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NOTES ON REGULAR MULTIPLIER HOPF ALGEBRAS

H. Abbasi[1] and G. A. HAGHIGHATDOOST[2]
[1] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran. Email: abbasi.makrani@gmail.com
[2] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran. Email: gorbanali@azaruniv.ac.ir

Abstract. In this paper, we associate canonically a precyclic module to a regular multiplier Hopf algebra endowed with a group-like projection and a modular pair in involution satisfying certain conditions.

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1. Introduction

The notion of a multiplier Hopf algebra is a natural generalization of the notion of a Hopf algebra. Multiplier Hopf algebras were introduced in [8] by Van Daele. In this framework, one can consider an algebra $\mathcal{H}$ over the field $\mathbb{C}$, with or without an identity, but with a non-degenerate multiplication map. There is a homomorphism Δ from $\mathcal{H}$ to the multiplier algebra $M(\mathcal{H}\otimes\mathcal{H})$ of $\mathcal{H}\otimes\mathcal{H}$. Certain conditions on Δ are imposed. The motivating example is the algebra of complex valued, finitely supported functions on an arbitrary group (see [8]). If $\mathcal{H}$ has an identity, then we have a Hopf algebra.

[1] Corresponding author: abbasi.makrani@gmail.com
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In [2, 3], introduced a cyclic module for any Hopf algebra endowed with a modular pair in involution, i.e., with a group-like element σ and a character such that the corresponding doubly twisted antipode has square the identity. The simplicial structure is associated to the comultiplication of the Hopf algebra and the group-like element σ, while the cyclic structure makes use of the product and of the twisted antipode.

Note that the group-like element σ is a non-zero element in the Hopf algebra $(\mathcal{H}, \Delta)$ satisfying $\Delta(\sigma) = \sigma \otimes \sigma$. This definition is not very useful when $(\mathcal{H}, \Delta)$ is a multiplier Hopf algebra. In this paper, a group-like element is a non-zero element σ in the multiplier algebra $M(\mathcal{H})$ of $\mathcal{H}$ satisfying

$$\Delta(\sigma) = \sigma \otimes \sigma.$$

This makes sense because Δ has a unique extension to the multiplier algebra $M(\mathcal{H})$.

In [5], introduced the notion of a group-like projection for any multiplier Hopf *-algebra. In this paper, we do not require the existence of an involution and we will work with a regular multiplier Hopf algebra $(\mathcal{H}, \Delta)$. By a group-like projection we mean a non-zero idempotent element γ in $\mathcal{H}$ satisfying

$$\Delta(\gamma)(1 \otimes \gamma) = \gamma \otimes \gamma = \Delta(\gamma)(\gamma \otimes 1).$$

The notion of a group-like projection should not be confused with the notion of a group-like element. A group-like element σ in a Hopf algebra is always invertible and its inverse is $S(\sigma)$, where S is the antipode of the Hopf algebra. If an idempotent element in a unital algebra is invertible, then it is itself identity.

In this paper, we associate a precyclic module to a regular multiplier Hopf algebra $(\mathcal{H}, \Delta)$ endowed with a matched pair (χ, γ) and a modular pair in involution (χ, σ) such that γ and σ are compatible, i.e., with a group-like projection γ, a non-zero algebra homomorphism $\chi : \mathcal{H} \to \mathbb{C}$, a group-like element $\sigma \in M(\mathcal{H})$ and a modular pair in involution (χ, σ) such that $\chi(\gamma) = 1$ and $\gamma\sigma = \sigma\gamma$.

The presimplicial structure is associated to the comultiplication Δ, the multiplication, the group-like projection γ and the group-like element σ. Note that the group-like projection γ is a basic object in our paper and allows us to define the face operators. It is important to mention that if $\mathcal{H}$ is a Hopf algebra and $\gamma = 1$, then the face operators in this paper are the same with the face operators of Connes-Moscovici for Hopf algebras (see [2, 3]).

The standard references for Hopf algebras are [1, 6, 7]. For the basic theory of multiplier Hopf algebras, we refer the reader to [8].

2. Preliminaries and Basic Concepts

Throughout this paper, all vector spaces will be spaces over the complex field $\mathbb{C}$. We will use the convention that $V \otimes W$ represents the algebraic tensor product in which V and W are vector spaces.

An algebra $\mathcal{A}$ is called non-degenerate if the product in $\mathcal{A}$ is non-degenerate, i.e., if $ab = 0$ for all b implies $a = 0$ and $ab = 0$ for all a implies $b = 0$. It is clear that any unital algebra is a non-degenerate algebra. Using Lemma $A.2$ in [8], the algebraic tensor product of two non-degenerate algebras is again an algebra with a non-degenerate product.

Let $\mathcal{A}$ be a non-degenerate algebra. The multiplier algebra $M(\mathcal{A})$ is defined as the set of pairs (l, r) of linear maps of $\mathcal{A}$ into $\mathcal{A}$ satisfying:

$$r(a)b = al(b),$$

for all $a, b \in \mathcal{A}$. It is easy to see that this set $M(\mathcal{A})$ can be made into an associative algebra. It always contains a unit, and that $\mathcal{A}$ has a natural imbedding as an essential two-sided ideal in $M(\mathcal{A})$. There are natural imbeddings

$$\mathcal{A} \otimes \mathcal{A} \to M(\mathcal{A}) \otimes M(\mathcal{A}) \to M(\mathcal{A} \otimes \mathcal{A}).$$

We will write xa for $l(a)$ and ax for $r(a)$ when $x = (l, r)$ is an element of $M(\mathcal{A})$ and $a \in \mathcal{A}$. If $\mathcal{A}, \mathcal{B}$ are non-degenerate algebras, then a homomorphism φ from $\mathcal{A}$ to $M(\mathcal{B})$ is said to be non-degenerate if and only if $\varphi(\mathcal{A})\mathcal{B} = \mathcal{B} = \mathcal{B}\varphi(\mathcal{A})$. Using Proposition $A.5$ in [8], such a non-degenerate homomorphism has a unique extension to a unital homomorphism $M(\mathcal{A}) \to M(\mathcal{B})$.

For the following definition, we refer the reader to [8].

Definition 2.1. Assume that $\mathcal{H}$ is a non-degenerate algebra and Δ from $\mathcal{H}$ to $M(\mathcal{H} \otimes \mathcal{H})$ is a non-degenerate homomorphism. The pair $(\mathcal{H}, \Delta)$ is called a multiplier Hopf algebra if

- (i) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$,
- (ii) $\Delta(h)(1 \otimes k)$ and $(h \otimes 1)\Delta(k)$ are elements in $\mathcal{H} \otimes \mathcal{H}$,
- (iii) the linear mappings $T_1, T_2 : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, defined by

$$T_1(h \otimes k) = \Delta(h)(1 \otimes k), \qquad T_2(h \otimes k) = (h \otimes 1)\Delta(k),$$

for all $h, k \in \mathcal{H}$ are bijective.

Since Δ is a non-degenerate homomorphism, so $\Delta \otimes \iota$ can be extended to the multiplier algebra $M(\mathcal{H} \otimes \mathcal{H})$. Similarly, $\iota \otimes \Delta$ can be extended to the multiplier algebra $M(\mathcal{H} \otimes \mathcal{H})$. The homomorphism Δ is called a comultiplication on $\mathcal{H}$. We can consider the opposite comultiplication Δ' obtained from Δ by composing it with the flip operator on $\mathcal{H} \otimes \mathcal{H}$.

A multiplier Hopf algebra $(\mathcal{H},\Delta)$ is said to be regular if $(\mathcal{H},\Delta')$ also is a multiplier Hopf algebra.

