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Infinitely many Solutions for a Class of P-Biharmonic Problems with Neumann Boundary Conditions

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Abstract. The existence of infinitely many solutions is established for a class of nonlinear functionals involving the p-biharmonic operator with nonhomogeneous Neumann boundary conditions. Using a recent critical-point theorem for nonsmooth functionals and under appropriate behavior of the nonlinear term and nonhomogeneous Neumann boundary conditions, we obtain the result.

Keywords: Infinitely many solutions, P-biharmonic type operators, locally Lipshitz functionals, Neumann boundary value problem.

2000 Mathematics subject classification: 31A30, 35J35, 49J40, 47J20.

1. Introduction

In order to develop a realistic model for physical phenomena from mechanics and engineering, P. D. Panagiotopoulos ([19], [20]), developed the theory of the hemivariational inequalities. Such inequalities appear in the mathematical modeling whose relevant energy functionals

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are locally Lipchitz and not everywhere differentiable and are closely connected to the notion of the generalized gradient of Clarke [5].

In the present paper, we study the following problem involving the p-biharmonic operator and express the constraint by requiring that the solutions are nonnegative.

$$\begin{cases} \Delta_p^2 u \in \lambda\alpha(x)\partial F(u) & in\ \Omega \\ u \geqslant 0 & in\ \Omega \\ \frac{\partial}{\partial n}(|\Delta u|^{p-2}\Delta u) \in -\mu\beta(x)\partial G(u) & on\ \partial\Omega \end{cases} \tag{1.1}$$

In problem (1.1), Ω is an open bounded subset of $\mathbb{R}^N$, $N \geqslant 1$, with a smooth enough boundary $\partial\Omega$, $p > \max\{1, \frac{N}{2}\}$. Also, $\alpha \in L^1(\Omega)$, $\beta \in L^1(\partial\Omega)$, with $\alpha(x) \geqslant 0$ for a.e $x \in \Omega$, $\alpha \not\equiv 0$, $\beta(x) \geqslant 0$ for a.e $x \in \partial\Omega$ and λ, μ are real parameters, with $\lambda > 0$ and $\mu \geqslant 0$.

Here, ∂F and ∂G are Clarke's generalized gradients of locally Lipschitz functions $F, G : \mathbb{R} \to \mathbb{R}$ given by $F(\xi) = \int_0^\xi f(t)dt$, $G(\xi) = \int_0^\xi g(t)dt$, $\xi \in \mathbb{R}$ with $f, g : \mathbb{R} \to \mathbb{R}$ measurable and locally bounded functions in $\mathbb{R}$.

Motivated also by the fact that such kind of fourth-order problems have attracted much attention in recent years due to their direct applications in mechanics, engineering and differential equations, we refer the reader to [4,8,10,13-15] and references therein.

The works mentioned above mostly obtain the existence of at least one solutions using a variational approach based on the nonsmooth critical point theory or at least three solutions using a three critical points theorem for non-differential functions by Morano and Motreanu in [16].

On the other hand, some authors in [9], [11] and [12] studied $(p_1, \cdots, p_n)$-biharmonic systems and established the existence of multiple solutions.

Moreover, the authors in [1-3,6,7] motivated by [1], proved the existence of infinitely many solutions for several hemivariational inequalities, but, in most of them, the problems driven by the p-laplacian.

Here, due to importance of fourth-order problems in describing a large class of elastic deflection or modeling to study travalling waves in suspension bridge, we treat a nonhomogenous Neumann-type problem driven by the p-biharmonic operator and in our main result, Theorem 3.1, under some hypotheses on the behaviour of F and G with the existence of a precise interval for λ, establish problem (1.1) admits infinitely many solutions. Also, some consequences and applications are pointed out.

2. PRELIMINARIES

This section is devoted to definition of generalized gradient of locally Lipschitz functions and related results.

Let $(X, ||.||)$ denote a real Banach space and $(X^*, ||.||_*)$ its topological dual.

Definition 2.1. A functional $H : X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exists a neighborhood U of u and $L > 0$ such that for every $v, w \in U$,

$$|H(v) - H(w)| \leqslant L||v - w||.$$

Definition 2.2. Let $H : X \to \mathbb{R}$ be a locally Lipschitz functional, $u, v \in X$: the generalized derivative of H in u along the direction v is

$$H^0(u; v) = \limsup_{w \to u \ t \to 0^+} \frac{H(w + tv) - H(w)}{t};$$

the generalized gradient of H in u is the set

$$\partial H(u) = \{u^* \in X^* :< u^*, v > \leqslant H^0(u; v) \ for \ all \ v \in X\}.$$

Clearly, these definitions extend those of the Gâteaux directional derivative and gradient.

