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Sufficient Conditions for Density in Extended Lipschitz Algebras

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Abstract. Let (X, d) be a compact metric space and let K be a nonempty compact subset of X. Let $\alpha \in (0, 1]$ and let $\mathrm{Lip}(X, K, d^\alpha)$ denote the Banach algebra of all continuous complex-valued functions f on X for which $p_{\alpha,K}(f) = \sup\{\frac{|f(x)-f(y)|}{d^\alpha(x,y)} : x, y \in K, x \neq y\} < \infty$ when equipped the algebra norm $||f||_{\mathrm{Lip}(X,K,d^\alpha)} = ||f||_X + p_{\alpha,K}(f)$, where $||f||_X = \sup\{|f(x)| : x \in X\}$. We denote by $\mathrm{lip}(X, K, d^\alpha)$ the closed subalgebra of $\mathrm{Lip}(X, K, d^\alpha)$ consisting of all $f \in \mathrm{Lip}(X, K, d^\alpha)$ for which $\frac{|f(x)-f(y)|}{d^\alpha(x,y)} \to 0$ as $d(x, y) \to 0$ with $x, y \in K$. In this paper we obtain a sufficient condition for density of a linear subspace or a subalgebra of $\mathrm{Lip}(X, K, d^\alpha)$ in $(\mathrm{Lip}(X, K, d^\alpha), || \cdot ||_{\mathrm{Lip}(X,K,d^\alpha)})$ ($\mathrm{lip}(X, K, d^\alpha)$ in $(\mathrm{lip}(X, K, d^\alpha), || \cdot ||_{\mathrm{Lip}(X,K,d^\alpha)})$, respectively). In particular, we show that the Lipschitz algebra $\mathrm{Lip}(X, d^\alpha)$ is dense in $(\mathrm{Lip}(X, K, d^\alpha), \| \cdot \|_{\mathrm{Lip}(X,K,d^\alpha)})$ for $\alpha \in (0, 1]$ and $\mathrm{Lip}(X, d)$ and the little Lipschitz algebra $\mathrm{lip}(X, d^\alpha)$ are dense in $(\mathrm{lip}(X, K, d^\alpha), \| \cdot \|_{\mathrm{Lip}(X,K,d^\alpha)})$ for $\alpha \in (0, 1)$.

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1. Introduction and priliminaries

Let Ω be a locally compact Hausdorff space. The algebra of all continuous (bounded continuous) complex-valued functions on Ω is denoted by $C(\Omega)$ ($C^b(\Omega)$). It is known that $C^b(\Omega)$ under the uniform norm on Ω, that is,

$$\|h\|_\Omega = \sup\{|h(w)| : w \in \Omega\} \ \ (h \in C^b(\Omega)),$$

is a commutative unital Banach algebra. The set of all f in $C(\Omega)$ which vanish at infinity, is denoted by $C_0(\Omega)$, which is a closed subalgebra of $(C^b(\Omega), \|\cdot\|_\Omega)$. Clearly, $C_0(\Omega) = C^b(\Omega) = C(\Omega)$, whenever Ω is compact.

Let X be a compact Hausdorff space. A Banach function algebra on X is a subalgebra B of $C(X)$ such that contains the constant function 1 on X, separates the points of X and it is a unital Banach algebra with an algebra norm $||\cdot||$.

Let X be a compact Hausdorff space and let K be a nonempty compact subset of X. We denote by $CZ(X, K)$ the set of all $f \in C(X)$ for which $f|_K = 0$. Then $CZ(X, K)$ is a closed subalgebra of $(C(X), \|\cdot\|_X)$. It is known [6, Theorem 3.2] that, there exists an isometrical isomorphism from $(CZ(X, K), \|\cdot\|_X)$ onto $(C_0(X\backslash K), \|\cdot\|_{X\backslash K})$.

Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we denote

$$S_{(X,d)}(x, r) = \{y \in X : d(y, x) = r\},$$
$$B_{(X,d)}(x, r) = \{y \in X : d(y, x) < r\},$$
$$B_{(X,d)}[x, r] = \{y \in X : d(y, x) \leq r\}.$$

Let $\alpha \in (0, 1]$. Then the map $d^\alpha : X \times X \to \mathbb{R}$ defined by $d^\alpha(x, y) = (d(x, y))^\alpha$ is a metric on X. Moreover, for each $x \in X$ and every $\epsilon > 0$ we have

$$B_{(X,d^\alpha)}(x, \epsilon^\alpha) \subseteq B_{(X,d)}(x, \epsilon),$$
$$B_{(X,d)}(x, \epsilon^{\frac{1}{\alpha}}) \subseteq B_{(X,d^\alpha)}(x, \epsilon).$$

Therefore, the induced topology by d^α on X coincides to the induced topology by d on X.

