The Bernoulli Ritz-collocation method to the solution of modelling the pollution of a system of lakes

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Abstract. Pollution has become a very serious threat to our environment. Monitoring pollution is the first step toward planning to save the environment. The use of differential equations, monitoring pollution has become possible. In this paper, a Ritz-collocation method is introduced to solve modelling the pollution of a system of lakes by a system of differential equations. The method is based upon Bernoulli polynomials. These polynomials are first presented. The Bernoulli Ritz-collocation method is then utilized to reduce modelling the pollution of a system of lakes to the solution of algebraic equations. An illustrative example is included to demonstrate the validity and applicability of the proposed method.

Keywords: Bernoulli polynomials; modelling the pollution of a system of lakes; Ritz-collocation method.

2000 Mathematics subject classification: xxxx, xxxx; Secondary xxxx.

1. Introduction

Most physics and biology events can be modeled by the differential, integral and integro-differential equations. Since few of these equations cannot be solved analytically, so numerical method is requested.

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Polynomial series and orthogonal functions are incredibly useful mathematical tools for solving these problems. In recent years, the Taylor, Chebyshev, Legendre, Bernoulli and Bessel (Ritz-collocation and matrix) methods have been used to find the approximate solutions of differential, integral and integro-differential-difference equations and their systems [9, 10, 11, 12, 13, 14, 15].

In this paper, we introduce the Bernoulli Ritz-collocation method for solving modelling the pollution of a system of lakes. In 2006, Biazar et al. analyzed this model [16]. Recently, this model was solved using the variational iteration method [17], the homotopy perturbation method [18], the Bessel collocation method [19] and the modified differential transformation method [20].

**Figure 1.** System of three lakes with interconnecting channels.

The system of three lakes are showed in Fig. 1. We consider each lake to be a large compartment and the interconnecting channel as pipes between the compartments. We indicate the direction of flow in the channels or pipes by the arrows in the Fig. 1. A pollutant is introduced into the first lake where \( p(t) \) denotes the rate at which the pollutant enters the lake per unit time. The function \( p(t) \) may be constant or vary with time. We are interested to obtain the levels of pollution in each lake at any time. We denote \( x_i(t) \) the amount of the pollutant in lake \( i \) at any time \( t \geq 0 \), for \( i = 1, 2, 3 \). We assume the pollutant in each lake to be uniformly distributed throughout the lake by some mixing process. We consider that the volume of water \( V_i \) in lake \( i \) remain constant for each lakes. Then the concentration of the pollutant in lake \( i \) at any time
is obtained by
\[ c_i(t) = \frac{x_i(t)}{V_i}. \]

Each lake initially is assumed to be free of any contaminant, so \( x_i(0) = 0 \) for \( i = 1, 2, 3 \). To model the dynamic behavior of the system of lakes, we denote constant \( F_{ji} \) the flow rate from lake, \( i \) to lake, \( j \). These flow rates, which could be measured in gallons per minute or any other convenient units, are indicated in Fig. 1. Since there is no channel allowing any flow from lake 2 to lake 1, Note that \( F_{12} = 0 \). The flux of pollutant flowing from lake \( i \) into lake \( j \) at any time, written by \( r_{ji}(t) \), is defined by
\[ r_{ji}(t) = F_{ji} c_i(t) = \frac{F_{ji}}{V_i} x_i(t) \]
So \( r_{ji}(t) \) measures the rate at which the concentration of pollutant in lake \( i \) flows into lake \( j \) at time \( t \). We will observe that
Rate of change of pollutant = Input rate − Output rate.
Using the above principle to each lake results in the following system of first order equations modelling the dynamic behavior of the lakes system:
\[
\begin{align*}
\frac{dx_1(t)}{dt} &= \frac{F_{13}}{V_3} x_3(t) + p(t) - \frac{F_{31}}{V_1} x_1(t) - \frac{F_{21}}{V_1} x_1(t), \\
\frac{dx_2(t)}{dt} &= \frac{F_{21}}{V_1} x_1(t) - \frac{F_{32}}{V_2} x_2(t), \quad 0 \leq t \leq 1 \\
\frac{dx_3(t)}{dt} &= \frac{F_{31}}{V_1} x_1(t) + \frac{F_{32}}{V_2} x_2(t) - \frac{F_{13}}{V_3} x_3(t),
\end{align*}
\]
(1.1)
with the initial conditions:
\[ x_i(0) = 0, \quad i = 1, 2, 3. \]
(1.2)
In order for the volume of each lake to remain constant, the flow rate into each lake must balance the flow out of the lake. Thus we assume the following conditions:
- Lake 1: \( F_{13} = F_{21} + F_{31} \),
- Lake 2: \( F_{21} = F_{32} \),
- Lake 3: \( F_{31} + F_{32} = F_{13} \).
In this paper, we approximate the solution of system (1.1) in terms of the truncated Ritz series as
\[ x_i(t) \cong x_{i,m}(t) = \sum_{j=0}^{m} b_{i,j}^* t B_j(t) = \sum_{j=0}^{m} b_{i,j}^* \phi_j(t), \quad i = 1, 2, 3, \quad 0 \leq t \leq 1 \]
(1.3)
where \( b_{i,j}^* \ (j = 0,1,...,m) \) are unknown Ritz series coefficients and \( B_j(t) \ (j = 0,1,...,m) \) are the Bernoulli polynomials are introduced in section 2.