Proposition 2.3. *Let $H : X \to \mathbb{R}$ be a locally Lipschitz functional. Then, $H^0 : X \times X \to \mathbb{R}$ is upper semicontinuous and for all $\lambda \geqslant 0,\ u, v \in X$, one has*

$$(\lambda H)^0(u; v) = \lambda H^0(u; v)$$

Moreover, if $H_1, H_2 : X \to \mathbb{R}$ are locally Lipschitz functionals, then

$$(H_1 + H_2)^0(u, v) \leqslant H_1^0(u) + H_2^0(v),\ \forall u, v \in X. \tag{2.1}$$

(see [18, Chapter 3]).

Definition 2.4. Let $H : X \to \mathbb{R}$ be a locally Lipschitz functional and $j : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function, then, u is a critical point of $H + j$ if for every $v \in X$,

$$H^0(u; v - u) + j(v) - j(u) \geqslant 0.$$

From now on, assume that X is a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive real parameter, $j : X \to \mathbb{R} \cup \{+\infty\}$ is a convex, proper and lower semicontinuous functional and $D(j)$ is the effective domain of j. Write $\Psi := \Upsilon - j,\ I_\lambda := \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j$. We also assume that Φ is coercive and

$$D(j) \cap \Phi^{-1}(]-\infty, r[) \neq \emptyset, \tag{2.2}$$

for all $r > \inf_X \Phi$. Moreover, by (2.2) and provided $r > \inf_X \Phi$, we can define

$$\varphi(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v)) - \Psi(u)}{r - \Phi(u)}$$

and

$$\varphi^{+} := \liminf_{r\to+\infty} \varphi(r), \qquad \varphi^{-} := \liminf_{r\to(\inf_X \Phi)^{+}} \varphi(r)$$

If Φ and Υ are also locally Lipschitz continuous functionals, in [1] it is proved the following result, which is a version of [17, Theorem 1.1].

The next theorem is the key tool to prove of our main result.

Theorem 2.5. *Under the above assumptions on X, Φ and Ψ, one can conclude that:*

(a) If $\varphi^{+} < +\infty$ then, for each $\lambda \in]0, \frac{1}{\varphi^{+}}[$, the following alternative holds:

either

(a_1) I_λ possesses a global minimum,

or

(a_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(b) If $\varphi^{-} < +\infty$ then, for each $\lambda \in]0, \frac{1}{\varphi^{-}}[$, the following alternative holds:

either

(b_1) there is a global minimum of Φ which is also a local minimum of I_λ,

or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.

3. MAIN RESULT

In this section, we present an application of Theorem 2.5 to a Neumann-type problem involving the p-biharmonic.

Here and throughout, X will denote the Sobolev space $W^{2,p}(\Omega)$ equipped with the following norm

$$||u|| := (\int_\Omega |\Delta u(x)|^p dx)^{\frac{1}{p}}.$$

Let

$$k := \sup_{x\in X\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{||u||}. \tag{3.1}$$

From (3.1), we infer at once that

$$||u||_\infty \leqslant k||u||. \tag{3.2}$$

Suppose that $\tau := \sup dist_{x\in\Omega}(x, \partial\Omega)$. Simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0, \tau) \subseteq \Omega$, where $B(x_0, \tau)$ denotes the open ball of center x_0 and radius τ. Now consider $\bar{\eta} \in [0,1[$ and put

$$L := \frac{\Gamma(1+\frac{1}{N})}{\pi^{\frac{N}{2}} k^p((N-1)^p \tau^{N-p}(1-\bar{\eta}^{N-p}))}, \tag{3.3}$$

where Γ denotes the Gamma function and k is defined in (3.1). Finally, assume that

$$\begin{aligned} A &:= \liminf_{\xi\to+\infty} \tfrac{\max_{|t|\leqslant\xi}(-F(t))}{\xi^p}, \quad & B &:= \limsup_{\xi\to+\infty} \tfrac{-F(\xi)}{\xi^p}, \\ \lambda_1 &:= \tfrac{1}{pk^p L||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))} B}, & \lambda_2 &:= \tfrac{1}{pk^p||\alpha||_{L^1(\Omega)} A}. \end{aligned} \tag{3.4}$$

We now illustrate our main result in the following theorem.