Let $(\mathcal{H},\Delta)$ be a regular multiplier Hopf algebra. There is a unique non-zero homomorphism ε from $\mathcal{H}$ to $\mathbb{C}$ such that

$$(\varepsilon\otimes\iota)\Delta=\iota=(\iota\otimes\varepsilon)\Delta.$$

The map ε is called the counit of $(\mathcal{H},\Delta)$. Also there is a unique invertible anti-homomorphism S from $\mathcal{H}$ to $\mathcal{H}$ that satisfies the conditions

$$m(S\otimes\iota)(\Delta(h)(1\otimes k))=\varepsilon(h)k,$$

$$m(\iota\otimes S)((k\otimes 1)\Delta(h))=\varepsilon(h)k,$$

for all $h,k\in\mathcal{H}$, where m is the multiplication map. The map S is called the antipode of $(\mathcal{H},\Delta)$ (see [8]).

We use the generalized Sweedler notation for the comultiplication Δ. We can define $\Delta^{(n)}$ for any $n\geq -1$ according to

$$\Delta^{(-1)}:=\varepsilon,$$

$$\Delta^{(n)}:=(\Delta^{(n-1)}\otimes\iota)\Delta,$$

for all $n\geq 0$. We have the following formula as a direct consequence of the properties of Δ (see [1, 4, 6, 7]):

$$\Delta^{(n+m+r)}=(\iota^{\otimes(n)}\otimes\Delta^{(m)}\otimes\iota^{\otimes(r)})\Delta^{(n+r)},$$

for all $n,m,r\geq 0$. Let $h\in\mathcal{H}$. For all $k_1,\cdots,k_{n+1}$ in $\mathcal{H}$ we write

$$\Delta^{(n)}(h)(k_1\otimes\cdots\otimes k_{n+1}):=\sum h_{(1)}k_1\otimes\cdots\otimes h_{(n+1)}k_{n+1},$$

$$(k_1\otimes\cdots\otimes k_{n+1})\Delta^{(n)}(h):=\sum k_1h_{(1)}\otimes\cdots\otimes k_{n+1}h_{(n+1)}.$$

For all $h,k\in\mathcal{H}$ we put

$$\Delta(h)(1\otimes k):=\sum h_{(1)}1\otimes h_{(2)}k,$$

$$(1\otimes k)\Delta(h):=\sum 1h_{(1)}\otimes kh_{(2)},$$

$$\Delta(h)(k\otimes 1):=\sum h_{(1)}k\otimes h_{(2)}1,$$

$$(k\otimes 1)\Delta(h):=\sum kh_{(1)}\otimes 1h_{(2)}.$$

Throughout this paper, we fix the regular multiplier Hopf algebra $\mathcal{H}$, and we will use $m,\Delta,\Delta',\varepsilon,S$ for the multiplication, the comultiplication, the opposite comultiplication, the counit and the antipode, respectively. Also, τ will denote the flip operator $\tau:\mathcal{H}\otimes\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$.

3. Group-like projection and modular pair in involution

We begin with the definition of a group-like projection. This is essentially the basic object for the rest of the paper, in the classical case, it behaves like the characteristic function of a subgroup (see [5]).

In [5], introduced the notion of a group-like projection for any multiplier Hopf *-algebra. In this paper, we do not require the existence of an involution and therefore, the assumptions are a little different.

Definition 3.1. A non-zero element $\gamma \in \mathcal{H}$ is called a group-like projection if

$$\Delta(\gamma)(1 \otimes \gamma) = \gamma \otimes \gamma = \Delta(\gamma)(\gamma \otimes 1), \quad \gamma^2 = \gamma.$$

There are the following results, related with the group-like projections.

Lemma 3.2. *Let γ be a group-like projection in $\mathcal{H}$. Then $\varepsilon(\gamma) = 1$ and $S(\gamma) = \gamma$.*

Proof. Apply $\varepsilon \otimes \iota$ to the defining equation $\Delta(\gamma)(1 \otimes \gamma) = \gamma \otimes \gamma$. Then we get $\gamma\varepsilon(\gamma) = \gamma^2$. Since $\gamma^2 = \gamma$ and γ is assumed to be non-zero, so we obtain $\varepsilon(\gamma) = 1$. Using the properties of S one can get $S(\gamma)\gamma = \gamma$. Because $\mathcal{H}$ is a regular multiplier Hopf algebra, so S^{-1} is the antipode of $(\mathcal{H}, \Delta')$. Using $\Delta'(\gamma)(1 \otimes \gamma) = \gamma \otimes \gamma$ and the properties of S^{-1} one can get $S^{-1}(\gamma)\gamma = \gamma$. If we apply S, we get $S(\gamma)\gamma = S(\gamma)$. Hence, we have $S(\gamma) = \gamma$. □

Let $\chi : \mathcal{H} \to \mathbb{C}$ be a non-zero algebra homomorphism. Define a map

$$\widetilde{S} : \mathcal{H} \to L(\mathcal{H}),$$

by

$$\widetilde{S}(h)k := (\chi \otimes \iota)T_1^{-1}(h \otimes k),$$

where T_1 is defined by $T_1(h \otimes k) = \Delta(h)(1 \otimes k)$ and $L(\mathcal{H})$ is the space of all linear maps $l : \mathcal{H} \to \mathcal{H}$ such that $l(hk) = l(h)k$, for all $h, k \in \mathcal{H}$.

By definition of $\widetilde{S}$ we see that elements of the form $\widetilde{S}(h)k$ span $\mathcal{H}$.

Lemma 3.3. *We have $\widetilde{S}(\mathcal{H}) \subset \mathcal{H}$.*

Proof. Take an element $e \in \mathcal{H}$ such that $\chi(e) = 1$. Let $h \in \mathcal{H}$. For all $k \in \mathcal{H}$, we have

$$\widetilde{S}(h)k = (\chi \otimes \iota)((e \otimes 1)T_1^{-1}(h \otimes k)).$$

Using Lemma 4.2 in [8], we get

$$\begin{aligned}\widetilde{S}(h)k &= (\chi \otimes \iota)((\iota \otimes S)((e \otimes 1)\Delta(h))(1 \otimes k)) \\ &= (\chi \otimes S)((e \otimes 1)\Delta(h))k.\end{aligned}$$

Hence, we have $\widetilde{S}(h) = (\chi \otimes S)((e \otimes 1)\Delta(h)) \in \mathcal{H}$. □

The elementary properties of S imply that $\widetilde{S}$ is an algebra anti-homomorphism, i.e.,

$$\widetilde{S}(hk) = \widetilde{S}(k)\widetilde{S}(h),$$

for all $h, k \in \mathcal{H}$. Also, it satisfies

$$\varepsilon \circ \widetilde{S} = \chi.$$

For the following result, we refer the reader to [8].

Lemma 3.4. *For all $h, k \in \mathcal{H}$, we have*

$$(1 \otimes S(k))\Delta(S(h)) = (S \otimes S) \circ \tau(\Delta(h)(k \otimes 1)).$$

We obtain the similar results for $\widetilde{S}$ in the next lemma.