Let (X, d) be a metric space and K be a nonempty subset of X. Let $\alpha \in (0, 1]$ and let f be a complex-valued function on X. We define

$$p_{\alpha,K}(f) = \sup\{\frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in K, x \neq y\}.$$

Let (X, d) be a compact metric space and let $\alpha \in (0, 1]$. The complex algebra of all complex-valued functions f on X for which $p_{\alpha,X}(f) < \infty$, is called the Lipschitz algebra of order α on (X, d) and denoted by $\mathrm{Lip}(X, d^\alpha)$. We write $\mathrm{Lip}(X, d)$ instead of $\mathrm{Lip}(X, d^1)$. Clearly

$$\mathrm{Lip}(X, d) \subseteq \mathrm{Lip}(X, d^\alpha) \subseteq C(X),$$

$1 \in \mathrm{Lip}(X, d)$ and $\mathrm{Lip}(X, d)$ separates the point of X. The d^α-Lipschitz norm $\|\cdot\|_{\mathrm{Lip}(X,d^\alpha)}$ on $\mathrm{Lip}(X, d^\alpha)$ is defined by

$$||f||_{\mathrm{Lip}(X,d^\alpha)} = ||f||_X + p_{\alpha,X}(f) \quad (f \in \mathrm{Lip}(X, d^\alpha)).$$

Then $(\mathrm{Lip}(X, d^\alpha), ||\cdot||_{\mathrm{Lip}(X,d^\alpha)})$ is a Banach function algebra on (X, d). Moreover, $\mathrm{Lip}(X, d)$ is dense in $(C(X), \|\cdot\|_X)$ by Stone-Weierstrass theorem. The complex algebra of all complex-valued functions f on X for which

$$\frac{|f(x) - f(y)|}{d^\alpha(x, y)} \to 0 \;\; as \;\; d(x, y) \to 0,$$

is called the little Lipschitz algebra of order α on (X, d) and denoted by $\mathrm{lip}(X, d^\alpha)$. We write $\mathrm{lip}(X, d)$ instead of $\mathrm{lip}(X, d^1)$. The complex algebra $\mathrm{lip}(X, d^\alpha)$ is a closed subalgebra of $\mathrm{Lip}(X, d^\alpha)$ and contains 1. Moreover, $\mathrm{Lip}(X, d^\beta)$ is a subalgebra of $\mathrm{lip}(X, d^\alpha)$ whenever $0 < \alpha < \beta \leq 1$. Thus $(\mathrm{lip}(X, d^\alpha), ||\cdot||_{\mathrm{Lip}(X,d^\alpha)})$ is a Banach function algebra on (X, d) whenever $\alpha \in (0, 1)$. The Lipschitz algebras $\mathrm{Lip}(X, d^\alpha)$ and the little Lipschitz algebras $\mathrm{lip}(X, d^\alpha)$ were first studied by Sherbert in [8] and [9].

We define

$$\begin{aligned} \mathrm{Lip}_{\mathbb{R}}(X, d^\alpha) &= \{f \in \mathrm{Lip}(X, d^\alpha) : f \; is \; real - valued\}, \\ \mathrm{lip}_{\mathbb{R}}(X, d^\alpha) &= \{f \in \mathrm{lip}(X, d^\alpha) : f \; is \; real - valued\}. \end{aligned}$$

Then $\mathrm{Lip}_{\mathbb{R}}(X, d^\alpha)$ ($\mathrm{lip}_{\mathbb{R}}(X, d^\alpha)$, respectively) is a unital real closed subalgebra of $\mathrm{Lip}(X, d^\alpha)$ ($\mathrm{lip}(X, d^\alpha)$, respectively). Moreover,

$$\mathrm{Lip}_{\mathbb{R}}(X, d^\beta) \subseteq \mathrm{lip}_{\mathbb{R}}(X, d^\alpha) \subseteq \mathrm{Lip}_{\mathbb{R}}(X, d^\alpha)$$

whenever $0 < \alpha < \beta \leq 1$.

