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Biquaternions Lie Algebra and Complex-Projective Spaces

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Abstract. In this paper, Lie group and Lie algebra structures of unit complex 3-sphere $\boldsymbol{S}^3_{\mathbb{C}}$ are studied. In order to do this, adjoint representation of unit biquaternions (complexified quaternions) is obtained. Also, a correspondence between the elements of $\boldsymbol{S}^3_{\mathbb{C}}$ and the special bicomplex unitary matrices $\boldsymbol{SU}_{\mathbb{C}_2}(2)$ is given by expressing biquaternions as 2-dimensional bicomplex numbers $\mathbb{C}^2_2$. The relation $\boldsymbol{SO}(\mathbb{R}^3) \cong \boldsymbol{S}^3/\{\pm 1\} = \mathbb{R}\mathbf{P}^3$ among the special orthogonal group $\boldsymbol{SO}(\mathbb{R}^3)$, the quotient group of unit real quaternions $\boldsymbol{S}^3/\{\pm 1\}$ and the projective space $\mathbb{R}\mathbf{P}^3$ is known as the Euclidean-projective space [1]. This relation is generalized to the Complex-projective space and is obtained as $\boldsymbol{SO}(\mathbb{C}^3) \cong \boldsymbol{S}^3_{\mathbb{C}}/\{\pm 1\} = \mathbb{C}\mathbf{P}^3$.

Keywords: Bicomplex numbers, real quaternions, biquaternions (complexified quaternions), lie algebra, complex-projective space.

2000 Mathematics subject classification: 20G20, 32C15, 32M05.

1. Introduction

It is known that the special orthogonal group $\boldsymbol{SO}(3)$ forms the set of all the rotations in 3-dimensional Euclidean space $\mathbb{E}^3$, which preserves lenght and orientation [2]. Thus, Lie algebra of the Lie group $\boldsymbol{SO}(3)$ corresponds to the 3-dimensional Euclidean space $\mathbb{R}^3$, with the cross

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product operation. Killing bilinear form of the unit 3-sphere is the inner product in $\mathbb{E}^3$. Adjoint representation $ad\boldsymbol{O}(3)$ of the orthogonal group $\boldsymbol{O}(3)$ is the group of all isometries that preserves the inner product on the Lie algebra of $\boldsymbol{O}(3)$ [1]. Similar to $\boldsymbol{SO}(3)$, the special orthogonal group $\boldsymbol{SO}(\mathbb{C}^3)$ forms the set of all the rotations in $\mathbb{C}^3$, which preserves lenght and orientation.

In this paper, Lie group and Lie algebra structures of unit complex 3-sphere $\boldsymbol{S}^3_{\mathbb{C}}$ are studied. In order to do this, adjoint representation of unit biquaternions (complexified quaternions) is obtained. From the adjoint representation of $\boldsymbol{S}^3_{\mathbb{C}}$, it is obtained that the group $ad\boldsymbol{S}^3_{\mathbb{C}}$ is the group of all the complex isometries that preserves the complex metric tensor on the Lie algebra of $\boldsymbol{S}^3_{\mathbb{C}}$. Also, Killing bilinear form on the Lie algebra of the group $\boldsymbol{S}^3_{\mathbb{C}}$ is obtained. It is shown that the Killing bilinear form of $\boldsymbol{S}^3_{\mathbb{C}}$ can be given with the metric tensor in $\mathbb{E}^3$. It is obtained that $\boldsymbol{S}^3_{\mathbb{C}}$ is not compact, because the value of the Killing bilinear form of $\boldsymbol{S}^3_{\mathbb{C}}$ is obtained complex.

The relation $\boldsymbol{SO}(\mathbb{R}^3) \cong \boldsymbol{S}^3/\{\pm 1\} = \mathbb{R}\mathbf{P}^3$ among the special orthogonal group $\boldsymbol{SO}(\mathbb{R}^3)$, the quotient group of the unit real quaternions $\boldsymbol{S}^3/\{\pm 1\}$ and the projective space $\mathbb{R}\mathbf{P}^3$ is known as the Euclidean-projective space [1]. Finally, this relation is generalized to the Complex-projective space and is obtained as $\boldsymbol{SO}(\mathbb{C}^3) \cong \boldsymbol{S}^3_{\mathbb{C}}/\{\pm 1\} = \mathbb{C}\mathbf{P}^3$.

2. Preliminaries

In this section, initially we will present basics of lie group and lie algebra structures. Afterwards, we will present basics of real quaternions, bicomplex numbers and biquaternions (complexified quaternions).

2.1. Basics of Lie Group and Lie Algebra Structures.

Definition 2.1. Left and right translation on the group $\boldsymbol{G}$, determined by an element $a \in \boldsymbol{G}$, are the mappings

$$L_a \colon \boldsymbol{G} \to \boldsymbol{G} \text{ defined by } x \mapsto L_a(x) = ax$$

and

$$R_a \colon \boldsymbol{G} \to \boldsymbol{G} \text{ defined by } x \mapsto R_a(x) = xa$$

for all $x \in \boldsymbol{G}$, respectively [3].

Definition 2.2. Let M be a topological manifold with an atlas S. Then, M is called a differentiable manifold if transformations of coordinates have all partial derivatives of all orders. M is called an analytic manifold if transformations of coordinates are real analytic functions at every point [3].

Definition 2.3. Let $\boldsymbol{G}$ be a group. $\boldsymbol{G}$ is called a Lie group if $\boldsymbol{G}$ is an analytic manifold and the mapping $\phi\colon \boldsymbol{G}\times\boldsymbol{G}\to\boldsymbol{G}$ defined by $\phi(x,y)=xy$ is analytic, for all $x,y\in\boldsymbol{G}$ [3].

Definition 2.4. An algebra V is called a Lie algebra if the operation of multiplication in V denoted by $[\mathbf{u},\mathbf{v}]$ for $\mathbf{u},\mathbf{v}\in V$ satisfies

(1) $[\mathbf{u},\mathbf{v}]=-[\mathbf{v},\mathbf{u}]$ (antisymetric);
(2) $\big[\mathbf{u},[\mathbf{v},\mathbf{w}]\big]+\big[\mathbf{v},[\mathbf{w},\mathbf{u}]\big]+\big[\mathbf{w},[\mathbf{u},\mathbf{v}]\big]=0$ for all $\mathbf{w}\in V$ (Jacobi identity);

The operation $[\mathbf{u},\mathbf{v}]$ is called the bracket of vectors $\mathbf{u}$ and $\mathbf{v}$. First condition implies that $[\mathbf{u},\mathbf{u}]=0$ [3].