Lemma 3.5. *For all $h, k \in \mathcal{H}$, we have*

$$(1 \otimes \widetilde{S}(k))\Delta(\widetilde{S}(h)) = (S \otimes \widetilde{S}) \circ \tau(\Delta(h)(k \otimes 1)),$$

$$\Delta(\widetilde{S}(h))(S(k) \otimes 1) = (S \otimes \widetilde{S}) \circ \tau((1 \otimes k)\Delta(h)).$$

Proof. Take an element $e \in \mathcal{H}$ such that $\chi(e) = 1$. We have

$$(S\otimes\widetilde{S})\circ\tau(\Delta(h)(k\otimes 1)) = \sum (S\otimes\chi\otimes S)(h_{(2)}1\otimes(h_{(1)}k)_{(1)}e\otimes(h_{(1)}k)_{(2)}1),$$

for all $h, k \in \mathcal{H}$. Using the properties of Δ we get

$$(S \otimes \widetilde{S}) \circ \tau(\Delta(h)(k \otimes 1)) =$$

$$\sum (S \otimes \chi \otimes S)((h_{(2)}1)_{(2)}1 \otimes h_{(1)}(k_{(1)}e) \otimes (h_{(2)}1)_{(1)}(k_{(2)}1)),$$

for all $h, k \in \mathcal{H}$. Hence, we have

$$\begin{aligned}(S \otimes \widetilde{S}) \circ \tau(\Delta(h)(k \otimes 1)) &= \sum (1 \otimes S(k_{(2)}1))\chi(k_{(1)}e)\Delta(\widetilde{S}(h)) \\ &= (1 \otimes (\chi \otimes S)(\Delta(k)(e \otimes 1)))\Delta(\widetilde{S}(h)) \\ &= (1 \otimes \widetilde{S}(k))\Delta(\widetilde{S}(h)),\end{aligned}$$

and similarly for the other formula. □

Because $\mathcal{H}$ is a regular multiplier Hopf algebra, so the antipode S is invertible. We can check that a similar fact is true for $\widetilde{S}$ and its inverse is $\widetilde{S}^{-1} := (\chi \otimes S^{-1})\Delta'$.

Definition 3.6. Let $\chi : \mathcal{H} \to \mathbb{C}$ be a non-zero algebra homomorphism and γ be a group-like projection in $\mathcal{H}$. We call (χ, γ) is a matched pair if

$$\chi(\gamma) = 1.$$

It is easy to check that if (χ, γ) is a matched pair, then $\widetilde{S}(\gamma) = \gamma$.

A non-zero element $\sigma \in M(\mathcal{H})$ is called a group-like element if

$$\Delta(\sigma) = \sigma \otimes \sigma,$$

when we look at the extension of Δ to $M(\mathcal{H})$. It follows that $\varepsilon(\sigma) = 1$, σ is invertible and

$$S(\sigma) = \sigma^{-1}, \quad \Delta(\sigma^{-1}) = \sigma^{-1} \otimes \sigma^{-1}.$$

Definition 3.7. Let $\chi : \mathcal{H} \to \mathbb{C}$ be a non-zero algebra homomorphism and σ be a group-like element. We call (χ, σ) is a modular pair if

$$\chi(\sigma) = 1,$$

when we look at the extension of χ to $M(\mathcal{H})$.

It is easy to check that if (χ, σ) is a modular pair, then $\widetilde{S}(\sigma) = \sigma^{-1}$, when we look at the extension of $\widetilde{S}$ to $M(\mathcal{H})$. A modular pair (χ, σ) is called a modular pair in involution if

$$\sigma^{-1}\widetilde{S}^2(h)\sigma = h,$$

for all $h \in \mathcal{H}$.

Example 3.8. Let $\mathcal{G}$ be a discrete group and let $\mathcal{H}$ be the algebra of complex functions on $\mathcal{G}$ with finite support with pointwise operations. Consider the obvious comultiplication Δ on $\mathcal{H}$ given by

$$\Delta(f)(s,t) = f(st),$$

when $f \in \mathcal{H}$ and $s, t \in \mathcal{G}$. Then $\mathcal{H}$ is a regular multiplier Hopf algebra (see [8]). One has $\varepsilon(f) = f(e)$ and $S(f)(t) = f(t^{-1})$, where $f \in \mathcal{H}, t \in \mathcal{G}$ and e is the identity element of $\mathcal{G}$. Let γ be the characteristic function on $\{e\}$. Then (ε, γ) is a matched pair and $(\varepsilon, 1)$ is a modular pair in involution.

4. The precyclic module of a regular multiplier Hopf algebra

In the following, we define the precyclic module for $\mathcal{H}$ as a regular multiplier Hopf algebra which is associated to a matched pair (χ, γ) and a modular pair in involution (χ, σ) such that γ and σ are compatible, that is,

$$\gamma\sigma = \sigma\gamma.$$

We need a lemma before we can do this.

Lemma 4.1. *For all $h, k \in \mathcal{H}$, we have*

$$m(S^2 \otimes \widetilde{S}) \circ \tau((1 \otimes h)\Delta(k)) = \chi(k)S^2(h),$$
$$m(S^2 \otimes \widetilde{S}) \circ \tau((h \otimes 1)\Delta(k)) = \chi(k)\widetilde{S}(h).$$

Proof. Take an element $e \in \mathcal{H}$ such that $\chi(e) = 1$. For all $h, k \in \mathcal{H}$, we have

$$m(S^2 \otimes \widetilde{S}) \circ \tau((1 \otimes h)\Delta(k)) = \sum \chi((1k_{(1)})_{(1)}e)S^2(hk_{(2)})S((1k_{(1)})_{(2)}1).$$

Using the properties of Δ we obtain

$$\begin{aligned} m(S^2 \otimes \widetilde{S}) \circ \tau((1 \otimes h)\Delta(k)) &= \sum \chi(k_{(1)}e)S^2(h)\varepsilon(k_{(2)}1) \\ &= \chi((\iota \otimes \varepsilon)(\Delta(k)(e \otimes 1)))S^2(h) \\ &= \chi(k)S^2(h), \end{aligned}$$

and similarly for the other formula. □

We associate a presimplicial module to $\mathcal{H}$ by the comultiplication, the multiplication, the group-like projection γ and the group-like element σ such that γ and σ are compatible. The remaining features of the given data, namely the antipode and χ, are used to define the candidate for the cyclic operators.

Let $C^n(\mathcal{H}) = \mathcal{H}^{\otimes(n)}$, for all $n \geq 1$ and $C^0(\mathcal{H}) = \mathbb{C}$. First, we must define the face operators $\delta_i : C^{n-1}(\mathcal{H}) \to C^n(\mathcal{H})$, for all $0 \leq i \leq n$. We define

$$\delta_0(h^1 \otimes \cdots \otimes h^{n-1}) := \gamma \otimes h^1 \otimes \cdots \otimes h^{n-1}.$$

For every $1 \leq i \leq n-1$, we let

$$\delta_i(h^1 \otimes \cdots \otimes h^{n-1}) := h^1 \otimes \cdots \otimes \Delta(h^i)(1 \otimes \gamma) \otimes \cdots \otimes h^{n-1}.$$

Finally, we define

$$\delta_n(h^1 \otimes \cdots \otimes h^{n-1}) := \Delta^{(n-1)}(\gamma)(h^1 \otimes \cdots \otimes h^{n-1} \otimes \sigma).$$

Note that if $n = 1$, then $\delta_0, \delta_1 : \mathbb{C} \to \mathcal{H}$ are given by

$$\delta_0(1) := \gamma, \quad \delta_1(1) := \gamma\sigma.$$

For every $n \geq 1$, we define the cyclic operators $\tau_n : C^n(\mathcal{H}) \to C^n(\mathcal{H})$ as follows:

$$\tau_n(h^1 \otimes \cdots \otimes h^n) := \Delta^{(n-1)}(\widetilde{S}(h^1))(h^2 \otimes h^3 \otimes \cdots \otimes h^n \otimes \sigma),$$

If $n = 0$, then define $\tau_0 : \mathbb{C} \to \mathbb{C}$ by

$$\tau_0(1) := 1.$$

Theorem 4.2. *Assume that (χ, γ) is a matched pair and (χ, σ) is a modular pair in involution such that γ and σ are compatible. Then $\mathcal{H}^\bullet_{(\gamma,\chi,\sigma)} = \{C^n(\mathcal{H})\}_{n\geq 0}$ endowed with the above operators is a precyclic module. In other words, the operators δ_i satisfying in the following relations:*

$$\delta_j\delta_i = \delta_i\delta_{j-1}, \quad i < j,$$

and to obtain precyclic module the operators τ_n must satisfying in the relations:

$$\tau_n\delta_i = \delta_{i-1}\tau_{n-1}, \quad 1 \le i \le n,$$
$$\tau_n\delta_0 = \delta_n, \qquad \tau_n^{n+1} = \iota.$$