Theorem 3.1. *Let $\alpha \in L^1(\Omega)$ and $\beta \in L^1(\partial\Omega)$ with $\alpha(x) \geqslant 0$ for a.e $x \in \Omega$, $\alpha \not\equiv 0$, $\beta(x) \geqslant 0$ for a.e $x \in \partial\Omega$. Let $f : \mathbb{R} \to \mathbb{R}$ be a positive, measurable and locally bounded function in $\mathbb{R}$ and set $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$ that is locally Lipschitz. Assume that*

(f) $\sup_{\xi\in\Omega\setminus B(x_0,\bar{\eta}\tau)} F(\xi) = 0$,

$$A < LB. \tag{3.5}$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, where λ_1, λ_2 are given by (3.4), for each measurable and locally bounded function $g : \mathbb{R} \to \mathbb{R}$, where potential $G(\xi) = \int_0^\xi g(t)dt$, $\xi \in \mathbb{R}$ satisfies

$$G_\infty := \limsup_{\xi\to+\infty} \frac{\max_{|t|\leqslant\xi}(-G(t))}{\xi^p} < +\infty, \tag{3.6}$$

$$\sup_{\xi\geqslant 0} G(\xi) = 0 \tag{3.7}$$

and for every $\mu \in [0, \delta[$, where

$$\delta = \delta_{g,\lambda} := \frac{1}{||\beta||_{L^1(\partial\Omega)} G_\infty p k^p}(1 - \lambda p k^p ||\alpha||_{L^1(\Omega)} A),$$

the problem (1.1) admits a sequence of non-negative solutions which are unbounded in $W^{2,p}(\Omega)$.

Proof. Our aim is to apply part (a) of Theorem 2.5. To do this, fix $\bar{\lambda} \in]\lambda_1, \lambda_2[$ and let g be a measurable and locally bounded function satisfying our assumptions. Since $\bar{\lambda} < \lambda_2$, one can conclude that $\delta := \delta_{g,\bar{\lambda}} > 0$ and so we can consider $0 \leqslant \bar{\mu} < \delta$. It follows that $\bar{\lambda} p k^p ||\alpha||_{L^1(\Omega)} A + \bar{\mu}||\beta||_{L^1(\partial\Omega)} G_\infty k^p p < 1$, which implies

$$\bar{\lambda} < \frac{1}{pk^p||\alpha||_{L^1(\Omega)} A + \frac{\bar{\mu}}{\bar{\lambda}}||\beta||_{L^1(\partial\Omega)} G_\infty k^p p}. \tag{3.8}$$

Put $C := \{u \in X : u(x) \geqslant 0 \ for \ a.e. \ x \in \Omega\}$ which is a closed convex subset of $W^{2,p}(\Omega)$ containing the nonnegative functions. Assume that j is identically zero in C and let $\Phi, \Upsilon : X \to \mathbb{R}$ defined as follows:

$$\Phi(u) := \frac{1}{p}||u||^p, \ \Upsilon(u) := \int_\Omega \alpha(x)(-F(u(x))dx + \frac{\bar{\mu}}{\bar{\lambda}} \int_{\partial\Omega} \beta(x)(-G(u(x))d\sigma, \ \forall u \in X.$$

Since $\Psi = \Upsilon - j$, so we have the following equalities

$$I_{\bar{\lambda}}(u) = \frac{1}{p}||u||^p - \bar{\lambda}(\int_\Omega \alpha(x)(-F(u(x))dx + \frac{\bar{\mu}}{\bar{\lambda}} \int_{\partial\Omega} \beta(x)(-G(u(x))d\sigma - j(u))$$

$$= \frac{1}{p}||u||^p + \bar{\lambda} \int_\Omega \alpha(x)(F(u(x))dx + \frac{\bar{\mu}}{\bar{\lambda}} \int_{\partial\Omega} \beta(x)(G(u(x))d\sigma + \bar{\lambda} j(u). \quad (3.9)$$