Definition 2.5. Let $\boldsymbol{G}$ be a Lie group. A vector field X on $\boldsymbol{G}$ is called left-invariant if $L_g'(X)=X$ for all $g\in\boldsymbol{G}$, where L_g' is the differential of the mapping L_g [3].

Definition 2.6. Let e be the unit element of the Lie group $\boldsymbol{G}$ and $\xi,\eta\in T_e(\boldsymbol{G})$. Denote by X and Y the left-invariant vector fields determined by ξ and η. Then we define $[\xi,\eta]=[X,Y]_e$. Thus the tangent space $T_e(\boldsymbol{G})$ at the unit element of the group becomes a Lie algebra (which is the same as $\mathrm{X}^L(G)$, of course; we limit ourselves to the unit of the group for simplification only). This Lie algebra is called the Lie algebra of the group $T_e(\boldsymbol{G})$ and is denoted by $\mathfrak{G}$ [3].

Definition 2.7. Let a Lie group $\boldsymbol{G}$ be given and let g be a fixed choosen element of the group $\boldsymbol{G}$. For all $h\in\boldsymbol{G}$ the mapping

$$Int_g\colon \boldsymbol{G}\to\boldsymbol{G} \quad \text{defined by} \quad Int_g(h)=ghg^{-1}$$

is a differentiable isomorphism of the group $\boldsymbol{G}$ and we have $Int_g(e)=geg^{-1}=e$. This means that the differential of the mapping Int_g at the point e, i.e. $(Int_g)'_e$, maps $T_e(\boldsymbol{G})$ into $T_e(\boldsymbol{G})$. By denoting $(Int_g)'_e$ with ad_g then $ad_g\colon\mathfrak{G}\to\mathfrak{G}$. The mapping $ad\colon \boldsymbol{G}\to Hom(\mathfrak{G},\mathfrak{G})$ is called the adjoint representation of the group $\boldsymbol{G}$ [3].

Definition 2.8. Let $\mathfrak{G}$ be a Lie algebra and let us define the transformation

$$K\colon \mathfrak{G}\times\mathfrak{G}\to\mathbb{R} \quad \text{defined by} \quad K(X,Y)=Tr(AdX,AdY)$$

for all $X,Y\in\mathfrak{G}$. Then, the form $K(X,Y)$ is called the Killing bilinear form on $\mathfrak{G}$ and is a symmetric bilinear form where

$$AdX\colon\mathfrak{G}\to\mathfrak{G} \quad \text{defined by} \quad Y\mapsto AdX(Y)=[X,Y] \quad \text{for all} \quad Y\in\mathfrak{G},$$

and $Tr(AdX,AdY)$ stands for the trace of the mapping

$$AdX\cdot AdY\colon\mathfrak{G}\to\mathfrak{G} \quad \text{defined by} \quad Z\mapsto AdX\cdot AdY(Z)=\big[X,[Y,Z]\big].$$

2.2. **Basics of Real Quaternions.** Real quaternion algebra

$$\mathbb{H} = \{q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \mid w, x, y, z \in \mathbb{R}\}$$

is a four dimensional vector space over the field of real numbers $\mathbb{R}$ with a basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ invented by Hamilton in 1843. Multiplication is defined by the hypercomplex operator rules

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1,$$
$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k},\ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},\ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

Also, $\mathbb{H}$ is an associative and non-commutative division ring.

For any real quaternion $q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, we define the scalar part and the vector part of q as $S(q) = w$ and $\mathbf{V}(q) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, respectively, that is $q = S(q) + \mathbf{V}(q)$. The conjugate of $q = S(q) + \mathbf{V}(q)$ is defined as $\overline{q} = S(q) - \mathbf{V}(q)$, also known as quaternion conjugate. If $S(q) = 0$, then q is called pure real quaternion. Pure real quaternions set

$$\mathrm{Im}(\mathbb{H}) = \{q = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \mid w, x, y, z \in \mathbb{R}\}$$

is a linear subspace of $\mathbb{H}$ spanned by $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ [4, 5].

2.3. **Basics of Bicomplex Numbers.** The algebra of complex numbers can be generalized to bicomplex numbers

$$\mathbb{C}_2 = \{Z = A + \boldsymbol{J}B = (A, B)\colon A, B \in \mathbb{C})\}$$

where $\boldsymbol{J}$ is the complex imaginary operator (therefore $\boldsymbol{J}^2 = -1$) distinct from the complex numbers operator $\boldsymbol{I}$ satisfying $\boldsymbol{JI} = \boldsymbol{IJ}$. For any $Z = A + \boldsymbol{J}B \in \mathbb{C}_2$, we define the real part and imaginary part of Z as $\Re(Z) = A$ and $\Im(Z) = B$, respectively, that is $Z = A + \boldsymbol{J}B$. The bicomplex conjugate of Z is defined as $Z^* = \Re(Z) - \boldsymbol{J}\Im(Z)$. If $\Re(Z) = 0$, then Z is called pure bicomplex number. The set of pure bicomplex numbers

$$\mathrm{Im}\mathbb{C}_2 = \{Z = \boldsymbol{J}B\colon B \in \mathbb{C}\}$$

is a linear subspace of $\mathbb{C}_2$. Let $Z = A + \boldsymbol{J}B$ and $W = C + \boldsymbol{J}D$ be any two bicomplex numbers. Then, we define the sum and multiplication of Z and W as follows:

$$Z + W = (A + C) + \boldsymbol{J}(B + D) = (A + C, B + D),$$
$$ZW = (AC - BD) + \boldsymbol{J}(AD + BC) = (AC - BD, AD + BC).$$

In addition, we can write ZW in the matrix form as

$$ZW = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}.$$

Thus, $\mathbb{C}_2$ is a commutative ring with the unit element $1 = (1, 0)$.