The organization of this paper is as follows:
In the next section, we introduce the basic Bernoulli polynomials and numbers and some their properties. In section 3, we present the application of Bernoulli Ritz-collocation method to the solution of modelling the pollution of a system of lakes. In section 4, we apply the Bernoulli Ritz-collocation method to a numerical example to demonstrate the accuracy of the present numerical method and the last section is included to conclusions.

2. Bernoulli polynomials and some their properties

Bernoulli polynomials play an important role in various expansions and approximation formulas which are useful both in analytic theory of numbers and in classical and numerical analysis. These polynomials can be defined by various methods depending on the applications [21, 22, 23, 24, 25].

The Bernoulli polynomials of i-th degree are defined on the interval \([0,1]\) as

\[
B_i(x) = \sum_{k=0}^{i} \binom{i}{k} B_k x^{i-k}
\]

where \( B_k := B_k(0) \ (k = 0,1,...,i) \) are the Bernoulli numbers. Leopold Kronecker expressed the Bernoulli number \( B_i \) in the following form

\[
B_i = -\sum_{k=1}^{i+1} \frac{(-1)^k}{k} \binom{i+1}{k} \sum_{j=1}^{k} j^i, \quad i \geq 0, \quad i \neq 1.
\]

If \( i = 1 \), we defined \( B_1 = -\frac{1}{2} \). The Bernoulli polynomials and Bernoulli numbers are produced by the following generating functions, respectively,

\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}
\]

and

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.
\]

We have, in particular, \( B_0 = 1, \ B_1 = -\frac{1}{2}, \ B_2 = \frac{1}{6}, \ B_3 = 0 \), and \( B_{2k+1} = 0 \), for \( k = 2,3,..., \)

\[
B_0(x) = 1, \ B_1(x) = x - \frac{1}{2}, \ B_2(x) = x^2 - x + \frac{1}{6}, \ B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.
\]
If, we define vector $C_{i+1}$, as

$$C_{i+1} = \begin{bmatrix} \binom{i}{0} B_i & \binom{i}{1} B_{i-1} & \binom{i}{2} B_{i-2} & \cdots & \binom{i}{m-i} B_0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{1 \times (m+1)}$$

then $B_i(t) = C_{i+1} X^T(t)$, for $i=0, 1, \ldots, m$, where

$$X(t) = [1 \ t \ t^2 \ \ldots \ t^m].$$

Now, we can expand the matrix $B(t) = [B_0(t) \ B_1(t) \ \ldots \ B_m(t)]$ as

$$B(t) = X(t) C^T,$$  \hspace{1cm} (2.1)

where

$$C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_{m+1} \end{bmatrix} = \begin{bmatrix} B_0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{1}{1} B_1 & \binom{1}{0} B_0 & 0 & 0 & \cdots & 0 \\ \binom{2}{1} B_2 & \binom{2}{0} B_1 & \binom{2}{0} B_0 & 0 & \cdots & 0 \\ \binom{3}{1} B_3 & \binom{3}{0} B_2 & \binom{3}{0} B_1 & \binom{3}{0} B_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{m}{1} B_m & \binom{m}{0} B_{m-1} & \binom{m}{0} B_{m-2} & \binom{m}{0} B_{m-3} & \cdots & \binom{m}{0} B_0 \end{bmatrix}_{(m+1) \times (m+1)}.$$  \hspace{1cm}