In 1968, Hedberg obtained a Stone-Weierstrass theorem type in real little Lipschitz algebras $\mathrm{lip}_{\mathbb{R}}(X, d^\alpha)$ for $\alpha \in (0, 1)$ [4, Theorem 1] that can be modified in complex little Lipschitz algebras $\mathrm{lip}(X, d^\alpha)$ as the following.

Theorem 1.1. *Let (X, d) be a compact metric space and let $\alpha \in (0, 1)$. Let A be a self-adjoint subalgebra of* $\mathrm{lip}(X, d^\alpha)$ *which separates the points of X and contains the constant functions on X. Then A is dense in* $(\mathrm{lip}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$ *if for every $a \in X$, there are positive numbers M_a and δ_a such that for each δ with $0 < \delta < \delta_a$, there is a function f in A that satisfies $f(a) = 1$, $f(x) = 0$ for all $x \in S_{(X,d)}(a, \delta)$, and*

$$\sup\{\frac{|f(y) - f(z)|}{d^\alpha(y, z)} : y, z \in B_{(X,d)}[a, \delta], y \neq z\} < \frac{M_a}{\delta^\alpha}.$$

In 1987, Bade, Curtis and Dales [3] obtained a sufficient condition for density of a linear subspace P of $\mathrm{lip}(X, d^\alpha)$ in $(\mathrm{lip}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$, applying the measure theory and duality, and showed that $\mathrm{Lip}(X, d)$ is dense in $(\mathrm{lip}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$ as the following.

Theorem 1.2 (see [3, Theorem 3.6]). *Let* (X, d) *be a compact metric space and let* $\alpha \in (0, 1)$*. Let* P *be a linear subspace of* $\text{lip}(X, d^\alpha)$*. Suppose that there is a positive number* C *such that for each finite subset* E *of* X *and each* $f \in \text{lip}(X, d^\alpha)$*, there exists a function* g *in* P *with* $g|_E = f|_E$ *and with* $\|g\|_{\text{Lip}(X,d^\alpha)} \leq C\|f\|_{\text{Lip}(X,d^\alpha)}$*. Then* P *is dense in* $(\text{lip}(X, d^\alpha), \|\cdot\|_{\text{Lip}(X,d^\alpha)})$.

Theorem 1.3 (see [3, Corollary 3.7]). *Let* (X, d) *be a compact metric space and* $\alpha \in (0, 1)$*. Then* $\text{Lip}(X, d)$ *is dense in* $(\text{lip}(X, d^\alpha), \|\cdot\|_{\text{Lip}(X,d^\alpha)})$.

Definition 1.4. Let (X, d) be a compact metric space and $\alpha \in (0, 1]$. Let A be a subalgebra of $\text{Lip}(X, d^\alpha)$. It is said that A has the *separation property* with respect to X if there exists a constant $a > 1$ such that for every $x, y \in X$, there is a function f in A that satisfies $p_{(\alpha,X)}(f) \leq a$ and $|f(x) - f(y)| = d^\alpha(x, y)$.

In 1996, Weaver [10] obtained a sufficient condition for density of a subalgebra A of $\text{lip}(X, d)$ in $(\text{lip}(X, d), \|\cdot\|_{\text{Lip}(X,d)})$ as the following.

Theorem 1.5 (see [10, Theorem 1.4]). *Let* (X, d) *be a compact metric space. Suppose that* A *is a subalgebra of* $\text{lip}(X, d)$ *which contains the constant function* 1 *on* X*. If* A *has the separation property with respect to* X*, then* A *is dense in the little Lipschitz algebra* $(\text{lip}(X, d), \|\cdot\|_{\text{Lip}(X,d)})$.

Let (X, d) be a compact metric space, K be a nonempty compact subset of X and $\alpha \in (0, 1]$. We denote by $\text{Lip}(X, K, d^\alpha)$ ($\text{lip}(X, K, d^\alpha)$, respectively) the set of all $f \in C(X)$ for which $f|_K \in \text{Lip}(K, d^\alpha)$ ($f|_K \in \text{lip}(K, d^\alpha)$, respectively). Then $\text{Lip}(X, K, d^\alpha)$ ($\text{lip}(X, K, d^\alpha)$, respectively) is a complex subalgebra of $C(X)$ and $\text{lip}(X, K, d^\alpha)$ is a subset of $\text{Lip}(X, K, d^\alpha)$. The algebra $\text{Lip}(X, K, d^\alpha)$ ($\text{lip}(X, K, d^\alpha)$, respectively) is called the extended Lipschitz (little Lipschitz, respectively) algebra of order α on (X, d) with respect to K . Clearly, $\text{Lip}(X, d)$ is a subalgebra of $\text{Lip}(X, K, d^\alpha)$. Therefore, $\text{Lip}(X, K, d^\alpha)$ contains the constant function 1 on X and separates the points of X. It is easy to see that $\text{Lip}(X, K, d^\alpha)$ is a unital Banach algebra under the norm

