

On the determination of asymptotic formula of the nodal points for the Sturm-Liouville equation with one turning point

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ABSTRACT. In this paper, the asymptotic representation of the corresponding eigenfunctions of the eigenvalues has been investigated. Furthermore, we obtain the zeros of eigenfunctions.

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1. Introduction

In the literature review of mathematics, a large number of research studies has been devoted entirely or partially to the study of the Sturm-Liouville i.e.,

$$y''(x) + (\lambda\phi^2(x) - q(x))y = 0, \quad (1.1)$$

where $\lambda = \rho^2$ and the real valued functions ϕ^2 and q are said to be the coefficients of the problem, ϕ^2 is the weight and q is the potential function. The zeros of ϕ^2 are called turning points of (1). Differential equations with turning points play an important role in various areas of mathematics and other branches of natural sciences. For example in

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elasticity, optics, geophysics(see [6,8,11] and the references therein).

Inverse spectral problems consist in recovering operators from their spectral characteristics. The first spectral problem was given by Ambarzumyan [3]. Since 1946, various forms of the inverse problem have been considered by several authors [4,10]. In later years, these problems was studied for Sturm-Liouville operators with turning points (see [7,13]).

Recently, some researchers have paid attention to a new class of inverse problems. This is the so-called inverse nodal problem. Inverse nodal problems consist in recovering operators from given nodes (zeros) of their eigenfunctions.

In 1988, it seems that J.R. Mclaughlin [12] to be the first to consider this sort of inverse problem. She showed that the nodal set of the Dirichlet problem alone can determine the potential function of the Sturm-Liouville problem up to a constant. Yang [15] showed that this uniqueness result is valid for any q .

In resent years, some interesting results of inverse nodal problems of the Sturm-Liouville operators were obtained(for example, refer to[5,9,14]). In this work, we consider the following Sturm-Liouville equation

$$y''(x) + (\lambda x - q(x))y = 0, \quad -1 \leq x \leq 1, \quad (1.2)$$

where $q \in L[-1, 1]$ and λ is a real parameter.

In this paper, we obtain the eigenvalues and eigenfunctions corresponding to large modulus eigenvalues and we calculate an asymptotic of the nodal points.

2. Main result

Let $C(x, \lambda)$ is a solution for Eq.(2) with the initial conditions $C(-1, \lambda) = 0$, $C'(-1, \lambda) = 1$.

In [2], it was shown that

$$C(x, \lambda) = \begin{cases} \frac{1}{\sqrt{\lambda}}(-x)^{-\frac{1}{4}} \sinh(p(x)\sqrt{\lambda})(1 + O(\frac{1}{\sqrt{\lambda}})), & -1 \leq x < 0, \\ \frac{x^{-\frac{1}{4}}}{\sqrt{\lambda}} \{e^{\frac{2}{3}\sqrt{\lambda}} \cos(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda} - \frac{\pi}{4}) + e^{-\frac{2}{3}\sqrt{\lambda}\frac{1}{2}} \sin(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda} - \frac{\pi}{4})\} (1 + O(\frac{1}{\sqrt{\lambda}})), & x > 0, \end{cases}$$

where $p(x) = \int_{-1}^x \sqrt{-\nu} d\nu$.

We have the integral equations

$$C(x, \lambda) = \begin{cases} \frac{1}{\sqrt{\lambda}}(-x)^{-\frac{1}{4}} \sinh p(x)\sqrt{\lambda} \\ + \frac{1}{\sqrt{\lambda}} \int_{-1}^x (xt)^{-\frac{1}{4}} \sinh(p(x) - p(t))\sqrt{\lambda}q(t)C(t, \sqrt{\lambda})dt, & -1 \leq x < 0, \\ \frac{x^{-\frac{1}{4}}}{\sqrt{\lambda}} \{e^{\frac{2}{3}\sqrt{\lambda}} \cos(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda} - \frac{\pi}{4}) + e^{-\frac{2}{3}\sqrt{\lambda}\frac{1}{2}} \sin(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda} - \frac{\pi}{4})\} \\ + \frac{1}{\sqrt{\lambda}} \int_0^x (xt)^{-\frac{1}{4}} \sin(f(x) - f(t))\sqrt{\lambda}q(t)C(t, \sqrt{\lambda})dt, & x > 0, \end{cases} \quad (2.1)$$

where $f(x) = \int_0^x \sqrt{\nu} d\nu$.