Proof. We shall only use the basic properties of the multiplication, the comultiplication, the antipode, the group-like element and the group-like projection, and adhere to the standard notational conventions for the regular multiplier Hopf algebra calculus. Let $f = h^1 \otimes \cdots \otimes h^{n-1}$. We first look at the case $i = 0$ and $j = 1$. Thus

$$\delta_1\delta_0(f) = \delta_1(\gamma \otimes f) = \Delta(\gamma)(1 \otimes \gamma) \otimes f = \gamma \otimes \gamma \otimes f = \delta_0^2(f).$$

For $i = 1$ and $j = 2$, we have

$$\begin{aligned}
\delta_2\delta_1(f) &= \sum \delta_2(h^1_{(1)}1 \otimes h^1_{(2)}\gamma \otimes h^2 \otimes \cdots \otimes h^{n-1}) \\
&= \sum h^1_{(1)}1 \otimes \Delta(h^1_{(2)}\gamma)(1 \otimes \gamma) \otimes h^2 \otimes \cdots \otimes h^{n-1} \\
&= (\iota \otimes \Delta)(\Delta(h^1)(1 \otimes \gamma))(1 \otimes 1 \otimes \gamma) \otimes h^2 \otimes \cdots \otimes h^{n-1} \\
&= (\Delta \otimes \iota)(\Delta(h^1)(1 \otimes \gamma))(1 \otimes \gamma \otimes 1) \otimes h^2 \otimes \cdots \otimes h^{n-1} \\
&= \sum \Delta(h^1_{(1)}1)(1 \otimes \gamma) \otimes h^1_{(2)}\gamma \otimes h^2 \otimes \cdots \otimes h^{n-1} \\
&= \sum \delta_1(h^1_{(1)}1 \otimes h^1_{(2)}\gamma \otimes h^2 \otimes \cdots \otimes h^{n-1}) = \delta_1^2(f).
\end{aligned}$$

For $i = 0$ and $j = n + 1$, one has

$$\begin{aligned}
\delta_{n+1}\delta_0(f) &= \Delta^{(n)}(\gamma)(\gamma \otimes f \otimes \sigma) = (\iota \otimes \Delta^{(n-1)})(\Delta(\gamma))(\gamma \otimes f \otimes \sigma) \\
&= (\iota \otimes \Delta^{(n-1)})(\Delta(\gamma)(\gamma \otimes 1))(1 \otimes f \otimes \sigma) \\
&= \gamma \otimes \Delta^{(n-1)}(\gamma)(f \otimes \sigma) = \delta_0\delta_n(f).
\end{aligned}$$

For $i = 1$ and $j = n + 1$, we have

$$\begin{aligned}
\delta_{n+1}\delta_1(f) &= \Delta^{(n)}(\gamma)(\Delta(h^1)(1 \otimes \gamma) \otimes h^2 \otimes \cdots \otimes h^{n-1} \otimes \sigma) \\
&= (\Delta \otimes \iota^{\otimes(n-1)})(\Delta^{(n-1)}(\gamma))(\Delta(h^1)(1 \otimes \gamma) \otimes h^2 \otimes \cdots \\
&\otimes h^{n-1} \otimes \sigma) = (\Delta \otimes \iota^{\otimes(n-1)})(\Delta^{(n-1)}(\gamma))(\Delta \otimes \iota^{\otimes(n-1)})(f \\
&\otimes \sigma)(1 \otimes \gamma \otimes \cdots \otimes 1) \\
&= (\Delta \otimes \iota^{\otimes(n-1)})(\Delta^{(n-1)}(\gamma)(f \otimes \sigma))(1 \otimes \gamma \otimes \cdots \otimes 1) \\
&= \sum(\Delta \otimes \iota^{\otimes(n-1)})(\gamma_{(1)}h^1 \otimes \gamma_{(2)}h^2 \otimes \cdots \otimes \gamma_{(n-1)}h^{n-1} \\
&\otimes (\gamma_{(n)}1)\sigma)(1 \otimes \gamma \otimes \cdots \otimes 1) \\
&= \sum \Delta(\gamma_{(1)}h^1)(1 \otimes \gamma) \otimes \gamma_{(2)}h^2 \otimes \cdots \otimes \gamma_{(n-1)}h^{n-1} \\
&\otimes (\gamma_{(n)}1)\sigma = \delta_1\delta_n(f).
\end{aligned}$$

If $n = 1$, then we have

$$\delta_2\delta_0(1) = \delta_2(\gamma) = \Delta(\gamma)(\gamma \otimes \sigma) = \gamma \otimes \gamma\sigma = \delta_0\delta_1(1).$$

If $i = n$ and $j = n + 1$, then

$$\begin{aligned}\delta_{n+1}\delta_n(f) &= \delta_{n+1}(\Delta^{(n-1)}(\gamma)(f \otimes \sigma))\\ &= \Delta^{(n)}(\gamma)(\Delta^{(n-1)}(\gamma)(f \otimes \sigma) \otimes \sigma)\\ &= (\Delta^{(n-1)} \otimes \iota)(\Delta(\gamma))(\Delta^{(n-1)} \otimes \iota)(\gamma \otimes 1)(f \otimes \sigma \otimes \sigma)\\ &= (\Delta^{(n-1)} \otimes \iota)(\Delta(\gamma)(\gamma \otimes 1))(f \otimes \sigma \otimes \sigma)\\ &= (\Delta^{(n-1)} \otimes \iota)(\Delta(\gamma)(1 \otimes \gamma))(f \otimes \sigma \otimes \sigma)\\ &= \Delta^{(n)}(\gamma)(f \otimes \sigma \otimes \gamma\sigma).\end{aligned}$$

On the other hand

$$\begin{aligned}\delta_n\delta_n(f) &= \delta_n(\Delta^{(n-1)}(\gamma)(f \otimes \sigma))\\ &= \sum \delta_n(\gamma_{(1)}h^1 \otimes \gamma_{(2)}h^2 \otimes \cdots \otimes \gamma_{(n-1)}h^{n-1} \otimes (\gamma_{(n)}1)\sigma)\\ &= \sum \gamma_{(1)}h^1 \otimes \gamma_{(2)}h^2 \otimes \cdots \otimes \gamma_{(n-1)}h^{n-1} \otimes \Delta((\gamma_{(n)}1)\sigma)(1 \otimes \gamma)\\ &= \sum(\iota^{\otimes(n-1)} \otimes \Delta)(\gamma_{(1)}h^1 \otimes \gamma_{(2)}h^2 \otimes \cdots \otimes \gamma_{(n-1)}h^{n-1}\\ &\otimes (\gamma_{(n)}1)\sigma)(1 \otimes \cdots \otimes 1 \otimes \gamma)\\ &= (\iota^{\otimes(n-1)} \otimes \Delta)(\Delta^{(n-1)}(\gamma)(f \otimes \sigma))(1 \otimes \cdots \otimes 1 \otimes \gamma)\\ &= \Delta^{(n)}(\gamma)(f \otimes \Delta(\sigma)(1 \otimes \gamma)) = \Delta^{(n)}(\gamma)(f \otimes \sigma \otimes \gamma\sigma),\end{aligned}$$

where we used the relation $\sigma\gamma = \gamma\sigma$. Now, we shall verify the remaining relations. Starting with the compatibility with the face operators, one has

$$\tau_n\delta_0(f) = \tau_n(\gamma \otimes f) = \Delta^{(n-1)}(\widetilde{S}(\gamma))(f \otimes \sigma) = \delta_n(f).$$