$$||f||_{\text{Lip}(X,K,d^\alpha)} = ||f||_X + p_{\alpha,K}(f) \quad (f \in \text{Lip}(X, K, d^\alpha)),$$

and $\text{lip}(X, K, d^\alpha)$ is a closed unital subalgebra of $(\text{Lip}(X, K, d^\alpha), ||\cdot||_{\text{Lip}(X,K,d^\alpha)})$. Therefore, $(\text{Lip}(X, K, d^\alpha), ||\cdot||_{\text{Lip}(X,K,d^\alpha)})$ is a Banach function algebra on (X, d). Clearly, $\text{Lip}(X, K, d^\beta)$ is a subalgebra of $\text{lip}(X, K, d^\alpha)$ whenever $0 < \alpha < \beta \leq 1$. Therefore, $(\text{lip}(X, K, d^\alpha), ||\cdot||_{\text{Lip}(X,K,d^\alpha)})$ is a Banach function algebra on (X, d) whenever $\alpha \in (0, 1)$. We write $\text{Lip}(X, K, d)$ ($\text{lip}(X, K, d)$, respectively) instead of $\text{Lip}(X, K, d^1)$ ($\text{lip}(X, K, d^1)$, respectively). Note that $\text{Lip}(X, K, d^\alpha) = \text{Lip}(X, d^\alpha)$

and $\mathrm{lip}(X, K, d^\alpha) = \mathrm{lip}(X, d^\alpha)$, if $X \backslash K$ is finite. Also $\mathrm{Lip}(X, K, d^\alpha) = C(X)$ for $\alpha \in (0, 1]$ and $\mathrm{lip}(X, K, d^\alpha) = C(X)$ for $\alpha \in (0, 1)$, if K is finite. The extended Lipschitz algebras $\mathrm{Lip}(X, K, d^\alpha)$ and the extended little Lipschitz algebras $\mathrm{lip}(X, K, d^\alpha)$ were first introduced in [5].

Some properties of unital homomorphisms between extended Lipschitz algebras studied in [2].

In Section 2, we obtain sufficient conditions for density of linear subspaces and subalgebras of $\mathrm{lip}(X, K, d^\alpha)$ ($\mathrm{Lip}(X, K, d^\alpha)$, respectively) in $(\mathrm{lip}(X, K, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ $(\mathrm{Lip}(X, K, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$, respectively), and generalize mentioned theorems in Section 1.

2. The Density in Extended Lipschitz Algebras

Throughout this section we assume that (X, d) is a compact metric space and K is an infinite compact subset of X.

Theorem 2.1. *Suppose that $\alpha \in (0, 1]$, and $B = \mathrm{Lip}(X, K, d^\alpha)$ or $B = \mathrm{lip}(X, K, d^\alpha)$. Let P be a linear subspace of B. Then P is dense in $(B, ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$, if P satisfies the following conditions:*

- (i) *$CZ(X, K)$ is a subset of the closure of P in $(B, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$,*
- (ii) *$P|_K$ is dense in $(B|_K, ||\cdot||_{\mathrm{Lip}(K,d^\alpha)})$, where $S|_K = \{f|_K : f \in S\}$ for a subset S of B.*

Proof. By Tietze's extension theorem [7, Theorem 20.4], we have $B|_K = \mathrm{Lip}(K, d^\alpha)$ whenever $B = \mathrm{Lip}(X, K, d^\alpha)$ and $B|_K = \mathrm{lip}(K, d^\alpha)$ whenever $B = \mathrm{lip}(X, K, d^\alpha)$. Let $f \in B$ and let $\epsilon > 0$ be given. Then $f|_K \in B|_K$ and the density of $P|_K$ in $(B|_K, ||\cdot||_{\mathrm{Lip}(K,d^\alpha)})$ implies that there exists a function g in $P|_K$ such that

$$||g - f|_K||_{\mathrm{Lip}(K,d^\alpha)} < \frac{\epsilon}{2}. \tag{2.1}$$

Let $h = -f|_K + g$. Then $h \in B|_K$. By Tietze's extension theorem, there exists $H \in C(X)$ such that $H|_K = h$ and $\|H\|_X = \|h\|_K$. Therefore, $H \in B$ and

$$||H||_{\mathrm{Lip}(X,K,d^\alpha)} = ||h||_{\mathrm{Lip}(K,d^\alpha)}. \tag{2.2}$$