For any bicomplex number $Z = A + \boldsymbol{J}B$, where $A = a_0 + \boldsymbol{I}a_1$ and $B = b_0+\boldsymbol{I}b_1$, the norm and the modulus of Z are $N_Z = ||Z|| = A^2+B^2 = (a_0^2 + b_0^2) + \boldsymbol{J}(2a_0a_1 + 2b_0b_1)$ and $|Z| = \sqrt{N_Z} = \sqrt{||Z||}$, respectively. It is essential to note that $||Z|| \in \mathbb{C}$. If $||Z|| = 1$ then Z is called unit bicomplex number. Moreover, we can write Z in polar form as

$$Z = \sqrt{N_Z}(cos\varphi + \boldsymbol{J}sin\varphi) = \sqrt{N_Z}e^{\boldsymbol{J}\varphi},$$

where $\varphi \in \mathbb{C}$, $cos\varphi = A/\sqrt{N_Z}$ and $sin\varphi = B/\sqrt{N_Z}$. If Z is unit, then its polar form will be equal to $Z = cos\varphi + \boldsymbol{J}sin\varphi = e^{\boldsymbol{J}\varphi}$.

For any two unit bicomplex numbers in the polar form $Z = cos\varphi_Z + \boldsymbol{J}sin\varphi_Z = e^{\boldsymbol{J}\varphi_Z}$ and $W = cos\varphi_W + \boldsymbol{J}sin\varphi_W = e^{\boldsymbol{J}\varphi_W}$, we can write

$$\frac{W}{Z} = \frac{e^{\boldsymbol{J}\varphi_Z}}{e^{\boldsymbol{J}\varphi_W}} = e^{\boldsymbol{J}\varphi_W - \varphi_Z}.$$

By writing $\varphi = \varphi_W - \varphi_Z$, we get $W = e^{\boldsymbol{J}\varphi}Z$. Hence, the bicomplex operator is $\varphi \to e^{\boldsymbol{J}\varphi} = cos\varphi + \boldsymbol{J}sin\varphi$. Geometrically, multiplication of bicomplex number Z by $e^{\boldsymbol{J}\varphi}$ means a rotation of Z by the complex angle φ around the origin of the hypercomplex plane.

2.4. **Basics of Biquaternions (Complexified Quaternions).** The algebra of quaternions can be generalized to biquaternions (complexified quaternions) as

$$\mathbb{H}_\mathbb{C} = \{Q = W + X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} \mid W, X, Y, Z \in \mathbb{C}\}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are exactly the same in the real quaternions [6, 7, 8].

Let $Q = W + X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} \in \mathbb{H}_\mathbb{C}$, where the complex numbers are $W = \Re(W) + \boldsymbol{I}\Im(W),\ X = \Re(X) + \boldsymbol{I}\Im(X),\ Y = \Re(Y) + \boldsymbol{I}\Im(Y),\ Z = \Re(Z)+\boldsymbol{I}\Im(Z)$. Here $\boldsymbol{I}$ denotes the complex number operator (therefore $\boldsymbol{I}^2 = -1$) distinct from $\boldsymbol{i}, \Re()$ denotes the real part and $\Im()$ denotes the imaginary part of the complex number. Since reals commute with the quaternion operator, so do all complex numbers. Therefore $\boldsymbol{I}$ commutes with the quaternion operators, that is $\boldsymbol{iI} = \boldsymbol{Ii}, \boldsymbol{jI} = \boldsymbol{Ij}$ and $\boldsymbol{kI} = \boldsymbol{Ik}$. So, we can write $Q = \Re(Q)+\boldsymbol{I}\Im(Q)$, where $\Re(Q) = \Re(W) + \Re(X)\boldsymbol{i} + \Re(Y)\boldsymbol{j} + \Re(Z)\boldsymbol{k}$ and $\Im(Q) = \Im(W) + \Im(X)\boldsymbol{i} + \Im(Y)\boldsymbol{j} + \Im(Z)\boldsymbol{k}$ are real quaternions. We define the scalar part and vector part of Q as $S(Q) = W$ and $V(Q) = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$, respectively. The quaternion conjugate of Q is defined as $\overline{Q} = S(Q) - V(Q) = \overline{\Re(Q)} + \boldsymbol{I}\overline{\Im(Q)}$ [9]. If $S(Q) = 0$, then Q is called pure biquaternion. The set of pure biquaternions

$$\mathrm{Im}\mathbb{H}_\mathbb{C} = \{Q = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} \mid X, Y, Z \in \mathbb{C}\}$$

is a linear subspace of $\mathbb{H}_\mathbb{C}$ spanned by $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.

For biquaternions $Q = W_1 + X_1\boldsymbol{i} + Y_1\boldsymbol{j} + Z_1\boldsymbol{k}$ and $P = W_2 + X_2\boldsymbol{i} + Y_2\boldsymbol{j} + Z_2\boldsymbol{k}$, we define the sum of Q and P, multiplication of Q with a scalar $\lambda \in \mathbb{C}$ and the product of Q and P as follows:

$$Q + P = (W_1 + W_2) + (X_1 + X_2)\boldsymbol{i} + (Y_1 + Y_2)\boldsymbol{j} + (Z_1 + Z_2)\boldsymbol{k},$$

$$\lambda Q = \lambda W_1 + \lambda X_1\boldsymbol{i} + \lambda Y_1\boldsymbol{j} + \lambda Z_1\boldsymbol{k},$$

$$QP = S(Q)S(P) - \langle \mathbf{V}(Q), \mathbf{V}(P)\rangle + S(Q)\mathbf{V}(P) + S(P)\mathbf{V}(Q) + \mathbf{V}(Q)\wedge\mathbf{V}(P),$$

where $\langle \mathbf{V}(Q), \mathbf{V}(P)\rangle = X_1X_2 + Y_1Y_2 + Z_1Z_2$ and $\mathbf{V}(Q)\wedge\mathbf{V}(P) = (Y_1Z_2 - Z_1Y_2)\boldsymbol{i} - (X_1Z_2 - Z_1X_2)\boldsymbol{j} + (X_1Y_2 - Y_1X_2)\boldsymbol{k}$. It should be considered that the equations $\overline{PQ} = \overline{Q}\,\overline{P}$ and $\overline{P + Q} = \overline{P} + \overline{Q} = \overline{Q} + \overline{P}$ are valid.