By standard arguments, one can deduce that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative (at the point u) is the functional $\Phi'(u) \in X^*$ given by

$$\Phi'(u)(v) = \int_\Omega |\Delta u(x)|^{p-2} \Delta u(x).\Delta v(x)dx,$$

for all $v \in X$. Furthermore, Ψ is, in particular, sequentially weakly upper semicontinuous. Our goal is to prove that, under our hypotheses, there exists $\{\bar{u}_n\} \subset X$ of critical points for the functional $I_{\bar{\lambda}}(u)$, that is, every element $\bar{u}_n$ satisfies $I^\circ_{\bar{\lambda}}(\bar{u}_n, v - \bar{u}_n) + \bar{\lambda} j(v) - \bar{\lambda} j(\bar{u}_n) \geqslant 0$, for all $v \in X$. First of all, we show that $\varphi^+ < +\infty$. Hence, let $\{s_n\}$ be a real sequence such that $\lim_{n\to+\infty} s_n = +\infty$ and

$$\liminf_{n\to+\infty} \frac{\max_{|t|\leqslant s_n}(-F(t))}{s_n^p} = A. \quad (3.10)$$

Put $r_n := \frac{1}{p}(\frac{s_n}{k})^p$ and assume that $v \in \Phi^{-1}(]+\infty, r_n[)$, for every $n \in \mathbb{N}$. Taking into account formula (3.2) and $||v||^p < pr_n$, one has $|v(x)| \leqslant s_n$ for every $x \in \Omega$. Therefore, for every $n \in \mathbb{N}$, it follows that

$$\varphi(r_n) = \inf_{||u||^p < pr_n} \frac{(\sup_{||v||^p < pr_n} \Psi(v)) - \Psi(u)}{r_n - \frac{||u||^p}{p}} \leqslant \frac{\sup_{||v||^p < pr_n} \Psi(v)}{r_n} \leqslant \frac{\sup_{||v||^p < pr_n} \Upsilon(u)}{r_n}$$

$$\leqslant \frac{\sup_{||v||^p < pr_n} \int_\Omega \alpha(x)(-F(v(x)))dx}{r_n} + \frac{\bar{\mu}}{\bar{\lambda}} \frac{\sup_{||v||^p < pr_n} \int_{\partial\Omega} \beta(x)(-G(v(x)))d\sigma}{r_n}$$

$$\leqslant \frac{\int_\Omega \alpha(x) \max_{|t|\leqslant s_n}(-F(t))dx}{r_n} + \frac{\bar{\mu}}{\bar{\lambda}} \frac{\int_{\partial\Omega} \beta(x) \max_{|t|\leqslant s_n}(-G(t))d\sigma}{r_n}$$

Accordingly,

$$\varphi(r_n) \leqslant k^p p||\alpha||_{L^1(\Omega)} \frac{\max_{|t|\leqslant s_n}(-F(t))}{s_n^p} + \frac{\bar{\mu} k^p p}{\bar{\lambda}} ||\beta||_{L^1(\partial\Omega)} \frac{\max_{|t|\leqslant s_n}(-G(t))}{s_n^p} \ \forall n \in \mathbb{N}. \quad (3.11)$$

Therefore, due to (3.5), one has $A < +\infty$. So, from (3.6), we obtain

$$\lim_{n\to+\infty} \frac{\max_{|t|\leqslant s_n}(-G(t))}{s_n^p} = G_{+\infty}.$$

This implies that,

$$\varphi^+ \leqslant \liminf_{n\to+\infty} \varphi(r_n) \leqslant pk^p||\alpha||_{L^1(\Omega)}A + pk^p\frac{\bar{\mu}}{\bar{\lambda}}||\beta||_{L^1(\partial\Omega)}G_{+\infty} < +\infty,$$

and our claim is proved. Moreover, taking into account (3.8), yields

$$\varphi^+ < pk^p||\alpha||_{L^1(\Omega)}A + \frac{1-\bar{\lambda}pk^p||\alpha||_{L^1(\Omega)}A}{\bar{\lambda}}.$$

Thus,

$$\bar{\lambda} = \frac{1}{pk^p||\alpha||_{L^1(\Omega)}A + (1-\bar{\lambda}pk^p||\alpha||_{L^1(\Omega)}A)} < \frac{1}{\varphi^+}.$$

Next, we show that the function $I_{\bar{\lambda}}$ in (3.9) is unbounded from below. Let $\{d_n\}$ be a real sequence such that $\lim_{n\to+\infty} d_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{(-F(d_n))}{d_n^p} = B. \tag{3.12}$$