Since $g \in P|_K$, there exists a function G in P such that $G|_K = g$. Let $\varphi = f - G + H$. Then $\varphi \in B$ and $\varphi|_K = f|_K - g + h = 0$. So $\varphi \in CZ(X, K)$. Hence, φ belongs to the closure of P in $(B, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$, by (i). Therefore, $\varphi + G$ belongs to the closure of P in $(B, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ and so $f + H$ belongs to the closure of P in $(B, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$. This implies that there exists $\psi \in P$ such that

$$||f + H - \psi||_{\mathrm{Lip}(X,K,d^\alpha)} < \frac{\epsilon}{2}. \tag{2.3}$$

By (2.1), (2.2) and (2.3), we have

$$\begin{aligned}||\psi - f||_{\mathrm{Lip}(X,K,d^\alpha)} &\leq ||\psi - (f+H)||_{\mathrm{Lip}(X,K,d^\alpha)} + ||H||_{\mathrm{Lip}(X,K,d^\alpha)} \\ &< \frac{\epsilon}{2} + ||h||_{\mathrm{Lip}(K,d^\alpha)} \\ &= \frac{\epsilon}{2} + ||g - f|_K||_{\mathrm{Lip}(K,d^\alpha)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Thus, the proof is complete. □

We now give an extension of Theorem 1.2 applying Theorem 2.1 as the following.

Theorem 2.2. *Let $\alpha \in (0,1)$ and let P be a linear subspace of* $\mathrm{lip}(X,K,d^\alpha)$. *Then P is dense in* $(\mathrm{lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$, *if A satisfies the following conditions:*

(i) *$CZ(X,K)$ is a subset of $\bar{P}$, the closure of P in* $(\mathrm{lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$,

(ii) *there is a positive number C such that for each finite subset E of K and each $f \in \mathrm{lip}(X,K,d^\alpha)$, there exists a function g in P with $g|_E = f|_E$ and with $\|g\|_{\mathrm{Lip}(X,K,d^\alpha)} \leq C\|f\|_{\mathrm{Lip}(X,K,d^\alpha)}$.*

Proof. Clearly, $P|_K$ is a linear subspace of $\mathrm{lip}(K,d^\alpha)$. Suppose that E is a finite subset of K and $f \in \mathrm{lip}(K,d^\alpha)$. By Tietze's extension theorem, there exists a function F in $C(X)$ with $F|_K = f$ and with $\|F\|_X = \|f\|_K$. Then $F \in \mathrm{lip}(X,K,d^\alpha)$ and

$$||F||_{\mathrm{Lip}(X,K,d^\alpha)} = ||f||_{\mathrm{Lip}(K,d^\alpha)}. \tag{2.4}$$

Therefore, there exists a function G in P with $G|_E = F|_E$ and with

$$||G||_{\mathrm{Lip}(X,K,d^\alpha)} \leq C\|F\|_{\mathrm{Lip}(X,K,d^\alpha)}. \tag{2.5}$$

Let $g = G|_K$. Then $g \in P|_K$ and applying (2.4) and (2.5), we have

$$\begin{aligned}||g||_{\mathrm{Lip}(K,d^\alpha)} &\leq ||G||_{\mathrm{Lip}(X,K,d^\alpha)} \\ &\leq C||F||_{\mathrm{Lip}(X,K,d^\alpha)} \\ &= C||f||_{\mathrm{Lip}(K,d^\alpha)}.\end{aligned}$$

Hence, $P|_K$ is dense in $(\mathrm{lip}(K,d^\alpha), ||\cdot||_{\mathrm{Lip}(K,d^\alpha)})$ by Theorem 1.2. Therefore, P is dense in $(\mathrm{lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$ by Theorem 2.1. □

Note that, we have proved Theorem 2.2 in [1] using the measure theory and duality.