The norm and the modulus of a biquaternion $Q = W + X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ are defined, respectively, as

$$||Q|| = Q\overline{Q} = \overline{Q}Q = W^2 + X^2 + Y^2 + Z^2$$

and

$$|Q| = \sqrt{N_Q} = \sqrt{||Q||}.$$

It is important to emphasize that $||Q|| \in \mathbb{C}$. If $||Q|| = 1$ then Q is called unit biquaternion. The multiplicative inverse of Q is

$$Q^{-1} = \overline{Q}/||Q||.$$

Thus, the inverse of a nonzero biquaternion $Q = \Re(Q) + \boldsymbol{I}\Im(Q)$ is not defined when

$$||\Re(Q)|| = ||\Im(Q)|| \text{ and } \Re(Q)\overline{\Im(Q)} = -\Im(Q)\overline{\Re(Q)}.$$

This is an important difference between real quaternions and biquaternions, because every nonzero real quaternion has an inverse. Also, the algebra of biquaternions has zero divisors. So, $\mathbb{H}_\mathbb{C}$ is not a division algebra.

A nonzero biquaternion $Q = W + X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ can be written in polar form as

$$Q = \sqrt{N_Q}(cos\phi + \widehat{\boldsymbol{Q}}sin\phi) = \sqrt{N_Q}\,e^{\widehat{\boldsymbol{Q}}\phi}$$

where $\phi \in \mathbb{C}$, $\widehat{\boldsymbol{Q}} = (X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k})/\sqrt{X^2 + Y^2 + Z^2}$, $cos\phi = W/\sqrt{N_Q}$ and $sin\phi = \sqrt{X^2 + Y^2 + Z^2}/\sqrt{N_Q}$ [10]. Here $\widehat{\boldsymbol{Q}}$ is the unit pure biquaternion and the direction (or axis) of the vector part of the biquaternion Q in complex 3-space. The quaternion conjugate of Q in polar form is defined as $\overline{Q} = \sqrt{N_Q}(cos\phi - \widehat{\boldsymbol{Q}}sin\phi)$. In addition, we can write Q in complex form as

$$Q = A + \delta B,$$

where $A = W \in \mathbb{C}$, $\delta = (X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k})/\sqrt{X^2+Y^2+Z^2} \in \mathbb{C}^3$ and $B = \sqrt{X^2+Y^2+Z^2} \in \mathbb{C}$. The quaternion conjugate of Q in complex form is defined as

$$Q = A - \delta B.$$

Lastly, we can write Q as

$$Q = (W + X\boldsymbol{i}) + (Y + Z\boldsymbol{i})\boldsymbol{j} = C + D\boldsymbol{j},$$

where $C = W + X\boldsymbol{i}$, $D = Y + Z\boldsymbol{i} \in \mathbb{C}_2$. Thus, $\mathbb{H}_\mathbb{C}$ is isomorphic to $\mathbb{C}_2^2$; $\mathbb{H}_\mathbb{C} \cong \mathbb{C}_2^2$. The quaternion conjugate of Q is defined as

$$\overline{Q} = (W - X\boldsymbol{i}) - (Y + Z\boldsymbol{i})\boldsymbol{j} = \overline{C} - D\boldsymbol{j}.$$

The unit complex 3-dimensional sphere, i.e. the set of unit biquaternions, $S^3_\mathbb{C} = \{Q \in \mathbb{H}_\mathbb{C} \colon |Q| = 1\} \subset \mathbb{H}_\mathbb{C}$ constitutes a group under quaternion multiplication.

3. Matrix Representation of Unit Complex 3-Sphere $S^3_\mathbb{C}$

In this section, we will give a matrix representation of unit biquaternions. In terms of bicomplex variables, quaternionic multiplication of biquaternions $Q = Z_1 + W_1\boldsymbol{j}$ and $P = Z_2 + W_2\boldsymbol{j}$ can be written as

$$QP = (Z_1 + W_1\boldsymbol{j})(Z_2 + W_2\boldsymbol{j}) = Z_1Z_2 + Z_1W_2\boldsymbol{j} + W_1\boldsymbol{j}Z_2 + W_1\boldsymbol{j}W_2\boldsymbol{j}.$$

By using the equalities $\boldsymbol{j}Z_2 = \overline{Z_2}\boldsymbol{j}$, $\boldsymbol{j}W_2 = \overline{W_2}\boldsymbol{j}$ and $\boldsymbol{j}^2 = -1$, we get

$$QP = (Z_1Z_2 - W_1\overline{W_2}) + (Z_1W_2 + W_1\overline{Z_2})\boldsymbol{j}$$

so that, under the correspondence $\mathbb{H}_\mathbb{C} \cong \mathbb{C}_2^2$, multiplying Q by P from the right corresponds to the following matrix multiplication:

$$QP = \begin{bmatrix} Z_2 & -\overline{W_2} \\ W_2 & \overline{Z_2} \end{bmatrix} \begin{bmatrix} Z_1 \\ W_1 \end{bmatrix}.$$

By suppressing the incides, the right multiplication by $Q = Z + W\boldsymbol{j}$, corresponds to left multiplication by matrix $\mathbf{A}$ given as

$$\mathbf{A} = \begin{bmatrix} Z & -\overline{W} \\ W & \overline{Z} \end{bmatrix}.$$

Restricting biquaternion Q to unit complex 3-sphere $S^3_\mathbb{C}$, i.e. choosing Q a unit biquaternion, is equivalent to $||Q|| = ||W|| + ||Z|| = 1$ or $|Q|^2 = |W|^2 + |Z|^2$. In this instance, matrix $\mathbf{A}$ a special bicomplex unitary matrix that is

$$\mathbf{A}^* = \overline{\mathbf{A}}^T = \mathbf{A}^{-1} \text{ with } det(\mathbf{A}) = 1,$$

where $\mathbf{A}^*, \overline{\mathbf{A}}^T$ and $\mathbf{A}^{-1}$ are the adjoint matrix, the conjugate transpose matrix and the inverse matrix of the matrix $\mathbf{A}$, respectively. These matrices constitute the group $\boldsymbol{SU}_{\mathbb{C}_2}(2)$ of special bicomplex unitary 2×2 matrices.

