NOTES ON REGULAR MULTIPLIER HOPF ALGEBRAS

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Abstract. In this paper, we associate canonically a precyclic module to a regular multiplier Hopf algebra endowed with a group-like projection and a modular pair in involution satisfying certain conditions.

Keywords: Hopf algebra, Multiplier Hopf algebra, precyclic module.


1. Introduction

The notion of a multiplier Hopf algebra is a natural generalization of the notion of a Hopf algebra. Multiplier Hopf algebras were introduced in [8] by Van Daele. In this framework, one can consider an algebra $\mathcal{H}$ over the field $\mathbb{C}$, with or without an identity, but with a non-degenerate multiplication map. There is a homomorphism $\Delta$ from $\mathcal{H}$ to the multiplier algebra $M(\mathcal{H} \otimes \mathcal{H})$ of $\mathcal{H} \otimes \mathcal{H}$. Certain conditions on $\Delta$ are imposed. The motivating example is the algebra of complex valued, finitely supported functions on an arbitrary group (see [8]). If $\mathcal{H}$ has an identity, then we have a Hopf algebra.
In [2, 3], introduced a cyclic module for any Hopf algebra endowed with a modular pair in involution, i.e., with a group-like element $\sigma$ and a character such that the corresponding doubly twisted antipode has square the identity. The simplicial structure is associated to the comultiplication of the Hopf algebra and the group-like element $\sigma$, while the cyclic structure makes use of the product and of the twisted antipode.

Note that the group-like element $\sigma$ is a non-zero element in the Hopf algebra $(H, \Delta)$ satisfying $\Delta(\sigma) = \sigma \otimes \sigma$. This definition is not very useful when $(H, \Delta)$ is a multiplier Hopf algebra. In this paper, a group-like element is a non-zero element $\sigma$ in the multiplier algebra $M(H)$ of $H$ satisfying

$$\Delta(\sigma) = \sigma \otimes \sigma.$$ 

This makes sense because $\Delta$ has a unique extension to the multiplier algebra $M(H)$.

In [5], introduced the notion of a group-like projection for any multiplier Hopf $*$-algebra. In this paper, we do not require the existence of an involution and we will work with a regular multiplier Hopf algebra $(H, \Delta)$. By a group-like projection we mean a non-zero idempotent element $\gamma$ in $H$ satisfying

$$\Delta(\gamma)(1 \otimes \gamma) = \gamma \otimes \gamma = \Delta(\gamma)(\gamma \otimes 1).$$

The notion of a group-like projection should not be confused with the notion of a group-like element. A group-like element $\sigma$ in a Hopf algebra is always invertible and its inverse is $S(\sigma)$, where $S$ is the antipode of the Hopf algebra. If an idempotent element in a unital algebra is invertible, then it is itself identity.

In this paper, we associate a precyclic module to a regular multiplier Hopf algebra $(H, \Delta)$ endowed with a matched pair $(\chi, \gamma)$ and a modular pair in involution $(\chi, \sigma)$ such that $\gamma$ and $\sigma$ are compatible, i.e., with a group-like projection $\gamma$, a non-zero algebra homomorphism $\chi : H \to \mathbb{C}$, a group-like element $\sigma \in M(H)$ and a modular pair in involution $(\chi, \sigma)$ such that $\chi(\gamma) = 1$ and $\gamma \sigma = \sigma \gamma$.

The presimplicial structure is associated to the comultiplication $\Delta$, the multiplication, the group-like projection $\gamma$ and the group-like element $\sigma$. Note that the group-like projection $\gamma$ is a basic object in our paper and allows us to define the face operators. It is important to mention that if $H$ is a Hopf algebra and $\gamma = 1$, then the face operators in this paper are the same with the face operators of Connes-Moscovici for Hopf algebras (see [2, 3]).

The standard references for Hopf algebras are [1, 6, 7]. For the basic theory of multiplier Hopf algebras, we refer the reader to [8].
2. Preliminaries and Basic Concepts

Throughout this paper, all vector spaces will be spaces over the complex field \( \mathbb{C} \). We will use the convention that \( V \otimes W \) represents the algebraic tensor product in which \( V \) and \( W \) are vector spaces.

An algebra \( \mathcal{A} \) is called non-degenerate if the product in \( \mathcal{A} \) is non-degenerate, i.e., if \( ab = 0 \) for all \( b \) implies \( a = 0 \) and \( ab = 0 \) for all \( a \) implies \( b = 0 \). It is clear that any unital algebra is a non-degenerate algebra. Using Lemma A.2 in [8], the algebraic tensor product of two non-degenerate algebras is again an algebra with a non-degenerate product.

Let \( \mathcal{A} \) be a non-degenerate algebra. The multiplier algebra \( M(\mathcal{A}) \) is defined as the set of pairs \((l, r)\) of linear maps of \( \mathcal{A} \) into \( \mathcal{A} \) satisfying:

\[
  r(a)b = al(b),
\]

for all \( a, b \in \mathcal{A} \). It is easy to see that this set \( M(\mathcal{A}) \) can be made into an associative algebra. It always contains a unit, and that \( \mathcal{A} \) has a natural imbedding as an essential two-sided ideal in \( M(\mathcal{A}) \). There are natural imbeddings

\[
  \mathcal{A} \otimes \mathcal{A} \rightarrow M(\mathcal{A}) \otimes M(\mathcal{A}) \rightarrow M(\mathcal{A} \otimes \mathcal{A}).
\]

We will write \( xa \) for \( l(a) \) and \( ax \) for \( r(a) \) when \( x = (l, r) \) is an element of \( M(\mathcal{A}) \) and \( a \in \mathcal{A} \). If \( \mathcal{A}, \mathcal{B} \) are non-degenerate algebras, then a homomorphism \( \varphi \) from \( \mathcal{A} \) to \( M(\mathcal{B}) \) is said to be non-degenerate if and only if \( \varphi(\mathcal{A})\mathcal{B} = \mathcal{B} = \mathcal{B}\varphi(\mathcal{A}) \). Using Proposition A.5 in [8], such a non-degenerate homomorphism has a unique extension to a unital homomorphism \( M(\mathcal{A}) \rightarrow M(\mathcal{B}) \).

For the following definition, we refer the reader to [8].

**Definition 2.1.** Assume that \( \mathcal{H} \) is a non-degenerate algebra and \( \Delta \) from \( \mathcal{H} \) to \( M(\mathcal{H} \otimes \mathcal{H}) \) is a non-degenerate homomorphism. The pair \((\mathcal{H}, \Delta)\) is called a multiplier Hopf algebra if

(i) \( (\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta \),

(ii) \( \Delta(h)(1 \otimes k) \) and \( (h \otimes 1)\Delta(k) \) are elements in \( \mathcal{H} \otimes \mathcal{H} \),

(iii) the linear mappings \( T_1, T_2 : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} \), defined by

\[
  T_1(h \otimes k) = \Delta(h)(1 \otimes k), \quad T_2(h \otimes k) = (h \otimes 1)\Delta(k),
\]

for all \( h, k \in \mathcal{H} \) are bijective.

Since \( \Delta \) is a non-degenerate homomorphism, so \( \Delta \otimes \iota \) can be extended to the multiplier algebra \( M(\mathcal{H} \otimes \mathcal{H}) \). Similarly, \( \iota \otimes \Delta \) can be extended to the multiplier algebra \( M(\mathcal{H} \otimes \mathcal{H}) \). The homomorphism \( \Delta \) is called a comultiplication on \( \mathcal{H} \). We can consider the opposite comultiplication \( \Delta' \) obtained from \( \Delta \) by composing it with the flip operator on \( \mathcal{H} \otimes \mathcal{H} \).
A multiplier Hopf algebra \((\mathcal{H}, \Delta)\) is said to be regular if \((\mathcal{H}, \Delta')\) also is a multiplier Hopf algebra.