As another applications of Theorem 2.1, we will show that $\mathrm{Lip}(X,d)$ is dense in $(\mathrm{Lip}(X,K,d), ||\cdot||_{\mathrm{Lip}(X,K,d)})$ and $(\mathrm{lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$

for $\alpha \in (0,1)$. To prove these facts, we first show that $CZ(X,K)$ is a subset of the closure $\mathrm{Lip}(X,d)$ in $(\mathrm{Lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$ for $\alpha \in (0,1]$. To prove this result, we need the following lemma which is a modification of Sherbert's extension theorem [9, Proposition 1.4].

Lemma 2.3. *Let Y be a nonempty compact subset of X and let $f \in \mathrm{Lip}(Y,d)$. Then there exists a function $F \in \mathrm{Lip}(X,d)$ with $F|_Y = f$ such that $\|F\|_X \leq 2\|f\|_Y$ and $p_{\alpha,X}(F) \leq 2p_{\alpha,Y}(f)$.*

Theorem 2.4. *Let $\alpha \in (0,1]$. Then $CZ(X,K)$ is a subset of the closure $\mathrm{Lip}(X,d)$ in $(\mathrm{Lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$.*

Proof. Let $0 \neq f \in CZ(X,K)$ and let $\epsilon > 0$ be sufficiently small. Set

$$\begin{aligned} U &= \{x \in X : |f(x)| < \frac{\epsilon}{9}\}, \\ V &= \{x \in X : |f(x)| < \frac{\epsilon}{8}\}. \end{aligned}$$

Then U and V are open sets in X and $K \subseteq U \subseteq \overline{U} \subseteq V \neq X$, where $\overline{U}$ is the closure of U in the metric space (X,d). Since $f \in C(X)$ and $\mathrm{Lip}(X,d)$ is dense in $(C(X), \|\cdot\|_X)$, there exists a function $g \in \mathrm{Lip}(X,d)$ such that

$$\|g - f\|_X < \frac{\epsilon}{8}.$$

By Urysohn's lemma, there exists a function $h \in C(X)$ such that $0 \leq h(x) \leq 1$ for all $x \in X$, $h(x) = 0$ for all $x \in \overline{U}$ and $h(x) = 1$ for all $x \in X \backslash V$. Clearly, we have

$$p_{\alpha,K}(gh - f) = 0.$$

Let $Y = \overline{U} \cup (X \backslash V)$. Then Y is a nonempty compact subset of X and $\|h\|_Y = 1$. Let $\delta = \inf\{d(x,y) : x \in \overline{U}, y \in X\backslash V\}$. Then $\delta > 0$. It is easy to see that $p_{1,Y}(h) \leq \frac{1}{\delta}$. So $h|_Y \in \mathrm{Lip}(Y,d)$. By Lemma 2.3, there exists a function $H \in \mathrm{Lip}(X,d)$ with $H|_Y = h|_Y$ such that

$$\|H\|_X \leq 2\|h\|_Y.$$

Therefore, $gH \in \mathrm{Lip}(X,d)$ and we have

$$\begin{aligned}
\|gH-f\|_{\mathrm{Lip}(X,K,d^\alpha)} &= \|gH-f\|_X + p_{\alpha,K}(gH-f)\\
&= \|gH-f\|_X + p_{\alpha,K}(gh-f)\\
&= \|gH-f\|_X\\
&\le \|gH-f\|_{X\setminus V} + \|gH-f\|_{\overline{V}\setminus U} + \|gH-f\|_{\overline{U}}\\
&= \|gH-f\|_{X\setminus V} + \|gH-f\|_{\overline{V}\setminus U} + \|f\|_{\overline{U}}\\
&\le \|g-f\|_X + \|gH\|_{\overline{V}\setminus U} + \|f\|_{\overline{V}\setminus U} + \|f\|_{\overline{U}}\\
&\le \|g-f\|_X + \|g\|_{\overline{V}\setminus U}\|H\|_{\overline{V}\setminus U} + \|f\|_{\overline{V}\setminus U} + \|f\|_{\overline{U}}\\
&< \frac{\epsilon}{8} + \|g\|_{\overline{V}\setminus U}\ \|H\|_X + \frac{\epsilon}{8} + \frac{\epsilon}{8}\\
&\le \frac{3\epsilon}{8} + 2\|g\|_{\overline{V}\setminus U}\ \|h|_Y\|_Y\\
&= \frac{3\epsilon}{8} + 2\|g\|_{\overline{V}\setminus U}\\
&\le \frac{3\epsilon}{8} + 2(\|g-f\|_{\overline{V}\setminus U} + \|f\|_{\overline{V}\setminus U})\\
&\le \frac{3\epsilon}{8} + 2(\|g-f\|_X + \|f\|_{\overline{V}})\\
&< \frac{3\epsilon}{8} + 2(\frac{\epsilon}{8} + \frac{\epsilon}{8})\\
&< \epsilon.
\end{aligned}$$