We consider Eq.(2) with boundary conditions

$$y(-1, \lambda) = 0, \quad y'(-1, \lambda) = 1, \quad y(b, \lambda) = 0.$$

The problem corresponding to Eq.(2) on $[-1, b]$ where $b < 0$ is fixed, has an infinite number of negative eigenvalues $\{\lambda_n^{(1)}(b)\}$. The asymptotic distribution of each function $\lambda_n^{(1)}(b)$ is of the form

$$\sqrt{-\lambda_n^{(1)}(b)} = \frac{n\pi}{\int_{-1}^b \sqrt{-t} dt} + O\left(\frac{1}{n}\right), \quad b < 0. \tag{2.2}$$

For more details see [1].

For $b \in (0, 1]$, fixed, the problem for (2) on $[-1, b]$ has an infinite number of positive and negative eigenvalues which we denote by $\{\lambda_n^{(2)}(b)\}$, $\{\lambda_n^{(3)}(b)\}$, respectively.

The positive eigenvalues $\lambda_n^{(2)}(b)$ admit the asymptotic representation

$$\sqrt{\lambda_n^{(2)}(b)} = \frac{n\pi - \frac{\pi}{4}}{\int_0^b \sqrt{t} dt} + \frac{1}{2n\pi} T_1 + O\left(\frac{1}{n^2}\right), \tag{2.3}$$

where

$$T_1 = \frac{5}{72 \int_0^b \sqrt{t} dt} + \frac{1}{2} \int_0^b \frac{q(t)}{\sqrt{t}} dt.$$

Similarly, the negative eigenvalues, $\lambda_n^{(3)}(b)$, admit the asymptotic representation of the form

$$\sqrt{-\lambda_n^{(3)}(b)} = \frac{n\pi - \frac{\pi}{4}}{\int_{-1}^0 \sqrt{-t} dt} + \frac{1}{2n\pi} T_2 + O\left(\frac{1}{n^2}\right), \tag{2.4}$$

where

$$T_2 = \frac{5}{72 \int_{-1}^0 \sqrt{-t} dt} + \frac{1}{2} \int_{-1}^0 \frac{q(t)}{\sqrt{-t}} dt.$$

We now state a theorem which gives asymptotic approximation for the eigenfunctions of the Sturm-Liouville equation in one turning point case. Let $C(x, \lambda_n^{(i)})$ be the eigenfunction corresponding to the eigenvalue $\lambda_n^{(i)}$ where $i \in \{1, 2, 3\}$.

Theorem 2.1. *a) For $b \in [-1, 0)$ fixed, the corresponding eigenfunctions of the negative eigenvalues $\lambda_n^{(1)}(b)$, has asymptotic representation,*

$$C(x, \lambda_n^{(1)}(b)) = \frac{p(b)(-x)^{-\frac{1}{4}}}{n\pi} \sin \frac{n\pi p(x)}{p(b)}$$

$$- \frac{p^2(b)(-x)^{-\frac{1}{4}}}{n^2\pi^2} \cos \frac{n\pi p(x)}{p(b)} \int_{-1}^x (-t)^{-\frac{1}{2}} q(t) \sin^2 \frac{n\pi p(t)}{p(b)} dt + o\left(\frac{1}{n^2}\right). \quad (2.5)$$

b) For $b \in (0, 1]$ fixed, the corresponding eigenfunctions of the positive eigenvalues, $\lambda_n^{(2)}(b)$, admit asymptotic representation,