Let \((\mathcal{H}, \Delta)\) be a regular multiplier Hopf algebra. There is a unique non-zero homomorphism \(\varepsilon\) from \(\mathcal{H}\) to \(\mathbb{C}\) such that
\[
(\varepsilon \otimes \iota)\Delta = \iota = (\iota \otimes \varepsilon)\Delta.
\]
The map \(\varepsilon\) is called the counit of \((\mathcal{H}, \Delta)\). Also there is a unique invertible anti-homomorphism \(S\) from \(\mathcal{H}\) to \(\mathcal{H}\) that satisfies the conditions
\[
m(S \otimes \iota)(\Delta(h)(1 \otimes k)) = \varepsilon(h)k,
\]
\[
m(\iota \otimes S)((k \otimes 1)\Delta(h)) = \varepsilon(h)k,
\]
for all \(h, k \in \mathcal{H}\), where \(m\) is the multiplication map. The map \(S\) is called the antipode of \((\mathcal{H}, \Delta)\) (see \([8]\)).

We use the generalized Sweedler notation for the comultiplication \(\Delta\). We can define \(\Delta^{(n)}\) for any \(n \geq -1\) according to
\[
\Delta^{(-1)} := \varepsilon,
\]
\[
\Delta^{(n)} := (\Delta^{(n-1)} \otimes \iota)\Delta,
\]
for all \(n \geq 0\). We have the following formula as a direct consequence of the properties of \(\Delta\) (see \([1, 4, 6, 7]\)):
\[
\Delta^{(n+m+r)} = (\iota \otimes (n) \otimes \Delta^{(m)} \otimes \iota \otimes (r))\Delta^{(n+r)},
\]
for all \(n, m, r \geq 0\). Let \(h \in \mathcal{H}\). For all \(k_1, \ldots, k_{n+1}\) in \(\mathcal{H}\) we write
\[
\Delta^{(n)}(h)(k_1 \otimes \cdots \otimes k_{n+1}) := \sum h^{(1)}_1 k_1 \otimes \cdots \otimes h^{(n+1)}_{n+1} k_{n+1},
\]
\[
(k_1 \otimes \cdots \otimes k_{n+1})\Delta^{(n)}(h) := \sum k_1 h^{(1)}_1 \otimes \cdots \otimes k_{n+1} h^{(n+1)}_{n+1}.
\]
For all \(h, k \in \mathcal{H}\) we put
\[
\Delta(h)(1 \otimes k) := \sum h^{(1)}_1 \otimes h^{(2)}_1 k,
\]
\[
(1 \otimes k)\Delta(h) := \sum 1 h^{(1)}_1 \otimes kh^{(2)}_1,
\]
\[
\Delta(h)(k \otimes 1) := \sum h^{(1)}_1 k \otimes h^{(2)}_1,
\]
\[
(k \otimes 1)\Delta(h) := \sum kh^{(1)}_1 \otimes 1 h^{(2)}_1.
\]

Throughout this paper, we fix the regular multiplier Hopf algebra \(\mathcal{H}\), and we will use \(m, \Delta, \Delta', \varepsilon, S\) for the multiplication, the comultiplication, the opposite comultiplication, the counit and the antipode, respectively. Also, \(\tau\) will denote the flip operator \(\tau : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}\).
3. Group-like projection and modular pair in involution

We begin with the definition of a group-like projection. This is essentially the basic object for the rest of the paper, in the classical case, it behaves like the characteristic function of a subgroup (see [5]). In [5], introduced the notion of a group-like projection for any multiplier Hopf $^*$-algebra. In this paper, we do not require the existence of an involution and therefore, the assumptions are a little different.

**Definition 3.1.** A non-zero element $\gamma \in H$ is called a group-like projection if

$$\Delta(\gamma)(1 \otimes \gamma) = \gamma \otimes \gamma = \Delta(\gamma)(\gamma \otimes 1), \quad \gamma^2 = \gamma.$$  

There are the following results, related with the group-like projections.

**Lemma 3.2.** Let $\gamma$ be a group-like projection in $H$. Then $\varepsilon(\gamma) = 1$ and $S(\gamma) = \gamma$.

**Proof.** Apply $\varepsilon \otimes \iota$ to the defining equation $\Delta(\gamma)(1 \otimes \gamma) = \gamma \otimes \gamma$. Then we get $\gamma \varepsilon(\gamma) = \gamma^2$. Since $\gamma^2 = \gamma$ and $\gamma$ is assumed to be non-zero, so we obtain $\varepsilon(\gamma) = 1$. Using the properties of $S$ one can get $S(\gamma)\gamma = \gamma$. Because $H$ is a regular multiplier Hopf algebra, so $S^{-1}$ is the antipode of $(H, \Delta')$. Using $\Delta'(\gamma)(1 \otimes \gamma) = \gamma \otimes \gamma$ and the properties of $S^{-1}$ one can get $S^{-1}(\gamma)\gamma = \gamma$. If we apply $S$, we get $S(\gamma)\gamma = S(\gamma)$. Hence, we have $S(\gamma) = \gamma$. \qed

Let $\chi : H \to \mathbb{C}$ be a non-zero algebra homomorphism. Define a map

$$\tilde{S} : H \to L(H),$$

by

$$\tilde{S}(h)k := (\chi \otimes \iota)T_1^{-1}(h \otimes k),$$

where $T_1$ is defined by $T_1(h \otimes k) = \Delta(h)(1 \otimes k)$ and $L(H)$ is the space of all linear maps $l : H \to H$ such that $l(\eta k) = l(hk)$, for all $h, k \in H$.

By definition of $\tilde{S}$ we see that elements of the form $\tilde{S}(h)k$ span $H$.

**Lemma 3.3.** We have $\tilde{S}(H) \subset H$.

**Proof.** Take an element $e \in H$ such that $\chi(e) = 1$. Let $h \in H$. For all $k \in H$, we have

$$\tilde{S}(h)k = (\chi \otimes \iota)((e \otimes 1)T_1^{-1}(h \otimes k)).$$

Using Lemma 4.2 in [8], we get

$$\tilde{S}(h)k = (\chi \otimes \iota)((1 \otimes S)((e \otimes 1)\Delta(h))(1 \otimes k))
= (\chi \otimes S)((e \otimes 1)\Delta(h))k.$$  

Hence, we have $\tilde{S}(h) = (\chi \otimes S)((e \otimes 1)\Delta(h)) \in H$. \qed
The elementary properties of $S$ imply that $\widetilde{S}$ is an algebra anti-homomorphism, i.e.,

$$\widetilde{S}(hk) = \widetilde{S}(k)\widetilde{S}(h),$$

for all $h, k \in \mathcal{H}$. Also, it satisfies

$$\varepsilon \circ \widetilde{S} = \chi.$$

For the following result, we refer the reader to [8].

**Lemma 3.4.** For all $h, k \in \mathcal{H}$, we have

$$(1 \otimes S(k))\Delta(S(h)) = (S \otimes S) \circ \tau(\Delta(h)(k \otimes 1)).$$

We obtain the similar results for $\widetilde{S}$ in the next lemma.