Assume that $c_1, \cdots, c_{n-1}$ are arbitrary elements in $\mathcal{H}$ and let

$$\alpha = 1 \otimes S(c_{n-1}) \otimes \cdots \otimes S(c_2) \otimes \widetilde{S}(c_1).$$

Then we have

$$\begin{aligned}\alpha\tau_n\delta_1(f) &= \sum \alpha\Delta^{(n-1)}(\widetilde{S}(h^1_{(1)}1))(h^1_{(2)}\gamma \otimes h^2 \otimes \cdots \otimes h^{n-1} \otimes \sigma)\\ &= \sum \varepsilon((h^1_{(n-1)}1)_{(2)}1)\gamma \otimes S((h^1_{(n-1)}1)_{(1)}c_{n-1})h^2 \otimes \cdots\\ &\otimes S(h^1_{(2)}c_2)h^{n-1} \otimes \widetilde{S}(h^1_{(1)}c_1)\sigma\\ &= \sum \gamma \otimes S(c_{n-1})S(h^1_{(n-1)}1)h^2 \otimes \cdots\\ &\otimes S(h^1_{(2)}c_2)h^{n-1} \otimes \widetilde{S}(h^1_{(1)}c_1)\sigma = \alpha\delta_0\tau_{n-1}(f).\end{aligned}$$

Because S and $\widetilde{S}$ are invertible, we can cancel α and one obtains

$$\tau_n\delta_1 = \delta_0\tau_{n-1}.$$

Similarly, we can prove $\tau_n\delta_i = \delta_{i-1}\tau_{n-1}$, for all $2 \leq i \leq n-1$. If $n = 1$, then we have

$$\tau_1\delta_1(1) = \tau_1(\gamma\sigma) = \widetilde{S}(\gamma\sigma)\sigma = \sigma^{-1}\gamma\sigma = \gamma = \delta_0\tau_0(1),$$

where we used the relation $\sigma\gamma = \gamma\sigma$. Let $n = 2$ and c be an arbitrary element in $\mathcal{H}$. Then we have

$$\begin{aligned}(1\otimes\widetilde{S}(c))\tau_2\delta_2(h) &= \sum(1\otimes\widetilde{S}(c))\Delta(\widetilde{S}(\gamma_{(1)}h))((\gamma_{(2)}1)\sigma\otimes\sigma)\\ &= \sum S((\gamma_{(1)}h)_{(2)}1)(\gamma_{(2)}1)\sigma\otimes\widetilde{S}((\gamma_{(1)}h)_{(1)}c)\sigma\\ &= \sum S(h_{(2)}1)\varepsilon(\gamma_{(2)}1)\sigma\otimes\widetilde{S}(\gamma_{(1)}(h_{(1)}c))\sigma\\ &= \sum S(h_{(2)}1)\sigma\otimes\widetilde{S}(h_{(1)}c)\gamma\sigma\\ &= (1\otimes\widetilde{S}(c))\Delta(\widetilde{S}(h))(\sigma\otimes\gamma\sigma).\end{aligned}$$

Now, because $\widetilde{S}$ is invertible and since the formula holds for all c, we can cancel $1\otimes\widetilde{S}(c)$ and one obtains

$$\tau_2\delta_2(h) = \Delta(\widetilde{S}(h))(\sigma\otimes\gamma\sigma) = \delta_1(\widetilde{S}(h)\sigma) = \delta_1\tau_1(h),$$

where we used the relation $\sigma\gamma = \gamma\sigma$.

Similarly, we can prove $\tau_n\delta_n = \delta_{n-1}\tau_{n-1}$, for all $n \geq 3$. In the next step, we have to show that $\tau_n^{n+1} = \iota$. If $n = 1$, then

$$\tau_1^2(h) = \tau_1(\widetilde{S}(h)\sigma) = \sigma^{-1}\widetilde{S}^2(h)\sigma = h.$$

If $n = 2$, then we have

$$\tau_2(h^1\otimes h^2) = \Delta(\widetilde{S}(h^1))(S(a)\otimes\sigma) = \sum S(ah^1_{(2)})\otimes\widetilde{S}(1h^1_{(1)})\sigma,$$

where $a = S^{-1}(h^2)$. Let c be an arbitrary element in $\mathcal{H}$. It follows that

$$(1\otimes\widetilde{S}(S(c)))\tau_2^2(h^1\otimes h^2) = \sum S^2(1(ah^1_{(2)})_{(1)})\widetilde{S}(1h^1_{(1)})\sigma\otimes\widetilde{S}(S(c(ah^1_{(2)})_{(2)}))\sigma.$$

Using the properties of Δ we obtain

$$\begin{aligned}&(1\otimes\widetilde{S}(S(c)))\tau_2^2(h^1\otimes h^2) =\\ &\sum S^2((1a_{(1)})(1h^1_{(1)})_{(2)})\widetilde{S}(1(1h^1_{(1)})_{(1)})\sigma\otimes\widetilde{S}(S((ca_{(2)})h^1_{(2)}))\sigma.\end{aligned}$$

Using Lemma 4.1 and Lemma 3.5, we have

$$\begin{aligned}(1\otimes\widetilde{S}(S(c)))\tau_2^2(h^1\otimes h^2) &= \sum\chi(1h^1_{(1)})S^2(1a_{(1)})\sigma\otimes\widetilde{S}(S((ca_{(2)})h^1_{(2)}))\sigma\\ &= (S^2\otimes\widetilde{S}\circ S)((1\otimes c)\Delta(S^{-1}(h^2)))(\sigma\otimes\widetilde{S}^2(h^1)\sigma)\\ &= (1\otimes\widetilde{S}(S(c)))\Delta(\widetilde{S}(h^2))(\sigma\otimes\widetilde{S}^2(h^1)\sigma).\end{aligned}$$

Since the formula holds for all c and $\widetilde{S}$ and S are invertible, so we can cancel $1 \otimes \widetilde{S}(S(c))$ and one obtains

$$\tau_2^2(h^1 \otimes h^2) = \Delta(\widetilde{S}(h^2))(\sigma \otimes \widetilde{S}^2(h^1)\sigma) = \sum S(1h^2_{(2)})\sigma \otimes \widetilde{S}(bh^2_{(1)})\sigma,$$

where $b = \widetilde{S}(h^1)$. Hence, we have

$$\begin{aligned}\tau_2^3(h^1 \otimes h^2) &= \sum \Delta(\widetilde{S}(S(1h^2_{(2)})\sigma))(\widetilde{S}(bh^2_{(1)})\sigma \otimes \sigma)\\ &= (\sigma^{-1} \otimes \sigma^{-1})k(\sigma \otimes \sigma),\end{aligned}$$

where $k = \sum \Delta(\widetilde{S}(S(1h^2_{(2)})))(\widetilde{S}(bh^2_{(1)}) \otimes 1)$. Let c be an arbitrary element in $\mathcal{H}$. Then

$$(1 \otimes \widetilde{S}(S(c)))k = \sum S^2(1(1h^2_{(2)})_{(1)})\widetilde{S}(bh^2_{(1)}) \otimes \widetilde{S}(S(c(1h^2_{(2)})_{(2)})).$$

Using the properties of Δ and using Lemma 4.1 and Lemma 3.5, one has

$$\begin{aligned}(1 \otimes \widetilde{S}(S(c)))k &= \sum S^2(1(1h^2_{(1)})_{(2)})\widetilde{S}(b(1h^2_{(1)})_{(1)}) \otimes \widetilde{S}(S(ch^2_{(2)}))\\ &= \sum \chi(1h^2_{(1)})\widetilde{S}(b) \otimes \widetilde{S}(S(ch^2_{(2)}))\\ &= \widetilde{S}^2(h^1) \otimes \widetilde{S}(\widetilde{S}(h^2)S(c))\\ &= (1 \otimes \widetilde{S}(S(c)))(\widetilde{S}^2(h^1) \otimes \widetilde{S}^2(h^2)).\end{aligned}$$