Also, if we define vector $D_{i+1}$, as

$$D_{i+1} = \begin{bmatrix} 0 & \binom{i}{1} B_{i-1} & 2 \binom{i}{2} B_{i-2} & \cdots & (i-1) \binom{i}{i} B_1 & i \binom{i}{i} B_0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{1 \times (m+1)}$$

then $t \frac{dB_i(t)}{dt} = D_{i+1} X^T(t)$ for $i=0, 1, \ldots, m$, and we have

$$t \frac{dB(t)}{dt} = X(t) D^T$$  \hspace{1cm} (2.2)
where

\[
D = \begin{bmatrix}
D_1 & D_2 & D_3 & D_4 & \cdots & D_{m+1} \\
0 & (1^0)B_0 & 0 & 0 & \cdots & 0 \\
0 & (2^1)B_1 & 2(0^0)B_0 & 0 & \cdots & 0 \\
0 & (3^2)B_2 & 2(1^1)B_1 & 3(3^0)B_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & (m_{m-1})B_{m-1} & 2(m_{m-2})B_{m-2} & 3(m_{m-3})B_{m-3} & \cdots & m(0^m)B_0
\end{bmatrix}_{(m+1) \times (m+1)}
\]

By using relations (2.1) and (2.2), we get

\[
\frac{d\phi(t)}{dt} = B(t) + t \frac{dB(t)}{dt} = T(t)C^T + T(t)D^T = T(t)(C^T + D^T) = T(t)M,
\]

where \( \phi(t) = tB(t) = [tB_0(t) \ tB_1(t) \ \ldots \ tB_m(t)] \).

3. Bernoulli Ritz-collocation method

First, we can write the functions defined in relation (1.3) in the form

\[
x_{i,m}(t) = \phi(t)B_i^*, \quad i = 1, 2, 3
\]

where \( \phi(t) = [tB_0(t) \ tB_1(t) \ \ldots \ tB_m(t)] \), \( B_i^* = [b_{i,0}^* \ b_{i,1}^* \ \ldots \ b_{i,m}^*]^T \), \( i = 1, 2, 3 \).

Using the relations (2.3) and (3.1), we get recurrence relation

\[
\frac{dx_{i,m}(t)}{dt} = \frac{d\phi(t)}{dt} B_i^* = T(x)MB_i^*, \quad i = 1, 2, 3
\]

By substituting the relations (3.1) and (3.2) in the system (1.1), we have

\[
\begin{cases}
T(t)MB_1^* - \frac{F_{13}}{V_3} \phi(t)B_3^* + \frac{F_{31}}{V_1} \phi(t)B_1^* + \frac{F_{21}}{V_1} \phi(t)B_1^* = p(t), \\
T(t)MB_2^* - \frac{F_{21}}{V_1} \phi(t)B_1^* + \frac{F_{32}}{V_2} \phi(t)B_2^* = 0, \quad 0 \leq t \leq 1 \\
T(t)MB_3^* - \frac{F_{31}}{V_1} \phi(t)B_1^* - \frac{F_{32}}{V_2} \phi(t)B_2^* + \frac{F_{13}}{V_3} \phi(t)B_3^* = 0,
\end{cases}
\]
or briefly

\[ \tilde{T}(t) \tilde{M} + \theta \tilde{\phi}(t) \] \( B^* = \alpha(t) \) \hspace{1cm} (3.3)