The correspondence $\psi\colon S^3_{\mathbb{C}} \to \boldsymbol{SU}_{\mathbb{C}_2}(2)$ that associates to right multiplication by the unit biquaternion $Q \in S^3_{\mathbb{C}}$ the special bicomplex unitary matrix $\mathbf{A}$ is not quite an isomorphism, it is obviously one-to-one and onto, but satisfies the following equation for $P \in S^3_{\mathbb{C}}$:

$$\psi(QP) = \psi(P)\psi(Q).$$

We see that this follows by setting unit biquaternions $Q = Z_1 + W_1\boldsymbol{j}$ and $P = Z_2 + W_2\boldsymbol{j}$ and comparing the first column of the product

$$\psi(P)\psi(Q) = \begin{bmatrix} Z_2 & -\overline{W_2} \\ W_2 & \overline{Z_2} \end{bmatrix} \begin{bmatrix} Z_1 & -\overline{W_1} \\ W_1 & \overline{Z_1} \end{bmatrix}$$

with the bicomplex expression of the product QP above.

4. The Lie Algebra $\mathfrak{G}^3_{\mathbb{C}}$ of Unit Complex 3-Sphere $S^3_{\mathbb{C}}$

The group $S^3_{\mathbb{C}}$ is a Lie group of dimension 3. In fact, $S^3_{\mathbb{C}}$ is a complex 3-dimensional analytic manifold since the mapping

$$f\colon S^3_{\mathbb{C}} \mapsto \mathbb{C}$$

defined by

$$f(Q) = W^2 + X^2 + Y^2 + Z^2 \quad \text{for all} \quad Q = W + X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} \in S^3_{\mathbb{C}}$$

is differentiable and its regular value is 1. Furthermore, the mapping

$$\mu\colon S^3_{\mathbb{C}} \times S^3_{\mathbb{C}} \to S^3_{\mathbb{C}} \ \text{ defined by } \ \mu(Q,P) = QP \text{ for all } Q,P \in S^3_{\mathbb{C}}$$

is analytic.

Let us find the Lie algebra $\mathfrak{G}^3_{\mathbb{C}}$ of the Lie group $S^3_{\mathbb{C}}$, i.e. the tangent space of the unit element. Thus, let $\alpha(t) = A_0(t) + A_1(t)\boldsymbol{i} + A_2(t)\boldsymbol{j} + A_3(t)\boldsymbol{k}$ be a curve on $S^3_{\mathbb{C}}$ and let $\alpha(0) = 1$, i.e. $A_0(0) = 1, A_m(0) = 0$ for $m = 1, 2, 3$. By differentiation the equation

$$\sum_{m=0}^{3} [A_m(t)]^2 = 1$$

yields

$$\sum_{m=0}^{3} [A_m(t)A'_m(t)] = 0.$$

Substituting $t = 0$ we obtain $A_0(0) = 0$.

The Lie algebra $\mathfrak{G}^3_{\mathbb{C}}$ is thus constituted by vectors of the form

$$\xi = \xi^m(\partial/\partial_{A_m})\Big|_{\alpha=1}, \ m = 1, 2, 3.$$

The vector ξ is formally written in the form $\xi = \xi^1\boldsymbol{i} + \xi^2\boldsymbol{j} + \xi^3\boldsymbol{k}$. Thus, the Lie algebra $\mathfrak{G}^3_{\mathbb{C}}$ is equal to $\mathrm{Im}\mathbb{H}_{\mathbb{C}}$ and the tangent space $T_e(\boldsymbol{S}^3_{\mathbb{C}})$ is equal to situation transportation of the frame of complex 3-space $\mathbb{C}^3$ to the unit point e of the Lie group $\boldsymbol{S}^3_{\mathbb{C}}$, where $\mathbb{C}^3 = \{Q = (X, Y, Z)\colon \langle Q, Q\rangle = X^2 + Y^2 + Z^2; X, Y, Z \in \mathbb{C}\}$, $e = (1, 0, 0, 0) \in \boldsymbol{S}^3_{\mathbb{C}} \subset \mathbb{C}^4$ and $\mathbb{C}^4 = \{Q = (W, X, Y, Z)\colon \langle Q, Q\rangle = W^2 + X^2 + Y^2 + Z^2; W, X, Y, Z \in \mathbb{C}\}$. Hence, the Lie algebra of left-invariant vector fields $X^L(\boldsymbol{S}^3_{\mathbb{C}})$ correspondences to $\mathbb{C}^3$. Let us find the leftinvariant vector field X on $\boldsymbol{S}^3_{\mathbb{C}}$ for which $X_{\alpha=1} = \xi$. Let $\beta(t)$ be a curve on $\boldsymbol{S}^3_{\mathbb{C}}$ such that $\beta(0) = 1$ and $\beta'(t) = \xi$. Then $L_\alpha(\beta(t)) = \alpha\beta(t)$ is the left translation of the curve $\beta(t)$ by the unit biquaternion α and its tangent vector is $\alpha\beta'(0) = \alpha\xi$. In particular, denote by X_m those left invariant vector fields on $\boldsymbol{S}^3_{\mathbb{C}}$ for which

$$X_m\Big|_{\alpha=1} = \frac{\partial}{\partial_{A_m}}\Bigg|_{\alpha=1}, \ m = 1, 2, 3.$$

These three vector fields are represented at the point $\alpha = 1$, in biquaternion notation, by the quaternions $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. For the components of these vector fields at the point $\alpha = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$ we have $(X_1)_\alpha = \alpha\boldsymbol{i}, (X_2)_\alpha = \alpha\boldsymbol{j}, (X_3)_\alpha = \alpha\boldsymbol{k}$.