Now, because $\widetilde{S}$ are S are invertible and since the formula holds for all c, we can cancel $1 \otimes \widetilde{S}(S(c))$ and this implies that

$$k = \widetilde{S}^2(h^1) \otimes \widetilde{S}^2(h^2).$$

So, we have

$$\tau_2^3(h^1 \otimes h^2) = \sigma^{-1}\widetilde{S}^2(h^1)\sigma \otimes \sigma^{-1}\widetilde{S}^2(h^2)\sigma = h^1 \otimes h^2.$$

We now pass to the general case. Let $g = h^1 \otimes h^2 \otimes \cdots \otimes h^n$. Then

$$\begin{aligned}\tau_n(g) &= \Delta^{(n-1)}(\widetilde{S}(h^1))(S(a^2) \otimes \cdots \otimes S(a^n) \otimes \sigma)\\ &= (\iota \otimes \Delta^{(n-2)})(\Delta(\widetilde{S}(h^1))(S(a^2) \otimes 1))(1 \otimes S(a^3) \otimes \cdots \otimes S(a^n) \otimes \sigma)\\ &= \sum S(a^2h^1_{(2)}) \otimes \Delta^{(n-2)}(\widetilde{S}(1h^1_{(1)}))(S(a^3) \otimes \cdots \otimes S(a^n) \otimes \sigma)\\ &= \sum S(a^2h^1_{(n)}) \otimes S(a^3h^1_{(n-1)}) \otimes \cdots \otimes S(a^nh^1_{(2)}) \otimes \widetilde{S}(1h^1_{(1)})\sigma,\end{aligned}$$

where $a^2 = S^{-1}(h^2), \cdots, a^n = S^{-1}(h^n)$. Assume that $c_1, \cdots, c_{n-1}$ are arbitrary elements in $\mathcal{H}$. Let

$$\xi = 1 \otimes S^2(c_{n-1}) \otimes \cdots \otimes S^2(c_2) \otimes \widetilde{S}(S(c_1)).$$

Using Lemma 4.1 and Lemma 3.5, we have

$$\begin{aligned}
\xi\tau_n^2(g) &= \sum \xi\Delta^{(n-1)}(\widetilde{S}(S(a^2h^1_{(n)})))(S(a^3h^1_{(n-1)})\otimes\cdots\otimes S(a^nh^1_{(2)})\\
&\otimes \widetilde{S}(1h^1_{(1)})\sigma\otimes\sigma) = \sum S^2((1a^2_{(1)})h^1_{(n)})S(a^3h^1_{(n-1)})\\
&\otimes S^2((c_{n-1}a^2_{(2)})h^1_{(n+1)})S(a^4h^1_{(n-2)})\otimes\cdots\\
&\otimes S^2((c_2a^2_{(n-1)})h^1_{(2n-2)})\widetilde{S}(1h^1_{(1)})\sigma\otimes\widetilde{S}(S((c_1a^2_{(n)})h^1_{(2n-1)}))\sigma\\
&= \sum S(a^3S(1a^2_{(1)})\varepsilon(1h^1_{(n-1)}))\otimes S^2((c_{n-1}a^2_{(2)})h^1_{(n)})S(a^4h^1_{(n-2)})\otimes\\
&\cdots\otimes S^2((c_2a^2_{(n-1)})h^1_{(2n-3)})\widetilde{S}(1h^1_{(1)})\sigma\otimes\widetilde{S}(S((c_1a^2_{(n)})h^1_{(2n-2)}))\sigma\\
&= \sum S(a^3S(1a^2_{(1)}))\otimes S(a^4S(c_{n-1}a^2_{(2)}))\otimes\\
&\cdots\otimes S^2((c_2a^2_{(n-1)})(1h^1_{(2)}))\widetilde{S}(1h^1_{(1)})\sigma\otimes\widetilde{S}(S((c_1a^2_{(n)})h^1_{(3)}))\sigma\\
&= \sum S(a^3S(1a^2_{(1)}))\otimes S(a^4S(c_{n-1}a^2_{(2)}))\otimes\cdots\\
&\otimes S^2(c_2a^2_{(n-1)})\chi(1h^1_{(1)})\sigma\otimes\widetilde{S}(S((c_1a^2_{(n)})h^1_{(2)}))\sigma\\
&= \sum S^2(1a^2_{(1)})h^3\otimes S^2(c_{n-1}a^2_{(2)})h^4\otimes\cdots\\
&\otimes S^2(c_2a^2_{(n-1)})\sigma\otimes\widetilde{S}(S(c_1a^2_{(n)}))\widetilde{S}^2(h^1)\sigma\\
&= \xi\Delta^{(n-1)}(\widetilde{S}(h^2))(h^3\otimes h^4\otimes\cdots\otimes h^n\otimes\sigma\otimes\widetilde{S}^2(h^1)\sigma).
\end{aligned}$$

Since $\widetilde{S}$ are S are invertible, so we can cancel ξ and this implies that

$$\tau_n^2(g) = \Delta^{(n-1)}(\widetilde{S}(h^2))(h^3\otimes\cdots\otimes h^n\otimes\sigma\otimes\widetilde{S}^2(h^1)\sigma).$$

By induction, one obtains for any $j = 1,\cdots,n$

$$\tau_n^j(g) = \Delta^{(n-1)}(\widetilde{S}(h^j))(h^{j+1}\otimes\cdots\otimes\sigma\otimes\cdots\otimes\widetilde{S}^2(h^{j-1})\sigma).$$

This implies that

$$\begin{aligned}
\tau_n^{n+1}(g) &= \Delta^{(n-1)}(\widetilde{S}(\sigma))(\widetilde{S}^2(h^1)\sigma\otimes\cdots\otimes\widetilde{S}^2(h^n)\sigma)\\
&= \sigma^{-1}\widetilde{S}^2(h^1)\sigma\otimes\cdots\otimes\sigma^{-1}\widetilde{S}^2(h^n)\sigma = g,
\end{aligned}$$

where we used the modular pair in involution (χ,σ). □

Define the operators b_n, b'_n, λ_n and N_n as follows:

$$b_n := \sum_{i=0}^{n}(-1)^i\delta_i, \quad n\geq 1,$$

$$b'_n := \sum_{i=0}^{n-1}(-1)^i\delta_i, \quad n\geq 1,$$

$$\lambda_n := (-1)^n\tau_n, \qquad n\geq 0,$$

$$N_n := \sum_{i=0}^{n} (\lambda_n)^i. \qquad n \geq 0.$$

Using Theorem 4.2, one can check that $b_{n+1}b_n = 0$ and $b'_{n+1}b'_n = 0$. Also, we have

$$(1-\lambda_n)b_n = b'_n(1-\lambda_{n-1}), \qquad b_n N_{n-1} = N_n b'_n,$$

$$(1-\lambda_n)N_n = N_n(1-\lambda_n) = 0.$$

Hence, one obtains the bicomplex $CC^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H})$ as follows:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
b_3 \uparrow & & -b'_3 \uparrow & & b_3 \uparrow & & -b'_3 \uparrow & & \\
\mathcal{H}^{\otimes(2)} & \xrightarrow{1-\lambda_2} & \mathcal{H}^{\otimes(2)} & \xrightarrow{N_2} & \mathcal{H}^{\otimes(2)} & \xrightarrow{1-\lambda_2} & \mathcal{H}^{\otimes(2)} & \longrightarrow & \cdots \\
b_2 \uparrow & & -b'_2 \uparrow & & b_2 \uparrow & & -b'_2 \uparrow & & \\
\mathcal{H} & \xrightarrow{1-\lambda_1} & \mathcal{H} & \xrightarrow{N_1} & \mathcal{H} & \xrightarrow{1-\lambda_1} & \mathcal{H} & \longrightarrow & \cdots \\
b_1 \uparrow & & -b'_1 \uparrow & & b_1 \uparrow & & -b'_1 \uparrow & & \\
\mathbb{C} & \xrightarrow{1-\lambda_0} & \mathbb{C} & \xrightarrow{N_0} & \mathbb{C} & \xrightarrow{1-\lambda_0} & \mathbb{C} & \longrightarrow & \cdots
\end{array}$$