**Lemma 3.5.** For all $h, k \in \mathcal{H}$, we have

$$(1 \otimes \widetilde{S}(k))\Delta(\widetilde{S}(h)) = (S \otimes \widetilde{S}) \circ \tau(\Delta(h)(k \otimes 1)),$$

$$\Delta(\widetilde{S}(h))(S(k) \otimes 1) = (S \otimes \widetilde{S}) \circ \tau((1 \otimes k)\Delta(h)).$$

**Proof.** Take an element $e \in \mathcal{H}$ such that $\chi(e) = 1$. We have

$$(S \otimes \widetilde{S}) \circ \tau(\Delta(h)(k \otimes 1)) = \sum(S \otimes \chi \otimes S)(h(2)1 \otimes (h(1)k)(1) \otimes (h(2)1)k(2)1),$$

for all $h, k \in \mathcal{H}$. Using the properties of $\Delta$ we get

$$(S \otimes \widetilde{S}) \circ \tau(\Delta(h)(k \otimes 1)) =$$

$$\sum(S \otimes \chi \otimes S)((h(2)(1) \otimes h(1)(k(1) \otimes (h(2))1)(k(2)1)),$$

for all $h, k \in \mathcal{H}$. Hence, we have

$$(S \otimes \widetilde{S}) \circ \tau(\Delta(h)(k \otimes 1)) = \sum(1 \otimes S(k(1)1))\chi(k(1)1)\Delta(\widetilde{S}(h))$$

$$= (1 \otimes (\chi \otimes S)(\Delta(k)(e \otimes 1)))\Delta(\widetilde{S}(h))$$

$$= (1 \otimes \widetilde{S}(k))\Delta(\widetilde{S}(h)),$$

and similarly for the other formula. \qed

Because $\mathcal{H}$ is a regular multiplier Hopf algebra, so the antipode $S$ is invertible. We can check that a similar fact is true for $\widetilde{S}$ and its inverse is $\widetilde{S}^{-1} := (\chi \otimes S^{-1})\Delta'$.

**Definition 3.6.** Let $\chi : \mathcal{H} \to \mathbb{C}$ be a non-zero algebra homomorphism and $\gamma$ be a group-like projection in $\mathcal{H}$. We call $(\chi, \gamma)$ is a matched pair if

$$\chi(\gamma) = 1.$$
It is easy to check that if \((\chi, \gamma)\) is a matched pair, then \(\tilde{S}(\gamma) = \gamma\).
A non-zero element \(\sigma \in M(\mathcal{H})\) is called a group-like element if
\[
\Delta(\sigma) = \sigma \otimes \sigma,
\]
when we look at the extension of \(\Delta\) to \(M(\mathcal{H})\). It follows that \(\varepsilon(\sigma) = 1\), \(\sigma\) is invertible and
\[
S(\sigma) = \sigma^{-1}, \quad \Delta(\sigma^{-1}) = \sigma^{-1} \otimes \sigma^{-1}.
\]

**Definition 3.7.** Let \(\chi : \mathcal{H} \to \mathbb{C}\) be a non-zero algebra homomorphism and \(\sigma\) be a group-like element. We call \((\chi, \sigma)\) is a modular pair if
\[
\chi(\sigma) = 1,
\]
when we look at the extension of \(\chi\) to \(M(\mathcal{H})\).

It is easy to check that if \((\chi, \sigma)\) is a modular pair, then \(\tilde{S}(\sigma) = \sigma^{-1}\), when we look at the extension of \(\tilde{S}\) to \(M(\mathcal{H})\). A modular pair \((\chi, \sigma)\) is called a modular pair in involution if
\[
\sigma^{-1}\tilde{S}^2(h)\sigma = h,
\]
for all \(h \in \mathcal{H}\).

**Example 3.8.** Let \(\mathcal{G}\) be a discrete group and let \(\mathcal{H}\) be the algebra of complex functions on \(\mathcal{G}\) with finite support with pointwise operations. Consider the obvious comultiplication \(\Delta\) on \(\mathcal{H}\) given by
\[
\Delta(f)(s, t) = f(st),
\]
when \(f \in \mathcal{H}\) and \(s, t \in \mathcal{G}\). Then \(\mathcal{H}\) is a regular multiplier Hopf algebra (see \([8]\)). One has \(\varepsilon(f) = f(e)\) and \(S(f)(t) = f(t^{-1})\), where \(f \in \mathcal{H}, t \in \mathcal{G}\) and \(e\) is the identity element of \(\mathcal{G}\). Let \(\gamma\) be the characteristic function on \(\{e\}\). Then \((\varepsilon, \gamma)\) is a matched pair and \((\varepsilon, 1)\) is a modular pair in involution.

4. **The precyclic module of a regular multiplier Hopf algebra**

In the following, we define the precyclic module for \(\mathcal{H}\) as a regular multiplier Hopf algebra which is associated to a matched pair \((\chi, \gamma)\) and a modular pair in involution \((\chi, \sigma)\) such that \(\gamma\) and \(\sigma\) are compatible, that is,
\[
\gamma \sigma = \sigma \gamma.
\]
We need a lemma before we can do this.

**Lemma 4.1.** For all \(h, k \in \mathcal{H}\), we have
\[
m(S^2 \otimes \tilde{S}) \circ \tau((1 \otimes h)\Delta(k)) = \chi(k)S^2(h),
\]
\[
m(S^2 \otimes \tilde{S}) \circ \tau((h \otimes 1)\Delta(k)) = \chi(k)\tilde{S}(h).
\]
Proof. Take an element \( e \in H \) such that \( \chi(e) = 1 \). For all \( h, k \in H \), we have

\[
m(S^2 \otimes \tilde{S}) \circ \tau((1 \otimes h)\Delta(k)) = \sum \chi((1k(1))e)S^2(hk(2))S((1k(1))(2)1).
\]

Using the properties of \( \Delta \) we obtain

\[
m(S^2 \otimes \tilde{S}) \circ \tau((1 \otimes h)\Delta(k)) = \sum \chi(k(1)e)S^2(h)\varepsilon(k(2)1)
= \chi(k)S^2(h),
\]
and similarly for the other formula. \( \square \)

We associate a presimplicial module to \( H \) by the comultiplication, the multiplication, the group-like projection \( \gamma \) and the group-like element \( \sigma \) such that \( \gamma \) and \( \sigma \) are compatible. The remaining features of the given data, namely the antipode and \( \chi \), are used to define the candidate for the cyclic operators.

Let \( C^n(H) = H^\otimes(n) \), for all \( n \geq 1 \) and \( C^0(H) = \mathbb{C} \). First, we must define the face operators \( \delta_i : C^{n-1}(H) \rightarrow C^n(H) \), for all \( 0 \leq i \leq n \). We define

\[
\delta_0(h^1 \otimes \cdots \otimes h^{n-1}) := \gamma \otimes h^1 \otimes \cdots \otimes h^{n-1}.
\]

For every \( 1 \leq i \leq n-1 \), we let

\[
\delta_i(h^1 \otimes \cdots \otimes h^{n-1}) := h^1 \otimes \cdots \otimes \Delta(h^i)(1 \otimes \gamma) \otimes \cdots \otimes h^{n-1}.
\]

Finally, we define

\[
\delta_n(h^1 \otimes \cdots \otimes h^{n-1}) := \Delta^{(n-1)}(\gamma)(h^1 \otimes \cdots \otimes h^{n-1} \otimes \sigma).
\]

Note that if \( n = 1 \), then \( \delta_0, \delta_1 : \mathbb{C} \rightarrow H \) are given by

\[
\delta_0(1) := \gamma, \quad \delta_1(1) := \gamma \sigma.
\]

For every \( n \geq 1 \), we define the cyclic operators \( \tau_n : C^n(H) \rightarrow C^n(H) \) as follows:

\[
\tau_n(h^1 \otimes \cdots \otimes h^n) := \Delta^{(n-1)}(\tilde{S}(h^1))(h^2 \otimes h^3 \otimes \cdots \otimes h^n \otimes \sigma),
\]

If \( n = 0 \), then define \( \tau_0 : \mathbb{C} \rightarrow \mathbb{C} \) by

\[
\tau_0(1) := 1.
\]