For each $n \in \mathbb{N}$, define

$$w_n(x) := \begin{cases} 0 & if\ x \in \Omega\backslash B(x_0,\tau), \\ d_n l & if\ x \in B(x_0,\tau)\backslash B(x_0,\bar{\eta}\tau), \\ d_n & if\ x \in B(x_0,\bar{\eta}\tau). \end{cases}$$

where $l := \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$. Clearly $w_n \in W^{2,p}(\Omega)$ for each $n \in \mathbb{N}$. We have

$$\frac{\partial w_n(x)}{\partial x_i} = \begin{cases} 0 & if\ x \in (\Omega\backslash B(x_0,\tau)) \cup (B(x_0,\bar{\eta}\tau)), \\ d_n\frac{(x_i-x_i^0)}{l} & if\ x \in B(x_0,\tau)\backslash B(x_0,\bar{\eta}\tau). \end{cases}$$

and

$$\frac{\partial^2 w_n(x)}{\partial^2 x_i} \begin{cases} 0 & if\ x \in (\Omega\backslash B(x_0,\tau)) \cup (B(x_0,\bar{\eta}\tau)), \\ d_n(\frac{l^2-(x_i-x_i^0)^2}{l^3}) & if\ x \in B(x_0,\tau)\backslash B(x_0,\bar{\eta}\tau). \end{cases}$$

This implies that, $\sum_{i=1}^N \frac{\partial^2 w_n(x)}{\partial^2 x_i} = \frac{d_n}{l}(N-1)$, and so

$$||w_n||^p = \frac{\pi^{\frac{N}{2}} d_n^p}{\Gamma(1+\frac{1}{N})}((N-1)^p\tau^{N-p}(1-\bar{\eta}^{N-p})).$$

At this point, according to definition of $w_n(x)$ and hypothesis (f), it is routine to check that,

$$\int_\Omega (-\alpha(x))F(w_n(x))dx \geqslant \int_{B(x_0,\bar{\eta}\tau)} (-\alpha(x))F(d_n)dx \geqslant -F(d_n)||\alpha||_{L^1 B(x_0,\bar{\eta}\tau)},\ for\ any\ n \in \mathbb{N}$$

and one can conclude that

$$-\bar{\lambda}\int_{\Omega}(-\alpha(x))F(w_n(x))dx \leqslant \bar{\lambda}F(d_n)||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))} \tag{3.13}$$

Moreover, owing to (3.7), it follows that

$$-\bar{\mu}\int_{\partial\Omega}(-\beta(x))G(w_n(x))dx \leqslant 0. \tag{3.14}$$

Hence, inequalities (3.13) and (3.14) imply that

$$I_{\bar{\lambda}}(w_n) = \Phi(w_n) - \bar{\lambda}\Psi(w_n)$$

$$= \frac{||w_n||^p}{p} - \bar{\lambda}\int_{\Omega}(-\alpha(x))F(w_n(x))dx - \bar{\mu}\int_{\partial\Omega}(-\beta(x))G(w_n(x))dx +$$

$$\bar{\lambda}j(w_n)$$

$$= \frac{||w_n||^p}{p} - \bar{\lambda}\int_{\Omega}(-\alpha(x))F(w_n(x))dx - \bar{\mu}\int_{\partial\Omega}(-\beta(x))G(w_n(x))dx$$

$$\leqslant \frac{\pi^{\frac{N}{2}}d_n^p}{\Gamma(1+\frac{N}{2})p}((N-1)^p\tau^{N-p}(1-\bar{\eta}^{N-p})) + \bar{\lambda}F(d_n)||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))}$$

$$= \frac{d_n^p}{pk^pL} + \bar{\lambda}F(d_n||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))}) \quad \text{for every } n \in \mathbb{N}.$$

If $B < +\infty$, let $\varepsilon \in]\frac{1}{pk^pL\bar{\lambda}||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))}B}, 1[$. Due to (3.12), there exists v_ε such that

$$F(d_n) < -\varepsilon B d_n^p \qquad \forall n > v_\varepsilon.$$

Moreover,

$$I_{\bar{\lambda}}(w_n) \leqslant \frac{d_n^p}{pk^pL} - \bar{\lambda}||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))}\varepsilon B d_n^p = d_n^p(\frac{1}{pk^pL} - \bar{\lambda}||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))}\varepsilon B),\ \forall n > v_\varepsilon.$$

Taking into account the choice of ε, it follows that $\lim_{n\to+\infty} I_{\bar{\lambda}}(w_n) = -\infty$.