The cohomology of the total complex $Tot(CC^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H}))$ corresponding to the precyclic module $\mathcal{H}^{\bullet}_{(\gamma,\chi,\sigma)}$ is the precyclic cohomology $HC^{*}_{(\gamma,\chi,\sigma)}(\mathcal{H})$ of $\mathcal{H}$ relative to the matched pair (χ,γ), the modular pair in involution (χ,σ) and $\gamma\sigma = \sigma\gamma$. The Hochschild cohomology $HH^{*}_{(\gamma,\chi,\sigma)}(\mathcal{H})$ is the cohomology of the first column of the bicomplex $CC^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H})$ (see also Remark 5.4 below).

5. Some remarks

In this paper, the group-like projection γ plays an important role and allows us to define the face operators.

Remark 5.1. *If $\mathcal{H}$ is a Hopf algebra and $\gamma = 1$, then the face operators in this paper are the same with the face operators of Connes-Moscovici for Hopf algebras (see* [2, 3]*).*

Remark 5.2. *In Theorem 4.2, in addition, if there are the degeneracy operators*

$$\sigma_i : C^{n+1}(\mathcal{H}) \to C^n(\mathcal{H}), \;\; 0 \leq i \leq n,$$

with the following relations:

$$\sigma_j\sigma_i = \sigma_i\sigma_{j+1}, \quad i \leq j,$$

$$\tau_n\sigma_i = \sigma_{i-1}\tau_{n+1},\, 1 \leq i \leq n, \qquad \tau_n\sigma_0 = \sigma_n\tau_{n+1}^2,$$

$$\sigma_j\delta_i = \delta_i\sigma_{j-1}, \quad i < j, \qquad \sigma_j\delta_i = \delta_{i-1}\sigma_j, \quad i > j+1,$$

and

$$\sigma_j\delta_i = \iota, \;\; i = j \text{ or } \; i = j+1,$$

then $\mathcal{H}^{\bullet}_{(\gamma,\chi,\sigma)}$ *is a cyclic module in the sense of Connes. Using the work of Connes-Moscovici, our candidate is*

$$\sigma_i(h^1 \otimes \cdots \otimes h^{n+1}) = \varepsilon(h^{i+1})h^1 \otimes \cdots \otimes h^i \otimes h^{i+2} \otimes \cdots \otimes h^{n+1}.$$

Let $h \in \mathcal{H}$. *We have*

$$\begin{aligned}\sigma_1\delta_2(h) &= \sigma_1(\Delta(\gamma)(h \otimes \sigma))\\ &= \sum \sigma_1(\gamma_{(1)}h \otimes (\gamma_{(2)}1)\sigma)\\ &= \sum \varepsilon((\gamma_{(2)}1)\sigma)\gamma_{(1)}h = \gamma h.\end{aligned}$$

Hence, one has $\sigma_1\delta_2 \neq \iota$. *Therefore, our work will not imply that* $\mathcal{H}^{\bullet}_{(\gamma,\chi,\sigma)}$ *is a cyclic module.*

Remark 5.3. *It is known that if* $\mathcal{H}$ *is a Hopf algebra and* $\gamma = 1$, *then one has the vanishing of the cohomology groups of the complex* $(\mathcal{H}^{\bullet}, b'_{\bullet})$ *with the extra degeneracy operator* $s_n : \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes(n)}$ *as contracting homotopy, where*

$$s_n(h^1 \otimes h^2 \otimes \cdots \otimes h^{n+1}) = \Delta^{(n-1)}(\widetilde{S}(h^1))(h^2 \otimes h^3 \otimes \cdots \otimes h^{n+1}).$$

If $n = 0$, *then we have* $s_0(h) = \chi(h)$. *If* $n = 1$, *then we have* $s_1(h \otimes k) = \widetilde{S}(h)k$. *Consider the case when* $\mathcal{H}$ *is a regular multiplier Hopf algebra. In this case, one can check that* $s_0 b'_1 = \iota$, *and* $s_n b'_{n+1} + b'_n s_{n-1} = T_n \neq \iota$, *where*

$$T_n(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = \Delta^{(n-1)}(\gamma)(h^1 \otimes h^2 \otimes \cdots \otimes h^n).$$

However, we still have the vanishing of the cohomology groups of the complex $(\mathcal{H}^{\bullet}, b'_{\bullet})$. *Define* $\theta_n : \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes(n)}$ *by*

$$\theta_n(h^1 \otimes h^2 \otimes \cdots \otimes h^{n+1}) = (-1)^n\varepsilon(h^{n+1})(h^1 \otimes \cdots \otimes h^n).$$

Then $\theta_0 b'_1 = \iota$, *and* $\theta_n b'_{n+1} + b'_n\theta_{n-1} = \iota$. *Therefore, the result is still valid for regular multiplier Hopf algebras.*

Remark 5.4. *Consider the short exact sequence of bicomplexes*

$$0 \to \widetilde{CC}^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H}) \to CC^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H}) \to Coker(CC^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H})) \to 0$$

where $\widetilde{CC}^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H})$ *is the bicomplex*

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & b_3\uparrow & & -b'_3\uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{H}^{\otimes(2)} & \xrightarrow{1-\lambda_2} & \mathcal{H}^{\otimes(2)} & \xrightarrow{N_2} & \cdots \\
\uparrow & & \uparrow & & b_2\uparrow & & -b'_2\uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{H} & \xrightarrow{1-\lambda_1} & \mathcal{H} & \xrightarrow{N_1} & \cdots \\
\uparrow & & \uparrow & & b_1\uparrow & & -b'_1\uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{C} & \xrightarrow{1-\lambda_0} & \mathbb{C} & \xrightarrow{N_0} & \cdots
\end{array}$$

and $Coker(CC^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H}))$ *is the double complex associated to the first two columns of* $CC^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H})$*. Note that the complex* $Tot(\widetilde{CC}^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H}))$ *is* $Tot(CC^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H}))$ *switched two places. Also, the column with differential* $-b'$ *is acyclic (see Remark 5.3), then the Hochschild cohomology is isomorphic to the cohomology of the total complex associated to the double complex* $Coker(CC^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H}))$*. Therefore, the above short exact sequence induces a natural long exact sequence*

$$\cdots \to HC^{n}_{(\gamma,\chi,\sigma)}(\mathcal{H}) \to HH^{n}_{(\gamma,\chi,\sigma)}(\mathcal{H}) \to HC^{n-1}_{(\gamma,\chi,\sigma)}(\mathcal{H})$$
$$\to HC^{n+1}_{(\gamma,\chi,\sigma)}(\mathcal{H}) \to HH^{n+1}_{(\gamma,\chi,\sigma)}(\mathcal{H}) \to \cdots$$

A multiplier Hopf algebra is called of *discrete type* if it has a cointegral (see [9, 10]). A *left cointegral* is a non-zero element $a \in \mathcal{H}$ satisfying $ha = \varepsilon(h)a$ for all $h \in \mathcal{H}$. Remark that such cointegrals do not always exist and they are unique (up to a scalar) if they exist. Also in the Example 3.8, the group-like projection γ is a left cointegral. In general, however, left cointegrals and group-like projections are different. For example, consider the algebra $\mathcal{H}$ spanned by elements $\{e_p b^q \mid p \in \mathbb{Z}, q = 0, 1, 2, \ldots\}$ and where the elements e_p and b satisfy the relations $e_p e_q = \delta(p, q)e_p$ and $be_p = e_{p+1}b$. Choose any $\lambda \in \mathbb{C} - \{0\}$. Define a in $M(\mathcal{H})$ by $a = \sum_{k\in\mathbb{Z}} \lambda^k e_k$. Then define a comultiplication Δ on $\mathcal{H}$ by

$$\Delta(e_p) = \sum_{k\in\mathbb{Z}} e_k \otimes e_{p-k},$$

$$\Delta(b) = a \otimes b + b \otimes a^{-1}.$$

Remark that these infinite sums are well-defined in the strict topology on the multiplier algebra. Then $(\mathcal{H}, \Delta)$ is a regular multiplier Hopf algebra [10]. One can verify that no cointegrals exist for this example, but $\gamma = e_0$ is a group-like projection.