where

\[ \tilde{T}(t) = \begin{bmatrix} T(t) & 0 & 0 \\ 0 & T(t) & 0 \\ 0 & 0 & T(t) \end{bmatrix}_{3 \times 3}, \tilde{M} = \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix}_{3 \times 3}, \theta = \begin{bmatrix} F_{31} + F_{21} \frac{V_1}{V_3} & 0 & -F_{13} \frac{V_1}{V_3} \\ -F_{21} \frac{V_1}{V_3} & F_{32} \frac{V_2}{V_3} & 0 \\ -F_{31} \frac{V_1}{V_3} & -F_{32} \frac{V_2}{V_3} & F_{13} \frac{V_1}{V_3} \end{bmatrix}_{3 \times 3}, \]

\[ \tilde{\phi}(t) = \begin{bmatrix} \phi(t) & 0 & 0 \\ 0 & \phi(t) & 0 \\ 0 & 0 & \phi(t) \end{bmatrix}_{3 \times 3}, \alpha(t) = \begin{bmatrix} p(t) \\ 0 \end{bmatrix}_{3 \times 1}, \beta^* = \begin{bmatrix} \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{bmatrix}_{3 \times 1}. \]

Now, we substitute the collocation points \( t_s \) in relation (3.3) as

\[ \tilde{T}(t_s) \tilde{M} + \theta \tilde{\phi}(t_s) \] \( B^* = \alpha(t_s), \ s = 0, 1, ..., m \)

or briefly

\[ \tilde{T} \tilde{M} + \Theta \tilde{\phi} \] \( B^* = \alpha \) \hspace{1cm} (3.4)

where

\[ T = \begin{bmatrix} \tilde{T}(t_0) \\ \tilde{T}(t_1) \\ \vdots \\ \tilde{T}(t_m) \end{bmatrix}_{(m+1) \times 1}, \Theta = \begin{bmatrix} \theta & 0 & \cdots & 0 \\ 0 & \theta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta \end{bmatrix}_{(m+1) \times (m+1)}, \tilde{\phi} = \begin{bmatrix} \tilde{\phi}(t_0) \\ \tilde{\phi}(t_1) \\ \vdots \\ \tilde{\phi}(t_m) \end{bmatrix}_{(m+1) \times 1} \text{ and } \alpha = \begin{bmatrix} \alpha(t_0) \\ \alpha(t_1) \\ \vdots \\ \alpha(t_m) \end{bmatrix}_{(m+1) \times 1}. \]

The collocation points \( t_s \) \( (s = 0, 1, ..., m) \) are roots of the well-known shifted Chebyshev polynomials of order \( (m+1) \) on the interval \([0, 1]\) and also can be chosen as \( t_s = \frac{s}{m}, s = 0, 1, ..., m. \)

We can write the system (3.4) in the form \( FB^* = \alpha, \) where \( F = \tilde{T} \tilde{M} + \Theta \tilde{\phi}. \) This system is an algebraic system of equations with \( 3 \times (m+1) \) equations and \( 3 \times (m+1) \) unknown coefficients, \( b_{i,j}^* \) for \( i = 1, 2, 3 \) and \( j = 0, 1, ..., m. \)

4. Error analysis and numerical example

We can easily check the convergence behavior of the Bernoulli Ritz-collocation method to the solution of the pollution of a system of lakes, by error functions \( e_{i,m}(t) \) which come from putting the series solution \( x_{i,m}(t) \) in system (1.1) defined by

\[ e_{1,m}(t) = \left| \frac{dx_{1,m}(t)}{dt} - \frac{F_{13}}{V_3} x_{3,m}(t) - p(t) + \frac{F_{31}}{V_1} x_{1,m}(t) + \frac{F_{21}}{V_1} x_{1,m}(t) \right|, \]
4.1. Example. Consider the following system of equations \[16\]

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= \frac{38}{1180} x_3(t) + \{1 + \sin(t)\} - \frac{20}{2900} x_1(t) - \frac{18}{2900} x_1(t), \\
\frac{dx_2(t)}{dt} &= \frac{18}{2900} x_1(t) - \frac{18}{850} x_2(t), \\
\frac{dx_3(t)}{dt} &= \frac{20}{2900} x_1(t) + \frac{18}{850} x_2(t) - \frac{38}{1180} x_3(t),
\end{align*}
\]

with the initial conditions:

\[ x_i(0) = 0, \quad i = 1, 2, 3. \]

