = g_1(t),\ 0 \le t \le T, \tag{1.3}$$

$$u(L, t) = g_2(t),\ 0 \le t \le T, \tag{1.4}$$

and the overdetermination condition

$$\int_0^{L} u(x,t)dx = E(t),\ 0 \le t \le T, \tag{1.5}$$

This problem can be induced in the process of heat conduction [1, 6, 16]. If u represents a temperature distribution, then (1.1)-(1.5) can be regarded as a control problem with source control. In this process the condition (1.5) is the specification of energy over a certain part of a heat conductor relates physically to the specification of the relative heat content of a portion of the conductor. It is needed to identify the control function p(t) so that a desired thermal energy can be obtained on the spatial domain $[0, L]$. For diffusion problems the condition (1.5) is equivalent to the specification of the fluid mass in a portion of the diffusion domain [17].

The existence and uniqueness, and continuous dependence of the solution upon the data for the problem (1.1)-(1.5) are demonstrated in the following theorem [1].

Theorem 1.1. *If $g_1(t)$, $g_2(t)$, $E \in C[0,T]$, $u_0(x) \in C^1[0,L]$, $f \in C(Q)$, $g_1(0) = u_0'(0)$, $g_2(0) = u_0'(1)$, $E(0) = \int_0^1 u_0(x)dx$, $E(0) > 0$, $s(t), E(t) \in C^1[0,T]$, $g_1(t) \le 0$, $g_2(t) \ge 0$, $u_0(x) \ge 0$, $F(x,t) \ge 0$ (not all are identically zero), then there exists a unique solution pair $u(x,t)$ and $p(t)$ for problem (1.1)-(1.5), that this solution pair is depended continuously upon the data.*

Proof. Please refer to [1]. □

Our aim, in this paper, is to introduce an algorithm based on the weighted homotopy analysis method for solving the inverse problem (1.1)-(1.5).
Liao in 1992 employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM) [18-24]. After this, many types of nonlinear problems were solved with HAM [24-27]. In [25] a weighted algorithm based on the homotopy analysis method was introduced for solving inverse problems. We first introduce the basic idea of the HAM and the weighted homotopy analysis method (WHAM). Afterwards, the WHAM is employed to find the solution of the inverse source problem (1.1)-(1.5). Finally, some examples are handled.

2. The HAM solution

In order to describe the HAM, we consider the following differential equation in a general form

$$N[u(\mathbf{x},t)] = 0, \tag{2.1}$$

where N is a nonlinear operator, $\mathbf{x}$ and t denote two independent variables and u is an unknown function of the variables $\mathbf{x}$ and t. Now, we introduce the following definition.

Definition 2.1. Let ϕ be a function of the homotopy-parameter q, then

$$D_m(\phi) = \frac{1}{m!}\frac{\partial^m \phi}{\partial q^m}\bigg|_{q=0}, \tag{2.2}$$

is called the mth-order homotopy-derivative of ϕ, where $m \ge 0$ is an integer [18].

By means of the HAM, we first construct the so-called zero-order deformation equation

$$(1-q)L[\phi(\mathbf{x},t;q)-u_0(\mathbf{x},t)] = q\hbar H(\mathbf{x},t)N[\phi(\mathbf{x},t;q)], \tag{2.3}$$

where L is an auxiliary linear operator, $u_0(\mathbf{x},t)$ is an initial guess of $u(\mathbf{x},t)$, $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is convergence parameter, $H(\mathbf{x},t)$ is an non-zero auxiliary function and $\phi(\mathbf{x},t;q)$ is an unknown function. It is obvious that when $q=0$ and $q=1$, we have

$$\phi(\mathbf{x},t;0) = u_0(\mathbf{x},t), \tag{2.4}$$

$$\phi(\mathbf{x},t,1) = u(\mathbf{x},t), \tag{2.5}$$

respectively. The solution $\phi(\mathbf{x},t;q)$ varies from the initial guess $u_0(\mathbf{x},t)$ to the solution $u(\mathbf{x},t)$. Expanding $\phi(\mathbf{x},t;q)$ in Taylor series about the embedding parameter, one may find