Remark 5.5. *Assume that the group-like projection γ is a left cointegral. Define $D_n : \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes(n)}$ by*

$$D_n(h^1 \otimes h^2 \otimes \cdots \otimes h^{n+1}) = \varepsilon(h^1)(h^2 \otimes \cdots \otimes h^{n+1}).$$

Then $D_n b_{n+1} + b_n D_{n-1} = \iota$. This implies that $HH^n_{(\gamma,\chi,\sigma)}(\mathcal{H}) = 0$, for all $n \geq 1$. Also, since γ is a left cointegral, $\gamma^2 = \gamma$ and $\gamma\sigma = \sigma\gamma$, so one has $\gamma\sigma = \gamma$. Then $b_1(1) = \gamma - \gamma\sigma = 0$ and $HH^0_{(\gamma,\chi,\sigma)}(\mathcal{H}) = Ker(b_1) = \mathbb{C}$.

Remark 5.6. *Assume that γ is central in $\mathcal{H}$ and denote $\mathcal{H}_0 = \mathcal{H}\gamma$. Then $\mathcal{H}_0$ is a Hopf algebra when we define the comultiplication Δ_0 on $\mathcal{H}_0$ by $\Delta_0(h\gamma) = \Delta(h)(\gamma \otimes \gamma)$ (see [5]). The counit ε_0 for Δ_0 is simply the restriction of ε to $\mathcal{H}_0$ and the antipode S_0 is given by the restriction of S to $\mathcal{H}_0$. Also, γ is the identity in $\mathcal{H}_0$. Let (χ, γ) be a matched pair and (χ, σ) a modular pair in involution such that γ and σ are compatible. One can check that $(\chi, \gamma\sigma)$ is a modular pair in involution for the Hopf algebra $\mathcal{H}_0$, when we look at the restriction of χ to $\mathcal{H}_0$. Let $HC^*_{(\chi,\gamma\sigma)}(\mathcal{H}_0)$ be the cyclic cohomology of $\mathcal{H}_0$ relative to the modular pair in involution $(\chi, \gamma\sigma)$ in the sense of Connes-Moscovici for Hopf algebras and $HH^*_{(\chi,\gamma\sigma)}(\mathcal{H}_0)$ be its Hochschild cohomology. Then the following map is a precyclic map:*

$$T : \mathcal{H}^{\otimes(n)} \to \mathcal{H}_0^{\otimes(n)},$$

$$T(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = h^1\gamma \otimes h^2\gamma \otimes \cdots \otimes h^n\gamma.$$

Hence, T defines a canonical map from $HH^*_{(\gamma,\chi,\sigma)}(\mathcal{H})$ to $HH^*_{(\chi,\gamma\sigma)}(\mathcal{H}_0)$ and T defines a canonical map from $HC^*_{(\gamma,\chi,\sigma)}(\mathcal{H})$ to $HC^*_{(\chi,\gamma\sigma)}(\mathcal{H}_0)$.

Remark 5.7. *Assume that $h \in \mathcal{H}$ is a Hochschild 1-cocycle. Then we have*

$$\gamma \otimes h + \Delta(\gamma)(h \otimes \sigma) = \Delta(h)(1 \otimes \gamma).$$

If we apply $\iota \otimes \varepsilon$ and $\varepsilon \otimes \iota$, we get $\varepsilon(h) = 0$ and $h\gamma = h = \gamma h$. Remark that γ and $\gamma\sigma$ need not be the same. Now, assume that γ is central in $\mathcal{H}$. Then it is clear that $HH^0_{(\gamma,\chi,\sigma)}(\mathcal{H}) = HH^0_{(\chi,\gamma\sigma)}(\mathcal{H}_0)$ (see Remark 5.6). Also, using the argument above and Remark 5.6, one can verify that $HH^1_{(\gamma,\chi,\sigma)}(\mathcal{H})$ is isomorphic to $HH^1_{(\chi,\gamma\sigma)}(\mathcal{H}_0)$.

Remark 5.8. *Let $\mathcal{H}$ be an algebraic quantum group with a non-zero left invariant functional φ (see [9]). Assume that $\varphi(\gamma) \neq 0$. Let $h \in \mathcal{H}$ be a Hochschild 1-cocycle. Since*

$$\gamma \otimes h + \Delta(\gamma)(h \otimes \sigma) = \Delta(h)(1 \otimes \gamma),$$

$\gamma h = h$ (see Remark 5.7), so one has

$$\gamma \otimes h + h \otimes \gamma\sigma = \Delta(h)(1 \otimes \gamma).$$

Let $\phi = \varphi \circ S$, where S is the antipode of $\mathcal{H}$. If we apply $\phi \otimes \iota$, we get $\phi(\gamma)h = \phi(h)(\gamma - \gamma\sigma)$, where we used the right invariance of ϕ. Hence, we have $b_1(\frac{\phi(h)}{\phi(\gamma)}) = h$ and $HH^1_{(\gamma,\chi,\sigma)}(\mathcal{H}) = 0$. We refer to [5] *with examples for an illustration of the result.*

Remark 5.9. *An element $x \in M(\mathcal{H})$ is called* σ-primitive *if*

$$\Delta(x) = 1 \otimes x + x \otimes \sigma.$$

This makes sense because Δ has a unique extension to the multiplier algebra $M(\mathcal{H})$. Assume that x is σ-primitive. Then

$$b_2(\gamma x\gamma) = \gamma \otimes \gamma x\gamma - \Delta(\gamma)(1 \otimes x + x \otimes \sigma)(\gamma \otimes \gamma) + (\gamma \otimes \gamma)(x\gamma \otimes \sigma) = 0.$$

Therefore, $\gamma x\gamma$ is a Hochschild 1-cocycle.

Remark 5.10. *Let $\mathcal{H}$ be a Hopf algebra. Let (χ, γ) be a matched pair and (χ, σ) a modular pair in involution such that γ and σ are compatible. Assume that γ is central in $\mathcal{H}$. Let $\mathcal{H}^\bullet_{(\chi,\sigma)}$ be the cyclic module of $\mathcal{H}$ relative to the modular pair in involution (χ, σ) in the sense of Connes-Moscovici for Hopf algebras, and $HC^*_{(\chi,\sigma)}(\mathcal{H})$ be its cyclic cohomology. Then the following map is a precyclic map from $\mathcal{H}^\bullet_{(\chi,\sigma)}$ to $\mathcal{H}^\bullet_{(\gamma,\chi,\sigma)}$:*

$$\theta : \mathcal{H}^{\otimes(n)} \to \mathcal{H}^{\otimes(n)},$$
$$\theta(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = h^1\gamma \otimes h^2\gamma \otimes \cdots \otimes h^n\gamma.$$

Hence, θ defines a canonical map from $HC^*_{(\chi,\sigma)}(\mathcal{H})$ to $HC^*_{(\gamma,\chi,\sigma)}(\mathcal{H})$.

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