Also, in this example \( p(t) = 1 + \sin(t) \) and the parameter values have been fixed to \( V_1 = 2900 \text{ mi}^3, V_2 = 850 \text{ mi}^3, V_3 = 1180 \text{ mi}^3, F_{21} = 18 \text{ mi}^3/year, F_{32} = 18 \text{ mi}^3/year, F_{31} = 20 \text{ mi}^3/year, F_{13} = 38 \text{ mi}^3/year. \)

We solve the system of equations (4.1) by the Bernoulli Ritz-collocation method. The approximate solutions \( x_{i,m} (i = 1, 2, 3) \) of this system of equations are obtained for \( m = 4, 7, 9, \) respectively,

\[
\begin{align*}
x_{1,4}(t) &= (0.40465058683e-002) t^5 - (0.458432145597e-001) t^4 - (0.150994938452e-003) t^3 + 0.493097145228 t^2 + 1.0 t, \\
x_{1,7}(t) &= -(0.214732591743e-004) t^8 - (0.107307557937e-004) t^7 + (0.139751725613e-002) t^6 + (0.103363484193e-003) t^5 - (0.416486605518e-001) t^4 - (0.211867369508e-002) t^3 + 0.493448390148 t^2 + 1.0 t, \\
x_{1,9}(t) &= (0.238913116846e-006) t^{10} + (0.150111869601e-006) t^9 - (0.249642493912e-004) t^8 - (0.244247763703e-005) t^7 + (0.138825363085e-002) t^6 + (0.109138640518e-003) t^5
\end{align*}
\]
and the error functions \( e(0) \) of \( m = 4, 7, 9 \), respectively, of lake 1, 2 and 3. The condition number \( K_p(F) \) of \( p = 2, \infty \) and CPU time at various value of \( m \) of system (4.1) are shown in Table 4. Figs. 2, 3 and 4 give us the comparison of the error functions \( e_1(m), e_2(m), e_3(m) \) for \( m = 4, 7, 9 \) and 0 \( \leq t \leq 1 \), respectively, of lake 1, 2 and 3.

**Table 1:** Approximate solutions \( x_{1,m}(t) \) and the error functions \( e_{1,m}(t) \), for \( m = 4, 7, 9 \), of lake 1.

<table>
<thead>
<tr>
<th>( t_i )</th>
<th>( m = 4, x_{1,4}(t_i) )</th>
<th>( m = 4, e_{1,4}(t_i) )</th>
<th>( m = 7, x_{1,7}(t_i) )</th>
<th>( m = 7, e_{1,7}(t_i) )</th>
<th>( m = 9, x_{1,9}(t_i) )</th>
<th>( m = 9, e_{1,9}(t_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0000000</td>
<td>0.0000e-00</td>
<td>0.000000</td>
<td>0.0000e-00</td>
<td>0.000000</td>
<td>0.0000e-00</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2196506</td>
<td>9.7510e-06</td>
<td>0.2196545</td>
<td>2.5589e-01</td>
<td>0.2196545</td>
<td>3.7269e-01</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4776637</td>
<td>9.1587e-06</td>
<td>0.4776576</td>
<td>5.8930e-01</td>
<td>0.4776576</td>
<td>4.6978e-01</td>
</tr>
<tr>
<td>0.6</td>
<td>0.7715570</td>
<td>8.9919e-06</td>
<td>0.7718587</td>
<td>6.1398e-01</td>
<td>0.7718587</td>
<td>8.0520e-01</td>
</tr>
<tr>
<td>0.8</td>
<td>1.0980530</td>
<td>9.2274e-06</td>
<td>1.0980570</td>
<td>2.8829e-01</td>
<td>1.0980570</td>
<td>5.2182e-01</td>
</tr>
<tr>
<td>1.0</td>
<td>1.4511490</td>
<td>3.7076e-01</td>
<td>1.4511500</td>
<td>1.3810e-01</td>
<td>1.4511500</td>
<td>1.6909e-01</td>
</tr>
</tbody>
</table>

**Table 2:** Approximate solutions \( x_{2,m}(t) \) and the error functions \( e_{2,m}(t) \), for \( m = 4, 7, 9 \), of lake 2.
by taking m to be sufficiently large. The efficiency of the method for solving the system of lakes (1.1) is easily implemented and it is simple and efficient to approximate the pollution of the system (1.1). The numerical example is given, which show that the present method is then utilized to reduce modelling the pollution of the system (1.1). The solution of the system (1.1) can be obtained to arbitrarily high accuracy.