Theorem 4.2. Assume that \( (\chi, \gamma) \) is a matched pair and \( (\chi, \sigma) \) is a modular pair in involution such that \( \gamma \) and \( \sigma \) are compatible. Then \( H^*_\bigotimes(\gamma, \chi, \sigma) = \{ C^n(H) \}_{n \geq 0} \) endowed with the above operators is a precyclic module. In other words, the operators \( \delta_i \) satisfying in the following relations:

\[
\delta_j \delta_i = \delta_i \delta_{j-1}, \quad i < j,
\]
and to obtain precyclic module the operators \( \tau_n \) must satisfying in the relations:

\[
\begin{align*}
\tau_n \delta_i &= \delta_{i-1} \tau_{n-1}, & 1 \leq i \leq n, \\
\tau_n \delta_0 &= \delta_n, & \tau_n^{n+1} = 1.
\end{align*}
\]

**Proof.** We shall only use the basic properties of the multiplication, the comultiplication, the antipode, the group-like element and the grouplike projection, and adhere to the standard notational conventions for the regular multiplier Hopf algebra calculus. Let \( f = h^1 \otimes \cdots \otimes h^{n-1} \).

We first look at the case \( i = 0 \) and \( j = 1 \). Thus

\[
\delta_1 \delta_0(f) = \delta_1(\gamma \otimes f) = \Delta(\gamma)(1 \otimes \gamma) \otimes f = \gamma \otimes \gamma \otimes f = \delta_0^2(f).
\]

For \( i = 1 \) and \( j = 2 \), we have

\[
\begin{align*}
\delta_2 \delta_1(f) &= \sum \delta_2(h^{(1)}_1 1 \otimes h^{(2)}_1 \gamma \otimes h^2 \otimes \cdots \otimes h^{n-1}) \\
&= \sum h^{(1)}_1 1 \otimes \Delta(h^{(2)}_1 \gamma)(1 \otimes \gamma) \otimes h^2 \otimes \cdots \otimes h^{n-1} \\
&= (\iota \otimes \Delta)(\Delta(h^1_1)(1 \otimes \gamma))(1 \otimes 1 \otimes \gamma) \otimes h^2 \otimes \cdots \otimes h^{n-1} \\
&= (\Delta \otimes \iota)(\Delta(h^1_1)(1 \otimes \gamma))(1 \otimes \gamma \otimes 1) \otimes h^2 \otimes \cdots \otimes h^{n-1} \\
&= \sum \Delta(h^{(1)}_1 1)(1 \otimes \gamma) \otimes h^{(2)} \gamma \otimes h^2 \otimes \cdots \otimes h^{n-1} \\
&= \sum \delta_1(h^{(1)}_1 1 \otimes h^{(2)}_1 \gamma \otimes h^2 \otimes \cdots \otimes h^{n-1}) = \delta_0^2(f).
\end{align*}
\]

For \( i = 0 \) and \( j = n + 1 \), one has

\[
\begin{align*}
\delta_{n+1} \delta_0(f) &= \Delta^{(n)}(\gamma)(\gamma \otimes f \otimes \sigma) = (\iota \otimes \Delta^{(n-1)})(\Delta(\gamma))(\gamma \otimes f \otimes \sigma) \\
&= (\iota \otimes \Delta^{(n-1)})(\Delta(\gamma)(\gamma \otimes 1))(1 \otimes f \otimes \sigma) \\
&= \gamma \otimes \Delta^{(n-1)}(\gamma)(f \otimes \sigma) = \delta_0 \delta_n(f).
\end{align*}
\]

For \( i = 1 \) and \( j = n + 1 \), we have

\[
\begin{align*}
\delta_{n+1} \delta_1(f) &= \Delta^{(n)}(\gamma)(\Delta(h^1)(1 \otimes \gamma) \otimes h^2 \otimes \cdots \otimes h^{n-1} \otimes \sigma) \\
&= (\Delta \otimes \iota^{\otimes(n-1)})(\Delta^{(n-1)}(\gamma))(\Delta(h^1)(1 \otimes \gamma) \otimes h^2 \otimes \cdots \otimes h^{n-1} \otimes \sigma) \\
&= (\Delta \otimes \iota^{\otimes(n-1)})(\Delta^{(n-1)}(\gamma))(\Delta \otimes \iota^{\otimes(n-1)})(f \otimes \sigma)(1 \otimes \gamma \otimes \cdots \otimes 1) \\
&= (\Delta \otimes \iota^{\otimes(n-1)})(\Delta^{(n-1)}(\gamma)(f \otimes \sigma))(1 \otimes \gamma \otimes \cdots \otimes 1) \\
&= \sum (\Delta \otimes \iota^{\otimes(n-1)})(\gamma(h^1_1) \otimes \gamma(h^2_2) \otimes \cdots \otimes \gamma(h_{n-1})h^{n-1} \otimes (\gamma(n))_1 \sigma) \\
&= \sum \Delta(\gamma(h^1_1))(1 \otimes \gamma) \otimes \gamma(h^2_2) \otimes \cdots \otimes \gamma(h_{n-1})h^{n-1} \otimes (\gamma(n))_1 \sigma = \delta_1 \delta_n(f).
\end{align*}
\]
If \( n = 1 \), then we have
\[
\delta_2 \delta_0(1) = \delta_2(\gamma) = \Delta(\gamma)(\gamma \otimes \sigma) = \gamma \otimes \gamma \sigma = \delta_0 \delta_1(1).
\]

If \( i = n \) and \( j = n + 1 \), then
\[
\delta_{n+1} \delta_n(f) = \delta_{n+1}(\Delta^{(n-1)}(\gamma)(f \otimes \sigma))
\]
\[
= \Delta^{(n)}(\gamma)(\Delta^{(n-1)}(\gamma)(f \otimes \sigma) \otimes \sigma)
\]
\[
= (\Delta^{(n-1)} \otimes \iota)\left((\Delta(\gamma))(\Delta^{(n-1)} \otimes \iota)(\gamma \otimes 1)(f \otimes \sigma \otimes \sigma)\right)
\]
\[
= (\Delta^{(n-1)} \otimes \iota)(\Delta(\gamma)(\gamma \otimes 1))(f \otimes \sigma \otimes \sigma)
\]
\[
= (\Delta^{(n-1)} \otimes \iota)(\Delta(\gamma)(1 \otimes \gamma))(f \otimes \sigma \otimes \sigma)
\]
\[
= \Delta^{(n)}(\gamma)(f \otimes \sigma \otimes \gamma \sigma).
\]