$$\phi(\mathbf{x},t;q) = u_0(\mathbf{x},t) + \sum_{m=1}^{+\infty} u_m(\mathbf{x},t)q^m, \tag{2.6}$$

where

$$u_m(\mathbf{x},t) = D_m[\phi(\mathbf{x},t;q)].$$

If the auxiliary linear operator, the initial guess, the convergence parameter, and the auxiliary function are so properly chosen, the series (2.6) converges at $q=1$. Thus, according to the definition 2.1 one can represent $u(\mathbf{x},t)$ as

$$u(\mathbf{x},t) = u_0(\mathbf{x},t) + \sum_{m=1}^{+\infty} u_m(\mathbf{x},t), \tag{2.7}$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao [18]. Define the vectors

$$\vec{u}_n = \{u_0(\mathbf{x},t), u_1(\mathbf{x},t), ..., u_n(\mathbf{x},t)\}.$$

Operating on both sides of Eq. (2.3) with D_m, we get the following mth-order deformation equation

$$L[u_m(\mathbf{x},t) - \chi_m u_{m-1}(\mathbf{x},t)] = \hbar H(\mathbf{x},t)R_m(\vec{u}_{m-1},\mathbf{x},t), \tag{2.8}$$

where

$$R_m(\vec{u}_{m-1},\mathbf{x},t) = D_{m-1}\left(N[\phi(\mathbf{x},t;q)]\right), \tag{2.9}$$

and

$$\chi_m = \begin{cases} 0, m<2, \\ 1, m \geq 2. \end{cases}$$

The mth-order deformation equation (2.8) is linear and thus can be easily solved, especially by means of symbolic computation software such as Maple, Mathematica and so on.

3. The WHAM solution

In this section, the WHAM is employed for solving the Eq. (1.1) with the initial condition (1.2) and the boundary conditions (1.3) and (1.4). Integrating two sides of equation (1.1) with respect to x on the interval $[0, L]$ and using the overdetermination condition (1.5), one obtain

$$E'(t) = \int_0^L u_{xx}(x,t)dx + p(t)E(t) + \int_0^L F(x,t)dx,$$

thus

$$p(t) = \frac{E'(t) - \int_0^L u_{xx}(x,t)dx - \int_0^L F(x,t)dx}{E(t)}. \tag{3.1}$$

Substituting (3.1) into Eq. (1.1) yields

$$u_{xx}(x,t) - u_t(x,t) + \frac{E'(t) - \int_0^L u_{xx}(x,t)dx - \int_0^L F(x,t)dx}{E(t)} u(x,t) + F(x,t) = 0. \tag{3.2}$$

Now suppose that the general terms of the series solutions given by HAM for the Eq. (3.2) with (1.2), and for Eq. (3.2) with (1.3) and (1.4) are $\hat{u}_n(x,t)$ and $\check{u}_n(x,t)$ respectively. Then, for the Eq. (3.2) with conditions (1.2), (1.3) and (1.4), the approximate solution is

$$u_{approx[n]}(x,t,\hbar) = \alpha_n \hat{s}_n(x,t) + (1-\alpha_n)\check{s}_n(x,t), \tag{3.3}$$

where

$$\hat{s}_n(x,t) = \sum_{i=0}^{n} \hat{u}_i(x,t),$$
$$\check{s}_n(x,t) = \sum_{i=0}^{n} \check{u}_i(x,t),$$

and α_n is a function with respect to $\hbar$. The value of α_n is determined by the following theorem.