If $B = +\infty$, let us consider

$$M > \frac{1}{pk^pL\bar{\lambda}||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))}} \tag{3.15}$$

According to (3.12), there exists v_ε such that

$$F(d_n) < -M d_n^p \qquad \forall n > v_\varepsilon.$$

Then,

$$I_{\bar{\lambda}}(w_n) \leqslant \frac{d_n^p}{pk^pL} - \bar{\lambda}||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))}M d_n^p = d_n^p(\frac{1}{pk^pL} - \bar{\lambda}||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))}M),\ \forall n > v_\varepsilon.$$

Finally, inequality (3.15), implies that $\lim_{n\to+\infty} I_{\bar{\lambda}}(w_n) = -\infty$, which completes the proof that $I_{\bar{\lambda}}$ is unbounded from below. Thus, from part

(a) of Theorem 2.5, we know that the function $I_{\bar{\lambda}}$ admits a sequence of critical points $\{\bar{u_n}\} \subset X$ such that $\lim_{n\to+\infty} \Phi(\bar{u_n}) = +\infty$. Since Φ is bounded on bounded sets, then $\{\bar{u_n}\}$ has to be unbounded. Moreover, if $\bar{u_n} \in X$ is a critical point of $I_{\bar{\lambda}}$, clearly, by definition, one can conclude that

$$I^{\circ}_{\bar{\lambda}}(\bar{u_n}, v - \bar{u_n}) + \bar{\lambda} j(v) - \bar{\lambda} j(\bar{u_n}) \geqslant 0, \tag{3.16}$$

for each $v \in X$. From (3.16), and by $\{\bar{u_n}\}$ is unbounded, it follows that

$$I^{\circ}_{\bar{\lambda}}(\bar{u_n}, v - \bar{u_n}) = \Phi'(\bar{u_n}, v - \bar{u_n}) + \bar{\lambda}[-\Upsilon(\bar{u_n}, v - \bar{u_n})]^{\circ} \geqslant 0$$

for every $v \in C$.

Therefore,

$$\int_{\Omega} |\Delta\bar{u_n}(x)|^{p-2}\Delta\bar{u_n}(x).\Delta(v(x)-u_n(x))dx+\bar{\lambda}(-\Upsilon)^{\circ}(\bar{u_n}(x), v(x)-\bar{u_n})(x)) \geqslant 0, \ \forall v \in C. \tag{3.17}$$

By using (2.1) and formula (2) on [5, P. 77], we obtain

$$\bar{\lambda}(-\Upsilon)^{\circ}(\bar{u_n}(x), v(x) - \bar{u_n})(x)) \leqslant \bar{\lambda} \int_{\Omega} \alpha(x) F^{\circ}(\bar{u_n}(x); v(x) - u_n(x))dx$$
$$+\bar{\mu} \int_{\partial\Omega} \beta(x) G^{\circ}(\bar{u_n}(x); v(x)-u_n(x))dx.$$

Inserting this into (3.17), leads to
$\int_{\Omega} |\Delta\bar{u_n}(x)|^{p-2}\Delta\bar{u_n}(x).\Delta(v(x) - u_n(x))dx$
$+\bar{\lambda}[\int_{\Omega} \alpha(x)F^0(\bar{u_n}(x); v(x)-u_n(x))dx+\frac{\bar{\mu}}{\bar{\lambda}} \int_{\partial\Omega} \beta(x)G^0(\bar{u_n}(x); v(x)-u_n(x))dx] \geqslant$
0, for every $v \in C$, which completes the proof. □

Example 3.2. Set $a_n := 2n\pi$, $b_n := 2n\pi + \frac{\pi}{2}$, for every $n \in \mathbb{N}$ and define the positive (and discontinuous) function $h : \mathbb{R} \to \mathbb{R}$ as follows

$$h(t) = \begin{cases} t^{2p-1}(4p + 2\sin^2(t) + t\sin(2t)) & t \in \bigcup_{n\geqslant 0}]a_n, b_n[\\ 0 & O.W. \end{cases}$$