On the other hand
\[
\delta_n \delta_n(f) = \delta_n(\Delta^{(n-1)}(\gamma)(f \otimes \sigma))
\]
\[
= \sum \delta_n(\gamma(1)h^1 \otimes \gamma(2)h^2 \otimes \cdots \otimes \gamma(n-1)h^{n-1} \otimes (\gamma(n)1)\sigma)
\]
\[
= \sum \gamma(1)h^1 \otimes \gamma(2)h^2 \otimes \cdots \otimes \gamma(n-1)h^{n-1} \otimes \Delta((\gamma(n)1)\sigma)(1 \otimes \gamma)
\]
\[
= \sum (\iota \otimes (n-1) \otimes \Delta)(\gamma(1)h^1 \otimes \gamma(2)h^2 \otimes \cdots \otimes \gamma(n-1)h^{n-1}
\]
\[
\otimes (\gamma(n)1)\sigma)(1 \otimes \cdots \otimes 1 \otimes \gamma)
\]
\[
= (\iota \otimes (n-1) \otimes \Delta)(\Delta^{(n-1)}(\gamma)(f \otimes \sigma))(1 \otimes \cdots \otimes 1 \otimes \gamma)
\]
\[
= \Delta^{(n)}(\gamma)(f \otimes \Delta(\sigma)(1 \otimes \gamma)) = \Delta^{(n)}(\gamma)(f \otimes \gamma \sigma),
\]
where we used the relation \( \sigma \gamma = \gamma \sigma \). Now, we shall verify the remaining relations. Starting with the compatibility with the face operators, one has
\[
\tau_n \delta_0(f) = \tau_n(\gamma \otimes f) = \Delta^{(n-1)}(\tilde{S}(\gamma))(f \otimes \sigma) = \delta_n(f).
\]
Assume that \( c_1, \ldots, c_{n-1} \) are arbitrary elements in \( \mathcal{H} \) and let
\[
\alpha = 1 \otimes S(c_{n-1}) \otimes \cdots \otimes S(c_2) \otimes \tilde{S}(c_1).
\]
Then we have
\[
\alpha \tau_n \delta_1(f) = \sum \alpha \Delta^{(n-1)}(\tilde{S}(h^1_{(1)}(1))(h^1_{(2)})\gamma \otimes h^2 \otimes \cdots \otimes h^{n-1} \otimes \sigma)
\]
\[
= \sum \varepsilon((h^1_{(n-1)})(1))\gamma \otimes S((h^1_{(n-1)})(1)c_{n-1})h^2 \otimes \cdots \otimes S(h^1_{(2)}c_2)h^{n-1} \otimes \tilde{S}(h^1_{(1)}c_1)\sigma
\]
\[
= \sum \gamma \otimes S(c_{n-1})S(h^1_{(n-1)}(1))h^2 \otimes \cdots \otimes S(h^1_{(2)}c_2)h^{n-1} \otimes \tilde{S}(h^1_{(1)}c_1)\sigma = \alpha \delta_0 \tau_{n-1}(f).
\]
Because $S$ and $\widetilde{S}$ are invertible, we can cancel $\alpha$ and one obtains
\[ \tau_n \delta_1 = \delta_0 \tau_{n-1}. \]
Similarly, we can prove $\tau_n \delta_i = \delta_{i-1} \tau_{n-1}$, for all $2 \leq i \leq n - 1$. If $n = 1$, then we have
\[ \tau_1 \delta_1(1) = \tau_1(\gamma \sigma) = \widetilde{S}(\gamma \sigma) \sigma = \sigma^{-1} \gamma \sigma = \gamma = \delta_0 \tau_0(1), \]
where we used the relation $\sigma \gamma = \gamma \sigma$. Let $n = 2$ and $c$ be an arbitrary element in $\mathcal{H}$. Then we have
\[
(1 \otimes \widetilde{S}(c)) \tau_2 \delta_2(h) = \sum (1 \otimes \widetilde{S}(c)) \Delta(\widetilde{S}(\gamma(1)h))((\gamma(2))1) \sigma \otimes \sigma \\
= \sum S(\gamma(1))h_{(1)}(2)1(\gamma(2))1 \sigma \otimes \widetilde{S}(\gamma(1))(h_{(1)}c) \sigma \\
= \sum S(h_{(2)}1) \varepsilon(\gamma(2))1 \sigma \otimes \widetilde{S}(\gamma(1))(h_{(1)}c) \sigma \\
= \sum S(h_{(2)}1) \sigma \otimes \widetilde{S}(h_{(1)}c) \gamma \sigma \\
= (1 \otimes \widetilde{S}(c)) \Delta(\widetilde{S}(h))(\sigma \otimes \gamma \sigma).
\]
Now, because $\widetilde{S}$ is invertible and since the formula holds for all $c$, we can cancel $1 \otimes \widetilde{S}(c)$ and one obtains
\[
\tau_2 \delta_2(h) = \Delta(\widetilde{S}(h))(\sigma \otimes \gamma \sigma) = \delta_1(\widetilde{S}(h)\sigma) = \delta_1 \tau_1(h),
\]
where we used the relation $\sigma \gamma = \gamma \sigma$.