The computations yields

$$X_1 = -A_1\frac{\partial}{\partial_{A_0}} + A_0\frac{\partial}{\partial_{A_1}} + A_3\frac{\partial}{\partial_{A_2}} - A_2\frac{\partial}{\partial_{A_3}},$$

$$X_2 = -A_2\frac{\partial}{\partial_{A_0}} - A_3\frac{\partial}{\partial_{A_1}} + A_0\frac{\partial}{\partial_{A_2}} + A_1\frac{\partial}{\partial_{A_3}},$$

$$X_3 = -A_3\frac{\partial}{\partial_{A_0}} + A_2\frac{\partial}{\partial_{A_1}} - A_1\frac{\partial}{\partial_{A_2}} + A_0\frac{\partial}{\partial_{A_3}},$$

where all the partial derivatives are at the point α. Further, we obtain

$$[X_1, X_2] = 2X_3, \ [X_2, X_3] = 2X_1, \ [X_3, X_1] = 2X_2.$$

If we limit ourselves to the values at the point $\alpha = 1$, we obtain in quaternion notation

$$[\boldsymbol{i}, \boldsymbol{j}] = 2\boldsymbol{k}, \ [\boldsymbol{j}, \boldsymbol{k}] = 2\boldsymbol{i}, \ [\boldsymbol{k}, \boldsymbol{i}] = 2\boldsymbol{j}.$$

If we desire to replace $[,]$ by biquaternion multiplication, we have to omit the complex part since, e.g. $[\boldsymbol{i}, \boldsymbol{i}] = 0$ while $\boldsymbol{i}^2 = -1$. For $\xi, \eta \in \mathfrak{G}^3_{\mathbb{C}}$ we thus have $[\xi, \eta] = 2\xi\eta\big|_{\mathrm{mod}\mathbb{C}}$, i.e. the biquaternion without its complex part.

To every element $Q \in \boldsymbol{S}^3_{\mathbb{C}}$ the mapping $IntQ\colon \boldsymbol{S}^3_{\mathbb{C}} \to \boldsymbol{S}^3_{\mathbb{C}}$ defined by $IntQ(X) = QXQ^{-1}$ for all $X \in \boldsymbol{S}^3_{\mathbb{C}}$ is assigned. The mapping $IntQ$ is a differentiable isomorphism of the group $\boldsymbol{S}^3_{\mathbb{C}}$ and we have $\mathrm{Int}Q(e) = QeQ^{-1} = e$. This means that the differential of the mapping $IntQ$ at the point e, i.e. $(IntQ)'_e$, maps $T_e(\boldsymbol{S}^3_{\mathbb{C}})$ into $T_e(\boldsymbol{S}^3_{\mathbb{C}})$.

Definition 4.1. Denote $(IntQ)'_e = adQ$. Then the mapping

$$adQ \colon \mathfrak{G}^3_{\mathbb{C}} \to \mathfrak{G}^3_{\mathbb{C}}$$

is called the adjoint representation of the group $\boldsymbol{S}^3_{\mathbb{C}}$.

Theorem 4.2. *Let* $Q \in \boldsymbol{S}^3_{\mathbb{C}}$ *and* $\xi, \eta \in \mathfrak{G}^3_{\mathbb{C}}$. *Then* $adQ[\xi,\eta] = [adQ\xi, adQ\eta]$.

Theorem 4.3. *The adjoint representation of any unit biquaternion* $Q = W + X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ *corresponds to the following complex orthogonal matrix*

$$adQ = \begin{bmatrix} W^2+X^2-Y^2-Z^2 & 2(XY-WZ) & 2(WY+XZ) \\ 2(WZ+XY) & W^2-X^2+Y^2-Z^2 & 2(YZ-WX) \\ 2(XZ-WY) & 2(WX+YZ) & W^2-X^2-Y^2+Z^2 \end{bmatrix}.$$

Proof. We know that the adjoint representation of Q is

$$adQ = (IntQ)'_e \colon \mathfrak{G}^3_{\mathbb{C}} \to \mathfrak{G}^3_{\mathbb{C}},$$

which is defined by $adQ(\xi) = (IntQ)'_e(\xi) = d(IntQ)_e(\xi)$ for all $\xi \in \mathfrak{G}^3_{\mathbb{C}}$. Let $\alpha(t)$ be a curve on $\boldsymbol{S}^3_{\mathbb{C}}$ such that $\alpha_0 = 1$ and $\alpha'(0) = \xi$. Then,

$$\begin{aligned} adQ(\xi) = (IntQ)'_e(\xi) &= d(IntQ)_e\left((d\alpha)\left(\frac{d}{dt}\Big|_0\right)\right) \\ &= d(IntQ \circ \alpha)\big|_0\left(\frac{d}{dt}\Big|_0\right) \\ &= d(Q\alpha Q^{-1})\big|_0\left(\frac{d}{dt}\Big|_0\right) \\ &= Q(d\alpha)\big|_0\left(\frac{d}{dt}\Big|_0\right)Q^{-1} \\ &= Q\alpha'(0)Q^{-1} \\ &= Q\xi Q^{-1} \end{aligned}$$

Consequently, complex orthogonal matrix adQ corresponding to linear isomorphism has been obtained according to the bases $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. The adjoint reprsentation adQ satisfies the following equations:

$$adQ(\boldsymbol{i}) = Q\boldsymbol{i}Q^{-1},\ adQ(\boldsymbol{j}) = Q\boldsymbol{j}Q^{-1},\ adQ(\boldsymbol{k}) = Q\boldsymbol{k}Q^{-1}.$$

The complex orthogonality of the matrix adQ can be easily shown. Therefore, adjoint representation of the unit complex 3-sphere $ad\boldsymbol{S}^3_{\mathbb{C}}$ is an isometry. □

Theorem 4.4. *The Killing bilinear form of the unit complex 3-sphere* $\boldsymbol{S}^3_{\mathbb{C}}$ *is invariant.*

Proof. We know that the Killing bilinear form on $\mathfrak{G}^3_{\mathbb{C}}$ is

$$K \colon \mathfrak{G}^3_{\mathbb{C}} \times \mathfrak{G}^3_{\mathbb{C}} \to \mathbb{R}$$

which is defined by $K(X,Y) = Tr(AdX, AdY)$ for all $X, Y \in \mathfrak{G}^3_{\mathbb{C}}$. For every element $X = X^1\boldsymbol{i} + X^2\boldsymbol{j} + X^3\boldsymbol{k}$ and $Y = Y^1\boldsymbol{i} + Y^2\boldsymbol{j} + Y^3\boldsymbol{k}$ from $\mathfrak{G}^3_{\mathbb{C}}$, the following equations can be written:

$$[X,Y] = 2X \wedge Y \quad \text{and} \quad \langle X,Y\rangle = \sum_{m=1}^{3} X^m Y^m.$$