Moreover,

$$H(t) = \int_0^t h(\xi)d\xi = \begin{cases} t^{2p}(2 + \sin^2(t)) & t \in \bigcup_{n\geqslant 0}]a_n, b_n[\\ 0 & O.W. \end{cases}$$

One can conclude that $H(a_n) = 0, H(b_n) = \int_{a_n}^{b_n} h(\xi)d\xi = 3{b_n}^{2p} - 2{a_n}^{2p}$, then $\lim_{n\to+\infty} \frac{H(b_n)}{b_n^p} = +\infty$ and $\lim_{n\to+\infty} \frac{H(a_n)}{b_n^p} = 0$. Accordingly, we obtain

$$\liminf_{\xi\to+\infty} \frac{H(\xi)}{\xi^p} = 0, \qquad \limsup_{\xi\to+\infty} \frac{H(\xi)}{\xi^p} = +\infty.$$

Then, for every $(\lambda, \mu) \in]0, +\infty[\times[0, +\infty[$ and for every positive and locally bounded function $g : \mathbb{R} \to \mathbb{R}$ with potential G satisfying $\lim_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = 0$, the problem (1.1) possesses a sequence of solutions which are unbounded in $W^{2,p}(\Omega)$.

A similar argument, assures the existence of infinitely many solutions to problem (1.1) converging at zero. More precisely, the following theorem holds.

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable and locally bounded function and put $F(\xi) := \int_0^\xi f(t)dt$ for every $\xi \in \mathbb{R}$. Put*

$$\begin{aligned} A^\circ &:= \liminf_{\xi\to 0^+} \frac{\max_{|t|\leqslant \xi}(-F(t))}{\xi^p}, \quad B^\circ := \limsup_{\xi\to 0^+} \frac{-F(\xi)}{\xi^p}, \\ \lambda_1 &:= \frac{1}{pk^p L||\alpha||_{L^1(B(x_0,\bar{\eta}\tau))} B^\circ}, \qquad \lambda_2 := \frac{1}{pk^p||\alpha||_{L^1(\Omega)} A^\circ} \end{aligned} \tag{3.18}$$

Assume that condition (f) in Theorem 3.1, is satisfied and

$$\liminf_{\xi\to 0^+} \frac{\max_{|t|\leqslant \xi}(-F(\xi))}{\xi^p} < L \limsup_{\xi\to 0^+} \frac{(-F(\xi))}{\xi^p},$$

where L is given by (3.3). Then, for every $\lambda \in]\lambda_1^0, \lambda_2^0[$, for every locally essentially bounded function $g : \mathbb{R} \to \mathbb{R}$ whose potential $G(\xi) = \int_0^\xi g(t)dt$ for every $\xi \in \mathbb{R}$, satisfies

$$\lim_{\xi\to 0^+} \frac{\max_{|t|\leqslant \xi}(-G(\xi))}{\xi^p} < +\infty, \qquad \sup_{\xi\geqslant 0}(-G(\xi)) = 0$$

and for every $\mu \in [0, \mu^*_{g,\lambda}[$, where

$$\mu^*_{g,\lambda} := \frac{1}{||\beta||_{L^1(\partial\Omega)} G_{+\infty} pk^p}(1 - \lambda||\alpha||_{L^1(\Omega)} pk^p \liminf_{\xi\to 0^+} \frac{(-F(\xi))}{\xi^p})$$

the problem (1.1), posses a sequence of nonzero solutions which strongly converges to 0 in $W^{2,p}(\Omega)$.

Proof. Taking X, Φ and Ψ as in the proof of Theorem 3.1, fix $\bar{\lambda} \in]\lambda_1^0, \lambda_2^0[$, let g be a function that satisfy hypotheses in theorem and take $0 \leqslant \bar{\mu} < \bar{\mu}^*_{g,\lambda}$. Let $\{s_n\}$ be a sequence of positive numbers such that $s_n \to 0^+$ and

$$\lim_{n\to+\infty} \frac{\max_{|t|\leqslant s_n}(-F(t))}{s_n^p} = A^0 < +\infty$$