Hence $f \in \overline{\mathrm{Lip}(X,d)}$, the closure of $\mathrm{Lip}(X,d)$ in $(\mathrm{Lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$. Thus, the proof is complete. □

Theorem 2.5. *The Lipschitz algebra* $\mathrm{Lip}(X,d)$ *is dense in the extended Lipschitz algebra* $(\mathrm{Lip}(X,K,d), ||\cdot||_{\mathrm{Lip}(X,K,d)})$.

Proof. Let $P = \mathrm{Lip}(X,d)$ and $B = \mathrm{Lip}(X,K,d)$. Then P is a linear subspace of B. By Theorem 2.4, $CZ(X,K)$ is a subset of the closure P in $(B, ||\cdot||_{\mathrm{Lip}(X,K,d)})$. On the other hand, $P|_K = \mathrm{Lip}(K,d)$ by Lemma 2.3 and $B|_K = \mathrm{Lip}(K,d)$ using the Tietze's extension theorem. Thus, $P|_K$ is dense in $(B|_K, ||\cdot||_{\mathrm{Lip}(X,d)})$. Therefore, P is dense in $(B, ||\cdot||_{\mathrm{Lip}(X,K,d)})$ by Theorem 2.1. Thus, the proof is complete. □

Corollary 2.6. *Let* $\alpha \in (0,1]$. *Then the Lipcshitz algebra* $\mathrm{Lip}(X,d^\alpha)$ *is dense in the extended Lipschitz algebra* $(\mathrm{Lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$.

Proof. Since the induced topology by the metric d^α on X coincides to the induced topology by metric d on X, we conclude that K is a compact subset of X in the metric space (X,d^α). Thus, the result holds by Theorem 2.5. □

Since $\|f\|_{\mathrm{Lip}(X,K,d^\alpha)} \leq \|f\|_{\mathrm{Lip}(X,d^\alpha)}$ for all $f \in \mathrm{Lip}(X,K,d^\alpha)$, we obtain the following result as a consequence of Corollary 2.6.

Corollary 2.7. *Let* $\alpha \in (0,1]$ *and let* P *be a subset of* $\mathrm{Lip}(X,d^\alpha)$ *such that* P *is dense in* $(\mathrm{Lip}(X,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,d^\alpha)})$. *Then* P *is dense in* $(\mathrm{Lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$.

Theorem 2.8. *Let* $\alpha \in (0,1)$. *Then the Lipschitz algebra* $\mathrm{Lip}(X,d)$ *is dense in the extended little Lipschitz algebra* $(\mathrm{lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$.

Proof. Let $P = \mathrm{Lip}(X,d)$ and $B = \mathrm{lip}(X,K,d^\alpha)$. Then P is a linear subspace of B. By Theorem 2.4, $CZ(X,K)$ is a subset of the closure of P in $(\mathrm{Lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$. Since B is a closed set in $(\mathrm{Lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)}$, we deduce that $CZ(X,K)$ is a subset of the closure P in $(B, ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$.

On the other hand, $P|_K = \mathrm{Lip}(K,d)$ by Lemma 2.3 and $B|_K = \mathrm{lip}(K,d^\alpha)$ using the Tietze's extension theorem. Thus, $P|_K$ is dense in $(B|_K, ||\cdot||_{\mathrm{Lip}(K,d^\alpha)})$. Therefore, P is dense in $(B, ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$ by Theorem 2.1. Thus, the proof is complete. □

Corollary 2.9. *Let* $\alpha \in (0,1)$. *Then* $\mathrm{lip}(X,d^\alpha)$ *is dense in* $(\mathrm{lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$.

Proof. Since $\mathrm{Lip}(X,d)$ is a subset of $\mathrm{lip}(X,d^\alpha)$, the result holds by Theorem 2.8. □

Corollary 2.10. *Let* $\alpha \in (0,1)$. *If* P *is a subset of* $\mathrm{lip}(X,d^\alpha)$ *such that* P *is dense in* $(\mathrm{lip}(X,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,d^\alpha)})$, *then* P *is dense in* $(\mathrm{lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$.