Further, for $Z \in \mathfrak{G}^3_{\mathbb{C}}$ it can be obtained

$$\begin{aligned}\frac{1}{4}(AdX \cdot AdY(Z)) = \frac{1}{4}(AdX([Y,Z])) &= \frac{1}{4}\big[X,[Y,Z]\big]\\ &= X \wedge (Y \wedge Z)\\ &= \langle X,Z\rangle Y - \langle X,Y\rangle Z.\end{aligned}$$

From this relation, the matrix of the mapping $(AdX \cdot AdY(Z))/4$ can be written according to the bases $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ as follows:

$$\begin{bmatrix} -(X^2Y^2 + X^3Y^3) & X^2Y^1 & X^3Y^1 \\ X^1Y^2 & -(X^1Y^1 + X^3Y^3) & X^3Y^2 \\ X^1Y^3 & X^2Y^3 & -(X^1Y^1 + X^2Y^2) \end{bmatrix},$$

so that

$$Tr\left(\frac{1}{4}(AdX \cdot AdY(Z))\right) = -2(X^1Y^1 + X^2Y^2 + X^3Y^3) = -2\langle X,Y\rangle.$$

Hence, $K(X,Y) = Tr(AdX, AdY) = -8\langle X,Y\rangle$. The Killing bilinear form is the complex inner product in $\mathfrak{G}^3_{\mathbb{C}}$. This means that group $ad\boldsymbol{S}^3_{\mathbb{C}}$ preserves the complex inner product that can be shown as

$$\begin{aligned} K(X,Y) &= K(AdQ(X), AdQ(Y))\\ -8\langle X,Y\rangle &= -8\langle AdQ(X), AdQ(Y)\rangle\\ \langle X,Y\rangle &= \langle AdQ(X), AdQ(Y)\rangle. \end{aligned}$$

Consequently, the group $ad\boldsymbol{S}^3_{\mathbb{C}}$ is a subgroup of the complex orthogonal group $O(\mathbb{C}^3)$, i.e. $ad\boldsymbol{S}^3_{\mathbb{C}} \subset O(\mathbb{C}^3)$. □

$\boldsymbol{S}^3_{\mathbb{C}}$ is not compact because the value $-8\langle X,Y\rangle$ of the Killing bilinear form of $\boldsymbol{S}^3_{\mathbb{C}}$ is obtained complex, i.e. $-8\langle X,Y\rangle \in \mathbb{C}$. In other words, since $K(X,Y) = -8\langle X,Y\rangle$ is not negative semi-definete, $\boldsymbol{S}^3_{\mathbb{C}}$ is not compact.

Theorem 4.5. *The mapping adQ is a complex rotation of the vector space $\mathfrak{G}^3_{\mathbb{C}}$ about a certain axis through a certain complex angle.*

Proof. For $Q \in \boldsymbol{S}^3_{\mathbb{C}}$ it can be written $Q = cos\phi + \varepsilon_1 sin\phi$, where $\phi \in \mathbb{C}$ and ε_1 is a unit pure biquaternion. Therefore,

$$adQ(\varepsilon_1) = Q\varepsilon_1\overline{Q} = (cos\phi + \varepsilon_1 sin\phi)\varepsilon_1(cos\phi - \varepsilon_1 sin\phi) = \varepsilon_1$$

This means that to the biquaternion Q corresponds a complex rotation about the axis determined by ε_1.

Further, let us complete ε_1 to a right-hand orthonormal base by the biquaternions ε_2 and ε_3. For purely imaginary biquaternions ε and δ we have the relation

$$\varepsilon\delta = -\langle\varepsilon, \delta\rangle + \varepsilon \wedge \delta$$

so that

$$\varepsilon_1\varepsilon_2 = \varepsilon_3\varepsilon_2\varepsilon_3 = \varepsilon_1\varepsilon_3\varepsilon_1 = \varepsilon_3.$$

Now, let us find the image of the biquaternions ε_2 and ε_3 under the mapping adQ:

$$\begin{aligned} adQ(\varepsilon_2) = Q\varepsilon_2\overline{Q} &= (cos\phi + \varepsilon_1 sin\phi)\varepsilon_2(cos\phi - \varepsilon_1 sin\phi) \\ &= \varepsilon_2 cos(2\phi) + \varepsilon_3 sin(2\phi) \\ adQ(\varepsilon_3) = Q\varepsilon_3\overline{Q} &= (cos\phi + \varepsilon_1 sin\phi)\varepsilon_3(cos\phi - \varepsilon_1 sin\phi) \\ &= -\varepsilon_2 sin(2\phi) + \varepsilon_3 cos(2\phi) \end{aligned}$$

Thus, mapping adQ belonging to the biquaternion $Q = cos\phi + \varepsilon_1 sin\phi$ is the complex rotation in the complex plane about the axis determined by the unit vector ε_1 through the angle 2ϕ, see Fig. 1. □

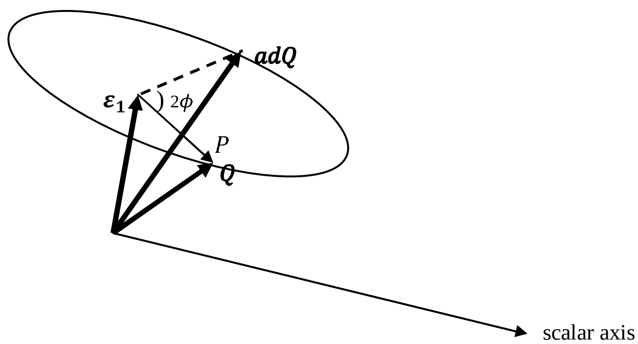

FIGURE 1. Rotation in complex plane

In Fig. 1, P is a unit pure biquaternion in the plane with normal ε_1, i.e. $\langle\varepsilon_1, P\rangle = 0$. In fact, P can be choosen as ε_2.