Putting $r_n := \frac{1}{p}(\frac{s_n}{k})^p$ for every $n \in \mathbb{N}$ and working as in the proof of Theorem 3.1, it follows that $\varphi^+ < +\infty$, $\bar{\lambda} < \frac{1}{\varphi^+}$.
We now claim that $\Phi - \bar{\lambda}\Psi$ has not a local minimum at zero. Indeed, let $\{d_n\}$ be a real sequence of positive numbers such that $\lim_{n\to+\infty} d_n = 0$ and $\lim_{n\to+\infty} \frac{(-F(d_n^p))}{d_n^p} = B^0$. Consider the sequence of functions $\{w_n\}$ defined in Theorem 3.1 and from hypotheses of g, it follows that $I_{\bar{\lambda}}(w_n) < 0$ for every integer sufficiently large. Since $I_{\bar{\lambda}}(0) = 0$ that implies claim in view of the fact $||w_n|| \to 0$, then, the unique global minimum of Φ is not a local minimum of the functional $I_{\bar{\lambda}}$. Due to the

part of (b) of Theorem 2.5, we obtain a sequence $\{v_n\} \subset X$ of critical points of $I_{\bar{\lambda}}$ such that $\lim_{n\to+\infty} ||v_n|| = 0$. Moreover, again as in the proof of theorem 3.1, one can show that every critical point of functional $I_{\bar{\lambda}}$ is also a solution of problem (1.1). Therefore, the proof is completed. $\square$

If $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $C = W^{2,p}(\Omega)$, we therefore can consider the following theorem that is a consequence of Theorem 3.1.

Theorem 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function with potential $F(\xi) = \int_0^\xi f(t)dt$, for every $\xi \in \mathbb{R}$. Assume that the following condition holds:*

(f') $\liminf_{\xi\to+\infty} \frac{(F(\xi))}{\xi^2} < 1/2 \limsup_{\xi\to+\infty} \frac{(F(\xi))}{\xi^2}$.

Then, for each

$$\lambda \in \left] \frac{4}{\limsup_{\xi\to+\infty} \frac{(F(\xi))}{\xi^2}}, \frac{2}{\liminf_{\xi\to+\infty} \frac{(F(\xi))}{\xi^2}} \right[, \tag{3.19}$$

for every non-negative, continuous function $g : \mathbb{R} \to \mathbb{R}$, whose potential $G(\xi) = \int_0^\xi g(t)dt$, $\xi \in \mathbb{R}$ satisfies

(g') $G_{+\infty} := \limsup_{\xi\to+\infty} \frac{(G(\xi))}{\xi^2} < +\infty$,

and for every $\mu \in [0, \delta[$, where

$$\delta := \frac{1}{G_{+\infty}}\left(2 - \lambda \liminf_{\xi\to+\infty} \frac{(F(\xi))}{\xi^2}\right), \tag{3.20}$$

the following problem

$$\begin{cases} u^{iv} = \lambda f(u) & in]0,1[, \\ u'''(0) = \mu g(u(0)), \\ -u'''(1) = \mu g(u(1)), \end{cases} \tag{3.21}$$

admits a sequence of pairwise distinct classical solutions.

To clarify the above theorem, we give the following example.

Example 3.5. Suppose that $\Omega =]0,1[$, $p = 2$ and define the non-negative continuous function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(t) = \begin{cases} 2t(t + \cos^2(\ln t)) & t > 0 \\ 0 & O.W. \end{cases}$$

$\Rightarrow \liminf_{\xi\to+\infty} \frac{\int_0^\xi f(t)dt}{\xi^2} = \frac{6-\sqrt{2}}{8}, \qquad \limsup_{\xi\to+\infty} \frac{\int_0^\xi f(t)dt}{\xi^2} = \frac{6+\sqrt{2}}{8}$.

One can conclude that $\liminf_{\xi\to+\infty} \frac{(F(\xi))}{\xi^2} < 1/2 \limsup_{\xi\to+\infty} \frac{(F(\xi))}{\xi^2}$. Due to Theorem 3.3, for each $\lambda \in]\frac{32}{6+\sqrt{2}}, \frac{16}{6+\sqrt{2}}[$, for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$ such that (*g*) $G_{+\infty} := \lim_{\xi\to+\infty} \frac{(G(\xi))}{\xi^2} < +\infty$,

holds and for every $\mu \in [0, \hat{\mu}[$ where $\hat{\mu} := \frac{1}{G_{+\infty}}(\frac{16-\lambda(6-\sqrt{2})}{8})$, the problem (3.21) possesses a sequence of weak solutions which are unbounded in $W^{2,p}(\Omega)$, as desired.

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