Similarly, we can prove $\tau_n \delta_n = \delta_{n-1} \tau_{n-1}$, for all $n \geq 3$. In the next step, we have to show that $\tau_n^{n+1} = \iota$. If $n = 1$, then
\[ \tau_1^2(h) = \tau_1(\widetilde{S}(h)\sigma) = \sigma^{-1} \widetilde{S}^2(h)\sigma = h. \]
If $n = 2$, then we have
\[ \tau_2(h^1 \otimes h^2) = \Delta(\widetilde{S}(h^1))(S(a) \otimes \sigma) = \sum S(ah_{(2)}^1) \otimes \widetilde{S}(h_{(1)}^1) \sigma, \]
where $a = S^{-1}(h^2)$. Let $c$ be an arbitrary element in $\mathcal{H}$. It follows that
\[ (1 \otimes \widetilde{S}(S(c))) \tau_2^2(h^1 \otimes h^2) = \sum S^2((a(1))(1h_{(1)}^1) )S(1h_{(1)}^1) \sigma \otimes \widetilde{S}(S(c(ah_{(2)})(2))) \sigma. \]
Using the properties of $\Delta$ we obtain
\[
(1 \otimes \widetilde{S}(S(c))) \tau_2^2(h^1 \otimes h^2) = \sum S^2((1a(1))(1h_{(1)}^1) )S(1h_{(1)}^1) \sigma \otimes \widetilde{S}(S((ca(2))(2))) \sigma.
\]
Using Lemma 4.4 and Lemma 3.5, we have
\[
(1 \otimes \widetilde{S}(S(c))) \tau_2^2(h^1 \otimes h^2) = \sum \chi((1h_{(1)}^1) )S^2(1a(1)) \sigma \otimes \widetilde{S}(S((ca(2)h_{(2)}^1))) \sigma \\
= (S^2 \otimes \widetilde{S} \circ S)((1 \otimes c) \Delta(S^{-1}(h^2))) \sigma \otimes \widetilde{S}^2(h^1) \sigma \\
= (1 \otimes \widetilde{S}(S(c))) \Delta(\widetilde{S}(h^2))(\sigma \otimes \widetilde{S}^2(h^1) \sigma).
\]
Since the formula holds for all $c$ and $\tilde{S}$ and $S$ are invertible, so we can cancel $1 \otimes \tilde{S}(S(c))$ and one obtains
\[
\tau_2^3(h^1 \otimes h^2) = \Delta(\tilde{S}(h^2))(\sigma \otimes \tilde{S}(h^1)\sigma) = \sum S(1h_{(2)}^2)\sigma \otimes \tilde{S}(bh_{(1)}^2)\sigma,
\]
where $b = \tilde{S}(h^1)$. Hence, we have
\[
\tau_2^3(h^1 \otimes h^2) = \sum \Delta(\tilde{S}(1h_{(2)}^2))((\tilde{S}(bh_{(1)}^2)\sigma \otimes \sigma) = (\sigma^{-1} \otimes \sigma^{-1})k(\sigma \otimes \sigma),
\]
where $k = \sum \Delta(\tilde{S}(1h_{(2)}^2))((\tilde{S}(bh_{(1)}^2)\otimes 1)$. Let $c$ be an arbitrary element in $\mathcal{H}$. Then
\[
(1 \otimes \tilde{S}(S(c)))k = \sum S^2((1h_{(2)}^2)_{(1)})\tilde{S}(bh_{(1)}^2) \otimes \tilde{S}(S(c(h_{(2)}^2)) = \sum \chi(1h_{(1)}^2)\tilde{S}(b) \otimes \tilde{S}(S(c(h_{(2)}^2)) = \tilde{S}(h^1) \otimes \tilde{S}(S(h^2)S(c)) = (1 \otimes \tilde{S}(S(c)))(\tilde{S}(h^1) \otimes \tilde{S}(h^2)).
\]
Now, because $\tilde{S}$ are $S$ are invertible and since the formula holds for all $c$, we can cancel $1 \otimes \tilde{S}(S(c))$ and this implies that
\[
k = \tilde{S}(h^1) \otimes \tilde{S}(h^2).
\]
So, we have
\[
\tau_2^3(h^1 \otimes h^2) = \sigma^{-1} \tilde{S}(h^1)\sigma \otimes \sigma^{-1} \tilde{S}(h^2)\sigma = h^1 \otimes h^2.
\]
We now pass to the general case. Let $g = h^1 \otimes h^2 \otimes \cdots \otimes h^n$. Then
\[
\tau_n(g) = \Delta^{(n-1)}(\tilde{S}(h^1))(S(a^2) \otimes \cdots \otimes S(a^n) \otimes \sigma)
\]
\[
= (\xi \otimes \Delta^{(n-2)}(\tilde{S}(h^1))(S(a^2) \otimes 1) \otimes \cdots \otimes S(a^n) \otimes \sigma)
\]
\[
= \sum S(a^2h_{(2)}^1) \otimes \Delta^{(n-2)}(\tilde{S}(h_{(1)}^1))(S(a^3) \otimes \cdots S(a^n) \otimes \sigma)
\]
\[
= \sum S(a^2h_{(a)}^1) \otimes S(a^3h_{(a-1)}^1) \otimes \cdots \otimes S(a^n h_{(1)}^2) \otimes \tilde{S}(h_{(1)}^1)\sigma,
\]
where $a^2 = S^{-1}(h^2), \cdots, a^n = S^{-1}(h^n)$. Assume that $c_1, \cdots, c_n$ are arbitrary elements in $\mathcal{H}$. Let
\[
\xi = 1 \otimes S^2(c_{n-1}) \otimes \cdots \otimes S^2(c_2) \otimes \tilde{S}(S(c_1)).
\]
Using Lemma 4.1 and Lemma 3.5 we have
\[ \xi \tau_n^2(g) = \sum \xi \Delta^{(n-1)}(\tilde{S}(S(a^2h_1^n))(S(a^3h_{1(n-1)})) \otimes \cdots \otimes S(a^n h_{1(2)})) \\
= \sum \xi \Delta^{(n-1)}(\tilde{S}(1h_{1(1)})) \otimes \sigma \otimes \sigma) = \sum S^2((1a^2_{1(1)}))h_{1(1)} \otimes S(a^3 h_{1(n-1)}) \\
\otimes S^2((c_{n-1}a^2_{(n-2)})h_{1(n-1)}) \otimes S(a^4 h_{1(n-2)}) \otimes \cdots \\
\otimes S^2((c_{2a^2_{(n-1)}})h_{1(2n-2)}) \otimes \tilde{S}(1h_{1(1)}) \otimes \tilde{S}(S(c_{a^2_{(n)}})h_{1(2n-2)})) \sigma \\
= \sum \sum S(a^3 S(1a^2_{1(1)})) \otimes S(a^4 S(c_{n-1}a^2_{(2)})) \otimes \cdots \\
\otimes S^2(c_{2a^2_{(n-1)}}) \chi(1h_{1(1)}) \otimes S(S(c_{a^2_{(n)}})h_{1(2)})) \sigma \\
= \sum \sum S^2(1a^2_{(1)})h^3 \otimes S^2(c_{n-1}a^2_{(2)})h^4 \otimes \cdots \\
\otimes S^2(c_{2a^2_{(n-1)}}) \sigma \otimes \tilde{S}(S(c_{a^2_{(n)}})) \tilde{S}^2(h_{1}(1)) \sigma \\
= \xi \Delta^{(n-1)}(\tilde{S}(h^2))(h^3 \otimes h^4 \otimes \cdots \otimes S^2(h_{1}) \sigma). \]

Since \( \tilde{S} \) are \( S \) are invertible, so we can cancel \( \xi \) and this implies that

\[ \tau_n^2(g) = \Delta^{(n-1)}(\tilde{S}(h^2))(h^3 \otimes \cdots \otimes \sigma \otimes \tilde{S}^2(h_{1}) \sigma). \]

By induction, one obtains for any \( j = 1, \cdots, n \)

\[ \tau_n^j(g) = \Delta^{(n-1)}(\tilde{S}(h^j))(h^{j+1} \otimes \cdots \otimes \sigma \otimes \cdots \otimes \tilde{S}^2(h^{j-1}) \sigma). \]

This implies that

\[ \tau_n^{n+1}(g) = \Delta^{(n-1)}(\tilde{S}(\sigma))(\tilde{S}^2(h_{1}) \sigma \otimes \cdots \otimes \tilde{S}^2(h^n) \sigma) = \sigma^{-1} \tilde{S}^2(h_{1}) \sigma \otimes \cdots \otimes \sigma^{-1} \tilde{S}^2(h^n) \sigma = g, \]

where we used the modular pair in involution \((\chi, \sigma)\). □

Define the operators \( b_n, b'_n, \lambda_n \) and \( N_n \) as follows:

\[ b_n := \sum_{i=0}^{n} (-1)^i \delta_i, \quad n \geq 1, \]

\[ b'_n := \sum_{i=0}^{n-1} (-1)^i \delta_i, \quad n \geq 1, \]

\[ \lambda_n := (-1)^n \tau_n, \quad n \geq 0, \]
\[ N_n := \sum_{i=0}^{n} (\lambda_n)^i, \quad n \geq 0. \]

Using Theorem 4.2, one can check that \( b_{n+1}b_n = 0 \) and \( b'_{n+1}b'_n = 0 \). Also, we have

\[
(1 - \lambda_n)b_n = b'_n(1 - \lambda_{n-1}), \quad b_nN_{n-1} = N_nb'_n,
\]
\[
(1 - \lambda_n)N_n = N_n(1 - \lambda_n) = 0.
\]

Hence, one obtains the bicomplex \( CC^*_{(\gamma,\chi,\sigma)}(H) \) as follows:

\[
\begin{array}{ccccccc}
& & & & & & \\
& b_3 \uparrow & -b'_3 \uparrow & b_3 \uparrow & -b'_3 \uparrow \\
\mathcal{H}^\otimes(2) & 1-\lambda_2 & \mathcal{H}^\otimes(2) & N_2 & \mathcal{H}^\otimes(2) & 1-\lambda_2 & \mathcal{H}^\otimes(2) & \rightarrow & \ldots \\
b_2 \uparrow & -b'_2 \uparrow & b_2 \uparrow & -b'_2 \uparrow \\
\mathcal{H} & 1-\lambda_1 & \mathcal{H} & N_1 & \mathcal{H} & 1-\lambda_1 & \mathcal{H} & \rightarrow & \ldots \\
b_1 \uparrow & -b'_1 \uparrow & b_1 \uparrow & -b'_1 \uparrow \\
\mathbb{C} & 1-\lambda_0 & \mathbb{C} & N_0 & \mathbb{C} & 1-\lambda_0 & \mathbb{C} & \rightarrow & \ldots \\
\end{array}
\]

The cohomology of the total complex \( \text{Tot}(CC^*_{(\gamma,\chi,\sigma)}(H)) \) corresponding to the precyclic module \( \mathcal{H}^*_{(\gamma,\chi,\sigma)} \) is the precyclic cohomology \( HC^*_{(\gamma,\chi,\sigma)}(H) \) of \( H \) relative to the matched pair \( (\chi, \gamma) \), the modular pair in involution \( (\chi, \sigma) \) and \( \gamma\sigma = \sigma\gamma \). The Hochschild cohomology \( HH^*_{(\gamma,\chi,\sigma)}(H) \) is the cohomology of the first column of the bicomplex \( CC^*_{(\gamma,\chi,\sigma)}(H) \) (see also Remark 5.4 below).