Table 3: Approximate solutions $x_{3,m}(t)$ and the error functions $e_{3,m}(t)$, for $m = 4, 7, 9$, of lake 3.

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>$m = 4, x_{3,4}(t_i)$</th>
<th>$m = 4, e_{3,4}(t_i)$</th>
<th>$m = 7, x_{3,7}(t_i)$</th>
<th>$m = 7, e_{3,7}(t_i)$</th>
<th>$m = 9, x_{3,9}(t_i)$</th>
<th>$m = 9, e_{3,9}(t_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0000000e - 000</td>
<td>0.0000000e - 000</td>
<td>0.0000000e - 000</td>
<td>0.0000000e - 000</td>
<td>0.0000000e - 000</td>
<td>0.0000000e - 000</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1320841e - 003</td>
<td>3.4378e - 008</td>
<td>0.1320999e - 003</td>
<td>2.9599e - 012</td>
<td>0.1320999e - 003</td>
<td>6.7759e - 015</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5597265e - 003</td>
<td>3.2815e - 008</td>
<td>0.5597436e - 003</td>
<td>4.5324e - 013</td>
<td>0.5597436e - 003</td>
<td>5.1636e - 015</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1327928e - 002</td>
<td>3.2815e - 008</td>
<td>0.1327949e - 002</td>
<td>6.5335e - 013</td>
<td>0.1327949e - 002</td>
<td>1.5133e - 014</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2477567e - 002</td>
<td>3.4378e - 008</td>
<td>0.2477594e - 002</td>
<td>2.9597e - 012</td>
<td>0.2477594e - 002</td>
<td>1.8487e - 015</td>
</tr>
<tr>
<td>1.0</td>
<td>0.4043712e - 002</td>
<td>9.8247e - 018</td>
<td>0.4043728e - 002</td>
<td>2.5713e - 016</td>
<td>0.4043728e - 002</td>
<td>2.6610e - 014</td>
</tr>
</tbody>
</table>

Table 4: Condition number $K_p(F)$ of $p = 2, \infty$ and CPU time at various value of $m$ of system (4.1).

<table>
<thead>
<tr>
<th>$m =$</th>
<th>4</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_2(F)$</td>
<td>5.9706e + 001</td>
<td>8.0873e + 003</td>
<td>4.8105e + 004</td>
<td>2.9455e + 005</td>
</tr>
<tr>
<td>$K_\infty(F)$</td>
<td>1.4708e + 002</td>
<td>1.7999e + 004</td>
<td>1.1239e + 005</td>
<td>7.2357e + 005</td>
</tr>
<tr>
<td>CPU time(s)</td>
<td>2.75</td>
<td>2.85</td>
<td>3.82</td>
<td>4.29</td>
</tr>
</tbody>
</table>

5. CONCLUSION

In this paper, the Bernoulli Ritz-collocation method is presented to the solution of modelling the pollution of a system of lakes. The presented method is then utilized to reduce modelling the pollution of a system of lakes to solution of algebraic equations. This method is easily implemented and it is simple and efficient to approximate the solution of system (1.1). The numerical example is given, which show the efficiency of the method for solving the system of lakes (1.1). In total, the presented method is easy and rapid to compute. Also, any solution of the system (1.1) can be obtained to arbitrarily high accuracy by taking m to be sufficiently large.
Figure 2. Comparison of the error functions $e_{1,m}$ for $m = 4, 7, 9$ and $0 \leq t \leq 1$ of lake 1.

Figure 3. Comparison of the error functions $e_{2,m}$ for $m = 4, 7, 9$ and $0 \leq t \leq 1$ of lake 2.

Figure 4. Comparison of the error functions $e_{3,m}$ for $m = 4, 7, 9$ and $0 \leq t \leq 1$ of lake 3.

References


The Bernoulli Ritz-collocation method