$$C(x, \lambda_n^{(2)}(b)) = \frac{x^{-\frac{1}{4}} e^{\frac{2}{3}\left(\frac{n\pi - \frac{\pi}{4}}{f(b)}\right)} \cos\left[f(x)\left(\frac{n\pi - \frac{\pi}{4}}{f(b)}\right) - \frac{\pi}{4}\right]}{\left(\frac{n\pi - \frac{\pi}{4}}{f(b)}\right)}$$

$$- \frac{x^{-\frac{1}{4}} e^{\frac{2}{3}\left(\frac{n\pi - \frac{\pi}{4}}{f(b)}\right)}}{\left(\frac{n\pi - \frac{\pi}{4}}{f(b)}\right)^2} \sin\left[f(x)\left(\frac{n\pi - \frac{\pi}{4}}{f(b)}\right) - \frac{\pi}{4}\right] \int_0^x t^{-\frac{1}{2}} q(t) \cos^2\left[f(t)\left(\frac{n\pi - \frac{\pi}{4}}{f(b)}\right) - \frac{\pi}{4}\right] dt + o\left(\frac{1}{n^2}\right) \quad (2.6)$$

c) For $b \in (0, 1]$ fixed, the corresponding eigenfunctions of the negative eigenvalues, $\lambda_n^{(3)}(b)$, admit asymptotic representation,

$$C(x, \lambda_n^{(3)}(b)) = \frac{2x^{-\frac{1}{4}} e^{(n\pi - \frac{\pi}{4})i} \cos\left[x^{\frac{3}{2}}(n\pi - \frac{\pi}{4})i - \frac{\pi}{4}\right]}{3(n\pi - \frac{\pi}{4})i}$$

$$- \frac{x^{-\frac{1}{4}} e^{(n\pi - \frac{\pi}{4})i}}{\left(\frac{n\pi - \frac{\pi}{4}}{3}\right)^2} \sin\left[x^{\frac{3}{2}}(n\pi - \frac{\pi}{4})i - \frac{\pi}{4}\right] \int_0^x t^{-\frac{1}{2}} q(t) \cos^2\left[t^{\frac{3}{2}}(n\pi - \frac{\pi}{4})i - \frac{\pi}{4}\right] dt + o\left(\frac{1}{n^2}\right) \quad (2.7)$$

Proof. a) In this case the eigenvalues are negative. Substituting the asymptotic form (4) in (3) and noting that $\sqrt{\lambda} = i\sqrt{-\lambda_n^{(1)}(b)}$ we can get

$$C(x, \lambda_n^{(1)}(b)) = \frac{1}{i\sqrt{-\lambda_n^{(1)}(b)}} (-x)^{-\frac{1}{4}} \sinh(ip(x)\sqrt{-\lambda_n^{(1)}(b)})$$

$$+ \frac{1}{i\sqrt{-\lambda_n^{(1)}(b)}} \int_{-1}^x (xt)^{-\frac{1}{4}} \sinh(i(p(x) - p(t))\sqrt{-\lambda_n^{(1)}(b)}) q(t) C(t, \lambda_n^{(1)}(b)) dt$$

$$= \frac{(-x)^{-\frac{1}{4}}}{\left(\frac{n\pi}{p(b)} + O\left(\frac{1}{n}\right)\right)} \left[\sin \frac{p(x)n\pi}{p(b)} \cos O\left(\frac{1}{n}\right) + \cos \frac{p(x)n\pi}{p(b)} \sin O\left(\frac{1}{n}\right) \right]$$

$$- \frac{(-x)^{-\frac{1}{4}}}{\left(\frac{n\pi}{p(b)} + O\left(\frac{1}{n}\right)\right)^2} \left(\int_{-1}^x (-t)^{-\frac{1}{2}} q(t) \left[\sin \frac{p(t)n\pi}{p(b)} \cos O\left(\frac{1}{n}\right) + \cos \frac{p(t)n\pi}{p(b)} \sin O\left(\frac{1}{n}\right) \right]^2 dt \right)$$

$$\left[\cos \frac{p(x)n\pi}{p(b)} \cos O\left(\frac{1}{n}\right) - \sin \frac{p(x)n\pi}{p(b)} \sin O\left(\frac{1}{n}\right) \right] + o\left(\frac{1}{n^2}\right)$$

Using the following facts for large n :

$$\cos O\left(\frac{1}{n}\right) = 1 + O\left(\frac{1}{n^2}\right), \quad \sin O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right)$$

we get the result.