5. Some remarks

In this paper, the group-like projection \( \gamma \) plays an important role and allows us to define the face operators.

**Remark 5.1.** If \( H \) is a Hopf algebra and \( \gamma = 1 \), then the face operators in this paper are the same with the face operators of Connes-Moscovici for Hopf algebras (see [2, 3]).

**Remark 5.2.** In Theorem 4.2, in addition, if there are the degeneracy operators

\[ \sigma_i : C^{n+1}(H) \to C^n(H), \quad 0 \leq i \leq n, \]

...
with the following relations:
\[ \sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \quad i \leq j, \]
\[ \tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}, \quad 1 \leq i \leq n, \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2, \]
\[ \sigma_j \delta_i = \delta_i \sigma_{j-1}, \quad i < j, \quad \sigma_j \delta_i = \delta_{i-1} \sigma_j, \quad i > j + 1, \]
and
\[ \sigma_j \delta_i = \iota, \quad i = j \text{ or } i = j + 1, \]
then \( \mathcal{H}_{(\gamma, \chi, \sigma)}^* \) is a cyclic module in the sense of Connes. Using the work of Connes-Moscovici, our candidate is
\[ \sigma_i(h^1 \otimes \cdots \otimes h^{n+1}) = \varepsilon(h^{i+1})h^1 \otimes \cdots \otimes h^i \otimes h^{i+2} \otimes \cdots \otimes h^{n+1}. \]

Let \( h \in \mathcal{H} \). We have
\[ \sigma_1 \sigma_2(h) = \sigma_1(\Delta(\gamma)(h \otimes \sigma)) \]
\[ = \sum \sigma_1(\gamma(1)h \otimes (\gamma(2)1)\sigma) \]
\[ = \sum \varepsilon((\gamma(2)1)\sigma)\gamma(1)h = \gamma h. \]
Hence, one has \( \sigma_1 \sigma_2 \neq \iota \). Therefore, our work will not imply that \( \mathcal{H}_{(\gamma, \chi, \sigma)}^* \) is a cyclic module.

**Remark 5.3.** It is known that if \( \mathcal{H} \) is a Hopf algebra and \( \gamma = 1 \), then one has the vanishing of the cohomology groups of the complex \( (\mathcal{H}^*, \mathcal{B}_n^*) \) with the extra degeneracy operator \( s_n : \mathcal{H}^\otimes(n+1) \to \mathcal{H}^\otimes(n) \) as contracting homotopy, where
\[ s_n(h^1 \otimes h^2 \otimes \cdots \otimes h^{n+1}) = \Delta^{(n-1)}(\tilde{S}(h^1))(h^2 \otimes h^3 \otimes \cdots \otimes h^{n+1}). \]
If \( n = 0 \), then we have \( s_0(h) = \chi(h) \). If \( n = 1 \), then we have \( s_1(h \otimes k) = \tilde{S}(h)k \). Consider the case when \( \mathcal{H} \) is a regular multiplier Hopf algebra. In this case, one can check that \( s_0 b'_1 = \iota \), and \( s_n b'_{n+1} + b'_n s_{n-1} = T_n \neq \iota \), where
\[ T_n(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = \Delta^{(n-1)}(\gamma)(h^1 \otimes h^2 \otimes \cdots \otimes h^n). \]
However, we still have the vanishing of the cohomology groups of the complex \( (\mathcal{H}^*, \mathcal{B}_n^*) \). Define \( \theta_n : \mathcal{H}^\otimes(n+1) \to \mathcal{H}^\otimes(n) \) by
\[ \theta_n(h^1 \otimes h^2 \otimes \cdots \otimes h^{n+1}) = (-1)^n \varepsilon(h^{n+1})(h^1 \otimes \cdots \otimes h^n). \]
Then \( \theta_0 b'_1 = \iota \), and \( \theta_n b'_{n+1} + b'_n \theta_{n-1} = \iota \). Therefore, the result is still valid for regular multiplier Hopf algebras.

**Remark 5.4.** Consider the short exact sequence of bicomplexes
\[ 0 \to \widetilde{CC}_{(\gamma, \chi, \sigma)}(\mathcal{H}) \to CC_{(\gamma, \chi, \sigma)}(\mathcal{H}) \to \text{Coker}(CC_{(\gamma, \chi, \sigma)}(\mathcal{H})) \to 0 \]
where $\widetilde{CC}^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H})$ is the bicomplex

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\uparrow & \uparrow & b_3 \uparrow & -b_3 \uparrow \\
0 & \to & 0 & \to \mathcal{H}^{\otimes (2)} \xrightarrow{1-\lambda_2} \mathcal{H}^{\otimes (2)} \xrightarrow{N_2} \cdots \\
\uparrow & \uparrow & b_2 \uparrow & -b_2 \uparrow \\
0 & \to & 0 & \to \mathcal{H} \xrightarrow{1-\lambda_1} \mathcal{H} \xrightarrow{N_1} \cdots \\
\uparrow & \uparrow & b_1 \uparrow & -b_1 \uparrow \\
0 & \to & 0 & \to \mathbb{C} \xrightarrow{1-\lambda_0} \mathbb{C} \xrightarrow{N_0} \cdots
\end{array}
\]

and $\text{Coker}(\widetilde{CC}^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H}))$ is the double complex associated to the first two columns of $\widetilde{CC}^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H})$. Note that the complex $\text{Tot}(\widetilde{CC}^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H}))$ is switched two places. Also, the column with differential $-b'$ is acyclic (see Remark 5.3), then the Hochschild cohomology is isomorphic to the cohomology of the total complex associated to the double complex $\text{Coker}(\widetilde{CC}^{*,*}_{(\gamma,\chi,\sigma)}(\mathcal{H}))$. Therefore, the above short exact sequence induces a natural long exact sequence

\[\cdots \to HC_{(\gamma,\chi,\sigma)}^{n}(\mathcal{H}) \to HH_{(\gamma,\chi,\sigma)}^{n}(\mathcal{H}) \to HC_{(\gamma,\chi,\sigma)}^{n-1}(\mathcal{H}) \to \cdots\]

\[\to HC_{(\gamma,\chi,\sigma)}^{n+1}(\mathcal{H}) \to HH_{(\gamma,\chi,\sigma)}^{n+1}(\mathcal{H}) \to \cdots\]

A multiplier Hopf algebra is called of discrete type if it has a cointegral (see [9, 10]). A left cointegral is a non-zero element $a \in \mathcal{H}$ satisfying $ha = \varepsilon(h)a$ for all $h \in \mathcal{H}$. Remark that such cointegrals do not always exist and they are unique (up to a scalar) if they exist. Also in the Example 3.8 the group-like projection $\gamma$ is a left cointegral. In general, however, left cointegrals and group-like projections are different. For example, consider the algebra $\mathcal{H}$ spanned by elements $\{e_pb^q \mid p \in \mathbb{Z}, q = 0, 1, 2, \ldots\}$ and where the elements $e_p$ and $b$ satisfy the relations $e_pe_q = \delta(p, q)e_p$ and $be_p = e_{p+1}b$. Choose any $\lambda \in \mathbb{C} - \{0\}$. Define $a$ in $M(\mathcal{H})$ by $a = \sum_{k \in \mathbb{Z}} \lambda^k e_k$. Then define a comultiplication $\Delta$ on $\mathcal{H}$ by

\[
\Delta(e_p) = \sum_{k \in \mathbb{Z}} e_k \otimes e_{p-k},
\]

\[
\Delta(b) = a \otimes b + b \otimes a^{-1}.
\]
Remark that these infinite sums are well-defined in the strict topology on the multiplier algebra. Then \((\mathcal{H}, \Delta)\) is a regular multiplier Hopf algebra [10]. One can verify that no cointegrals exist for this example, but \(\gamma = e_0\) is a group-like projection.