Theorem 3.1. *Suppose that $u_0(x) \in L^2[(0,1)]$, $c(t), d(t) \in L^2[(0,T)]$, and $\beta_{1n} = \|\hat{s}_n(0,t) - c(t)\|$, $\beta_{2n} = \|\hat{s}_n(1,t) - d(t)\|$ and $\beta_{3n} = \|\check{s}_n(x,t) - u_0(x)\|$ where $\|.\|$ denotes the $L^2 -$ norm. Then the best value for α_n in (3.3) is*

$$\alpha_n = \frac{\beta_{3n}^2}{\beta_{1n}^2 + \beta_{2n}^2 + \beta_{3n}^2}, \quad n \geq 0$$

Proof. Please refer to [28]. □

Corollary 3.2. *If $\alpha_n = 0$, then the exact solution of the Eq. (3.2) with conditions (1.2)-(1.4) is given by $\check{u}(x,t)$, and if $\alpha_n = 1$, then the exact solution of the Eq. (3.2) with conditions (1.2)-(1.4) is given by $\hat{u}(x,t)$.*

To obtain the $\hat{u}_n(x,t)$ and $\check{u}_n(x,t)$, $n = 1, 2, ...,$ we choose the auxiliary linear operators as follows

$$\hat{L} = \frac{\partial}{\partial t}, \tag{3.4}$$

$$\check{L} = \frac{\partial^2}{\partial x^2}, \tag{3.5}$$

The auxiliary linear operator (3.4) has the property $L(C) = 0$ where C is a function with respect to x and the auxiliary linear operator (3.5) has the property $L(C_1 + C_2 x) = 0$ where C_1 and C_2 are two functions with respect to t. For initial guesses, we choose

$$\hat{u}_0(x,t) = u_0(x),$$

and

$$\check{u}_0(x,t) = \frac{x}{L} g_2(t) + \frac{L - x}{L} g_1(t).$$

4. Numerical verification

To evaluate the efficiency of the described method in the two previous parts, the solution of the some parabolic inverse problems will be investigated. For following examples, we assume that $H(x,t) = 1$.

Example 4.1. Let us consider the inverse problem (1.1) − (1.5) on the region $0 \leq x \leq 1$ and $t > 0$ with

$$F(x,t) = (\pi^2 + 2t)e^t \cos(\pi x) + (2xt)e^t, \tag{4.1}$$

$$u_0(x) = (x + \cos(\pi x)), \tag{4.2}$$

$$g_1(t) = e^t, \tag{4.3}$$

$$g_2(t) = 0, \tag{4.4}$$

$$E(t) = \frac{1}{2} e^t, \tag{4.5}$$

The exact solution of this problem is

$$u(x,t) = (x + \cos(\pi x))e^t,$$

and

$$p(t) = 1 - 2t.$$

We get 12 terms of the approximation solution (3.3). Since $\alpha_{12}(-1) = 1$, thus the solution of the Eq. (1.1) with conditions (1.2)-(1.4) will be $\hat{u}(x,t)$. The exact solution and the approximation solution $u_{approx[12]}(-1)$ of the problem is shown in the Figure 1. Also, Figure 2 shows the relative errors of the obtained solution. Also, by using the approximation solution, from (3.1), we get $p(t) = 1 - 2t$.

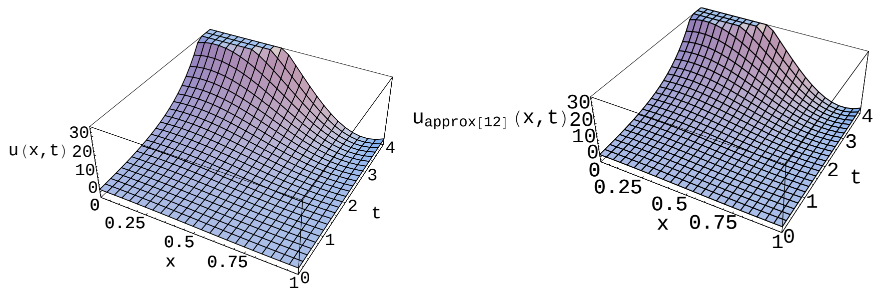

FIGURE 1. The exact solution and the approximation solution $u_{approx[12]}(-1)$ of example 4.1.