By inserting the asymptotic formulae (5) and (6) into (3) we get the results (b) and (c). \diamond

Suppose $\{x_j^{(i)n}\}$ is the j th nodal point of the eigenfunction $C(x, \lambda_n^{(i)})$ in $(-1, 1)$. In other words, $C(x_j^{(i)n}, \lambda_n^{(i)}) = 0$. Denote $X^{(i)} = \{x_j^{(i)n}\}_{n \geq 1, j = \overline{1, n}}$. $X^{(i)}$ is called the set of nodal points.

Theorem 2.2. *We take $x_j^{(i)n}, n \geq 1, j = \overline{1, n}$ as the nodal points of problem. Then*

$$\int_{-1}^{x_j^{(1)n}} \sqrt{-\nu} d\nu = \frac{jp(b)}{n} + \frac{p^2(b)}{n^2\pi^2} \int_{-1}^{x_j^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}} \sin^2\left(\frac{p(t)n\pi}{p(b)}\right) dt + o\left(\frac{1}{n^2}\right), \quad x < 0 \tag{2.8}$$

$$\begin{aligned} \int_0^{x_j^{(2)n}} \sqrt{\nu} d\nu &= \frac{(j - \frac{1}{4})f(b)}{n - \frac{1}{4}} \\ &+ \frac{f^2(b)}{(n\pi - \frac{\pi}{4})^2} \int_0^{x_j^{(2)n}} \frac{q(t)}{t^{\frac{1}{2}}} \cos^2\left(f(t)\left(\frac{n\pi - \frac{\pi}{4}}{f(b)}\right) - \frac{\pi}{4}\right) dt + o\left(\frac{1}{n^2}\right), \tag{2.9} > 0, \end{aligned}$$

as $n \rightarrow \infty$ uniformly in j .

Proof. Since $\{x_j^{(i)n}\}$ are zeros of eigenfunctions, in the case $x < 0$,

$$\begin{aligned} C(x_j^{(1)n}, \lambda_n^{(1)}(b)) &= \frac{p(b)(-x_j^{(1)n})^{-\frac{1}{4}}}{n\pi} \sin \frac{n\pi p(x_j^{(1)n})}{p(b)} \\ - \frac{p^2(b)(-x_j^{(1)n})^{-\frac{1}{4}}}{n^2\pi^2} \cos \frac{n\pi p(x_j^{(1)n})}{p(b)} \int_{-1}^{x_j^{(1)n}} (-t)^{-\frac{1}{2}} q(t) \sin^2 \frac{n\pi p(t)}{p(b)} dt + o\left(\frac{1}{n^2}\right) &= 0. \end{aligned}$$

Thus,

$$\tan \frac{n\pi p(x_j^{(1)n})}{p(b)} = \frac{p(b)}{n\pi} \int_{-1}^{x_j^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}} \sin^2 \frac{n\pi p(t)}{p(b)} dt + o\left(\frac{1}{n}\right).$$

Using Taylor's expansions for the arctangent, we obtain that

$$p(x_j^{(1)n}) = \frac{jp(b)}{n} + \frac{p^2(b)}{n^2\pi^2} \int_{-1}^{x_j^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}} \sin^2 \frac{n\pi p(t)}{p(b)} dt + o\left(\frac{1}{n^2}\right),$$

and hence formula (10) holds.

Using (8), by the same arguments as above, one can show that (11) holds. \diamond

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