**Remark 5.5.** Assume that the group-like projection \(\gamma\) is a left cointegral. Define \(D_n : \mathcal{H}^{\otimes (n+1)} \to \mathcal{H}^{\otimes (n)}\) by

\[
D_n(h^1 \otimes h^2 \otimes \cdots \otimes h^{n+1}) = \varepsilon(h^1)(h^2 \otimes \cdots \otimes h^{n+1}).
\]

Then \(D_n b_{n+1} + b_n D_{n-1} = \iota\). This implies that \(HH^0_{(\gamma,\chi,\sigma)}(\mathcal{H}) = 0\), for all \(n \geq 1\). Also, since \(\gamma\) is a left cointegral, \(\gamma^2 = \gamma\) and \(\gamma\sigma = \sigma\gamma\), so one has \(\gamma\sigma = \gamma\). Then \(b_1(1) = \gamma - \gamma\sigma = 0\) and \(HH^0_{(\gamma,\chi,\sigma)}(\mathcal{H}) = \ker(b_1) = \mathbb{C}\).

**Remark 5.6.** Assume that \(\gamma\) is central in \(\mathcal{H}\) and denote \(\mathcal{H}_0 = \mathcal{H}\gamma\). Then \(\mathcal{H}_0\) is a Hopf algebra when we define the comultiplication \(\Delta_0\) on \(\mathcal{H}_0\) by \(\Delta_0(h\gamma) = \Delta(h)(\gamma \otimes \gamma)\) (see [3]). The counit \(\varepsilon_0\) for \(\Delta_0\) is simply the restriction of \(\varepsilon\) to \(\mathcal{H}_0\) and the antipode \(S_0\) is given by the restriction of \(S\) to \(\mathcal{H}_0\). Also, \(\gamma\) is the identity in \(\mathcal{H}_0\). Let \((\chi,\gamma)\) be a matched pair and \((\chi,\sigma)\) a modular pair in involution such that \(\gamma\) and \(\sigma\) are compatible. One can check that \((\chi,\gamma\sigma)\) is a modular pair in involution for the Hopf algebra \(\mathcal{H}_0\), when we look at the restriction of \(\chi\) to \(\mathcal{H}_0\). Let \(HC^*_{(\chi,\gamma\sigma)}(\mathcal{H}_0)\) be the cyclic cohomology of \(\mathcal{H}_0\) relative to the modular pair in involution \((\chi,\gamma\sigma)\) in the sense of Connes-Moscovici for Hopf algebras and \(HH^*_{(\chi,\gamma\sigma)}(\mathcal{H}_0)\) be its Hochschild cohomology. Then the following map is a precyclic map:

\[
T : \mathcal{H}^{\otimes (n)} \to \mathcal{H}_0^{\otimes (n)}, \quad T(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = h^1\gamma \otimes h^2\gamma \otimes \cdots \otimes h^n\gamma.
\]

Hence, \(T\) defines a canonical map from \(HH^*_{(\gamma,\chi,\sigma)}(\mathcal{H})\) to \(HH^*_{(\chi,\gamma\sigma)}(\mathcal{H}_0)\) and \(T\) defines a canonical map from \(HC^*_{(\gamma,\chi,\sigma)}(\mathcal{H})\) to \(HC^*_{(\chi,\gamma\sigma)}(\mathcal{H}_0)\).

**Remark 5.7.** Assume that \(h \in \mathcal{H}\) is a Hochschild 1-cocycle. Then we have

\[
\gamma \otimes h + \Delta(\gamma)(h \otimes \sigma) = \Delta(h)(1 \otimes \gamma).
\]

If we apply \(\iota \otimes \varepsilon\) and \(\varepsilon \otimes \iota\), we get \(\varepsilon(h) = 0\) and \(h\gamma = h = \gamma h\). Remark that \(\gamma\) and \(\gamma\sigma\) need not be the same. Now, assume that \(\gamma\) is central in \(\mathcal{H}\). Then it is clear that \(HH^0_{(\gamma,\chi,\sigma)}(\mathcal{H}) = HH^0_{(\chi,\gamma\sigma)}(\mathcal{H}_0)\) (see Remark 5.6). Also, using the argument above and Remark 5.6, one can verify that \(HH^1_{(\gamma,\chi,\sigma)}(\mathcal{H})\) is isomorphic to \(HH^1_{(\chi,\gamma\sigma)}(\mathcal{H}_0)\).

**Remark 5.8.** Let \(\mathcal{H}\) be an algebraic quantum group with a non-zero left invariant functional \(\varphi\) (see [9]). Assume that \(\varphi(\gamma) \neq 0\). Let \(h \in \mathcal{H}\) be a Hochschild 1-cocycle. Since

\[
\gamma \otimes h + \Delta(\gamma)(h \otimes \sigma) = \Delta(h)(1 \otimes \gamma),
\]
$\gamma h = h$ (see Remark 5.7), so one has

$$\gamma \otimes h + h \otimes \gamma = \Delta(h)(1 \otimes \gamma).$$

Let $\phi = \varphi \circ S$, where $S$ is the antipode of $\mathcal{H}$. If we apply $\phi \otimes \iota$, we get $\phi(\gamma) h = \phi(h)(\gamma - \gamma \sigma)$, where we used the right invariance of $\phi$. Hence, we have $b_1(\phi(h)) = h$ and $HH^1_{(\gamma, \chi, \sigma)}(\mathcal{H}) = 0$. We refer to [5] with examples for an illustration of the result.

**Remark 5.9.** An element $x \in M(\mathcal{H})$ is called $\sigma$-primitive if

$$\Delta(x) = 1 \otimes x + x \otimes \sigma.$$ 

This makes sense because $\Delta$ has a unique extension to the multiplier algebra $M(\mathcal{H})$. Assume that $x$ is $\sigma$-primitive. Then

$$b_2(\gamma x \gamma) = \gamma \otimes \gamma x \gamma - \Delta(\gamma)(1 \otimes x + x \otimes \sigma)(\gamma \otimes \gamma) + (\gamma \otimes \gamma)(x \gamma \otimes \sigma) = 0.$$ 

Therefore, $\gamma x \gamma$ is a Hochschild 1-cocycle.

**Remark 5.10.** Let $\mathcal{H}$ be a Hopf algebra. Let $(\chi, \gamma)$ be a matched pair and $(\chi, \sigma)$ a modular pair in involution such that $\gamma$ and $\sigma$ are compatible. Assume that $\gamma$ is central in $\mathcal{H}$. Let $\mathcal{H}^{\bullet}_{(\chi, \sigma)}$ be the cyclic module of $\mathcal{H}$ relative to the modular pair in involution $(\chi, \sigma)$ in the sense of Connes-Moscovici for Hopf algebras, and $HC^\bullet_{(\chi, \sigma)}(\mathcal{H})$ be its cyclic cohomology. Then the following map is a precyclic map from $\mathcal{H}^{\bullet}_{(\chi, \sigma)}$ to $\mathcal{H}^{\bullet}_{(\gamma, \chi, \sigma)}$:

$$\theta : H^{\otimes(n)} \to H^{\otimes(n)},$$

$$\theta(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = h^1 \gamma \otimes h^2 \gamma \otimes \cdots \otimes h^n \gamma.$$ 

Hence, $\theta$ defines a canonical map from $HC^\bullet_{(\chi, \sigma)}(\mathcal{H})$ to $HC^\bullet_{(\gamma, \chi, \sigma)}(\mathcal{H})$.

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**References**