Consequently, the mapping adQ is an orientation preserving congruence since it maps a right-hand frame into a right-hand frame so that the determinant of this transformation is equal to 1. All the elements of $\boldsymbol{O}(\mathbb{C}^3)$ whose determinants are equal to 1 constitute a subgroup of the group $\boldsymbol{O}(\mathbb{C}^3)$. This subgroup is a Lie group of dimension 3 and has the same Lie algebra with the group $\boldsymbol{O}(\mathbb{C}^3)$. We call this subgroup the special orthogonal group of degree 3 and denote it by $\boldsymbol{SO}(\mathbb{C}^3)$. These facts imply that the groups $ad\boldsymbol{S}^3_{\mathbb{C}}$ and $\boldsymbol{SO}(\mathbb{C}^3)$ are isomorphic. Also, the mapping $ad : \boldsymbol{S}^3_{\mathbb{C}} \mapsto ad\boldsymbol{S}^3_{\mathbb{C}}$ is not one-to-one. Because, the same rotations corresponds to the biquaternions Q and $-Q$.

Theorem 4.6. *For an arbitrary biquaternion $Q = A + \delta B$, where $A, B \in \mathbb{C}$ and δ is unit pure biquaternion, the transformation $f_{\varepsilon_1}(Q) = ad_{\varepsilon_1}(Q) = \varepsilon_1 Q \varepsilon_1^{-1} = -\ \varepsilon_1 Q \varepsilon_1$, where ε_1 is any unit pure biquaternion, leaves the scalar part of Q (that is, A) invariant, and reflects the vector part of Q (that is, δB) in the line defined by the axis of ε_1 (i.e. a half turn rotation of δB in the complex plane) about the axis of ε_1* [9].

Theorem 4.7. *The mapping adQ that associates to each unit biquaternion $Q \in \boldsymbol{S}^3_{\mathbb{C}}$ the transformation $adQ(P) = QPQ^{-1}$ restricted to $\mathrm{Im}\mathbb{H}_{\mathbb{C}}$ is a surjective (or onto) group homomrphism*

$$adQ : \boldsymbol{S}^3_{\mathbb{C}} \mapsto \boldsymbol{SO}(\mathbb{C}^3)$$

with kernel

$$Ker(adQ) = \{\pm 1\}.$$

Proof. From Theorem 4.5, if $\pm 1 \in \boldsymbol{S}^3_{\mathbb{C}}$ then Q defines a rotation which is an element of $\boldsymbol{SO}(\mathbb{C}^3)$. It is clear that adQ is homomorphism of groups and ± 1 are in the kernel of adQ. Since all the elements in $\boldsymbol{SO}(\mathbb{C}^3)$ are rotations, the mapping adQ is onto. It remains to show that the kernel of adQ is exactly ± 1. Let $Q \in Ker(adQ)$ that is $QPQ^{-1} = P$ for all $P \in \mathrm{Im}\mathbb{H}_{\mathbb{C}}$. Equivalently, Q commutes with all pure biquaternions. Writing this condition out in terms of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$, we obtain that Q must be complex. Since it is in $\boldsymbol{S}^3_{\mathbb{C}}$, it must be -1 or $+1$. □

Theorem 4.6 implies that the group $\boldsymbol{S}^3_{\mathbb{C}}$ of unit biquaternions modulo the normal subgroup ± 1 is isomorphic with the group $\boldsymbol{SO}(\mathbb{C}^3)$ of direct spatial linear isometries. The quotient group $\boldsymbol{S}^3_{\mathbb{C}}/\{\pm 1\}$ is the group of right- (or left-) cosets of ± 1. A right-coset containing $Q \in \boldsymbol{S}^3_{\mathbb{C}}$ has the form $\{\pm 1\}Q = \{\pm 1 Q\}$. Thus, topologically $\boldsymbol{S}^3_{\mathbb{C}}/\{\pm 1\}$ can be considered as a model for the Complex-projective space $\mathbb{C}\mathbf{P}^3$. Also, by the Theorem 4.7, $\mathbb{C}\mathbf{P}^3$ can be identified by the group of direct spatial isometries $\boldsymbol{SO}(\mathbb{C}^3)$. Thus, we obtain $\boldsymbol{S}^3_{\mathbb{C}}/\{\pm 1\} = \mathbb{C}\mathbf{P}^3$. These relationships can be illustrated by the following diagram:

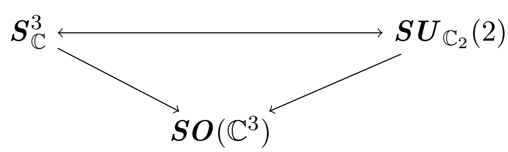

Diagram : The relationship among groups $\boldsymbol{S}^3_{\mathbb{C}}$, $\boldsymbol{SU}_{\mathbb{C}_2}(2)$ and $\boldsymbol{SO}(\mathbb{C}^3)$.

5. Conclusions

For illustrating the relationship among the groups unit complex 3-sphere $\boldsymbol{S}^3_{\mathbb{C}}$, special bicomplex unitary matrices $\boldsymbol{SU}_{\mathbb{C}_2}(2)$ and special complex orthogonal group $\boldsymbol{SO}(\mathbb{C}^3)$ it is firstly given a correspondence between the elements of $\boldsymbol{S}^3_{\mathbb{C}}$ and $\boldsymbol{SU}_{\mathbb{C}_2}(2)$ by expressing biquaternions $\mathbb{H}_{\mathbb{C}}$ as 2-dimensional bicomplex numbers $\mathbb{C}^2_2$. Secondly, $\boldsymbol{SO}(\mathbb{C}^3)$, which is a subgroup of the complex orthogonal group $\boldsymbol{O}(\mathbb{C}^3)$, is obtained by obtaining the Lie algebra of $\boldsymbol{S}^3_{\mathbb{C}}$. Finally, it is shown that the quaotient group of biquaternions $\boldsymbol{S}^3_{\mathbb{C}}/\{\pm 1\}$ can be considered topologically as a model for the Complex-projective space $\mathbb{C}\mathbf{P}^3$.

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