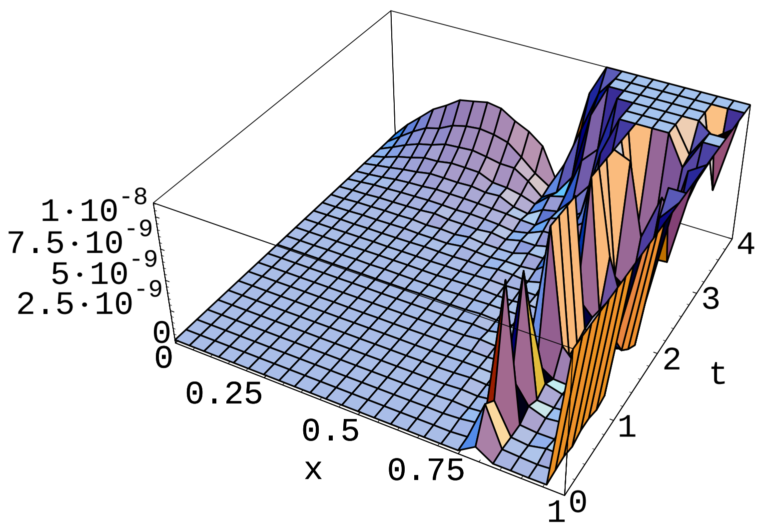

FIGURE 2. The relative errors for $u_{approx[12]}(-1)$ in the example 4.1.

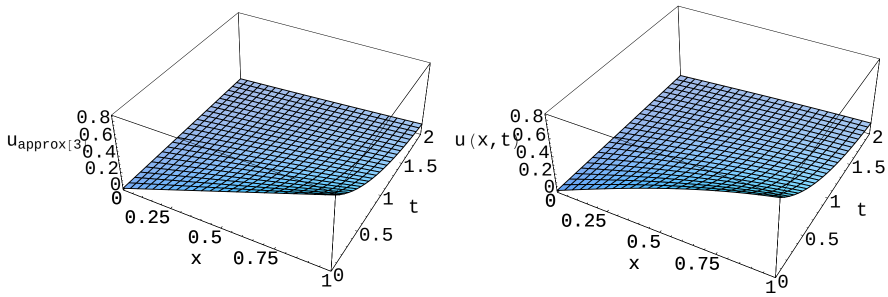

FIGURE 3. The exact solution and the approximation solution $u_{approx[3]}(-1)$ of example 4.2.

Example 4.2. Suppose that $L = 1$ and

$$F(x,t) = -e^{-t}t^2\sin(x), \tag{4.6}$$

$$u_0(x) = \sin(x), \tag{4.7}$$

$$g_1(t) = 0, \tag{4.8}$$

$$g_2(t) = e^{-t}\sin(1), \tag{4.9}$$

$$E(t) = 2e^{-t}(\sin(\frac{1}{2}))^2, \tag{4.10}$$

The exact solution of this problem is

$$u(x,t) = e^{-t}\sin(x),$$

and

$$p(t) = t^2.$$

We obtain three terms of (3.3). Since $\alpha_3(-1) = 1$, thus $\hat{u}(x,t)$ is the solution of the problem. The exact solution and the approximation solution $u_{approx[3]}(-1)$ of the problem is shown in the Figure 3. Table 1 shows the absolute errors of the obtained solution for various values of x and t.

Table 1. The absolute errors for $u_{approx[3]}(-1)$ in example 4.2.

x/t	2	4
0.1	3.40843×10^{-3}	2.64627×10^{-3}
0.2	6.56799×10^{-3}	5.05287×10^{-3}
0.3	8.73912×10^{-3}	6.99926×10^{-3}
0.4	9.38535×10^{-3}	8.30191×10^{-3}
0.5	1.12007×10^{-2}	8.82969×10^{-3}
0.6	9.62536×10^{-3}	8.51559×10^{-3}
0.7	8.90683×10^{-3}	7.36397×10^{-3}
0.8	8.63329×10^{-3}	5.45255×10^{-3}
0.9	4.93115×10^{-3}	2.92869×10^{-3}

5. conclusion

In this paper, the inverse problem of determining an unknown control parameter in a semilinear parabolic equation under an integral overdetermination condition is considered. By using the WHAM, an algorithm for solving the problem was introduced. In the last section, this algorithm was used for the some problems. The results show that the WHAM is a reliable technique for solving this type of inverse problems.

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