Corollary 2.11. *Let* $\alpha \in (0,1)$ *and* P *be a linear subspace of* $\mathrm{lip}(X,d^\alpha)$. *Suppose that there is a positive number* C *such that for each finite subset* E *of* X *and for each* $f \in \mathrm{lip}(X,d^\alpha)$, *there exists a function* $g \in P$ *with* $g|_E = f|_E$ *and* $\|g\|_{\mathrm{Lip}(X,d^\alpha)} \leq C\|f\|_{\mathrm{Lip}(X,d^\alpha)}$. *Then* P *is dense in* $(\mathrm{lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$.

Proof. By Theorem 1.2, P is dense in $(\mathrm{lip}(X,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,d^\alpha)})$. Thus, the proof holds by Corollary 2.10. □

Applying Theorem 2.1, we give an extension of Theorem 1.1 as the following.

Theorem 2.12. *Suppose that* $\alpha \in (0,1)$ *and* A *is a self-adjoint subalgebra of* $\mathrm{lip}(X,K,d^\alpha)$ *which separates the points of* X *and contains the constant functions on* X. *Then* A *is dense in* $(\mathrm{lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$, *if* A *satisfies the following conditions:*

(i) $CZ(X,K)$ *is a subset of the closure of* A *in* $(\mathrm{lip}(X,K,d^\alpha), ||\cdot||_{\mathrm{Lip}(X,K,d^\alpha)})$,

(ii) *for every* $a \in K$, *there are positive numbers* M_a *and* δ_a *such that for each* δ *with* $0 < \delta < \delta_a$, *there is a function* f *in* A *that satisfies* $f(a) = 1$, $f(x) = 0$ *for all* $x \in S_{(K,d)}(a, \delta)$, *and*

$$\sup\{\frac{|f(y) - f(z)|}{d^\alpha(y, z)} : y, z \in B_{(K,d)}[a, \delta], y \neq z\} < \frac{M_a}{\delta^\alpha}.$$

Proof. Clearly, $A|_K$ is a self-adjoint subalgebra of $\text{lip}(K, d^\alpha)$ which separates the points of K and contains the constant functions on K. From the condition (ii) and applying Theorem 1.1, we conclude that $A|_K$ is dense in $(\text{lip}(K, d^\alpha), ||\cdot||_{\text{Lip}(K,d^\alpha)})$. Therefore, A is dense in $(\text{lip}(X, K, d^\alpha), || \cdot ||_{\text{Lip}(X,K,d^\alpha)})$ by Theorem 2.1. □

Definition 2.13. Let $\alpha \in (0, 1]$ and let A be a subalgebra of $\text{Lip}(X, K, d^\alpha)$. We say that A has *separation property* with respect to K if there exists a constant $a > 1$ such that for every $x, y \in K$, there is a function f in A that satisfies $p_{(\alpha,K)}(f) \leq a$ and $|f(x) - f(y)| = d^\alpha(x, y)$.

We now give an extension of Theorem 1.5, applying Theorem 2.1 as the following.

Theorem 2.14. *Suppose that* A *is a subalgebra of* $\text{lip}(X, K, d)$ *which contains the constant functions* 1 *on* X. *Then* A *is dense in* $(\text{lip}(X, K, d), || \cdot ||_{\text{Lip}(X,K,d)})$, *if* A *satisfies the following conditions:*

(i) $CZ(X, K)$ *is a subset of* $\bar{A}$, *the closure of* A *in* $(\text{lip}(X, K, d), \| \cdot \|_{\text{Lip}(X,K,d)})$,

(ii) A *has the separation property with respect to* K.

Proof. Clearly, $A|_K$ is subalgebra of $\text{lip}(K, d)$ which contains the constant function 1 on K. The condition (ii) implies that there exists a constant $a > 1$ such that for every $x, y \in K$ there is a function f in A that satisfies $p_{1,K}(f) \leq a$ and $|f(x) - f(y)| = d(x, y)$. Let $x, y \in K$. Choose $f \in A$ such that $p_{1,K}(f) \leq a$ and $|f(x) - f(y)| = d(x, y)$. If $g = f|_K$, then $g \in A|_K$, $p_{1,K}(g) = p_{1,K}(f) \leq a$ and $|g(x) - g(y)| = |f(x) - f(y)| = d(x, y)$. Hence, $A|_K$ has the separation property with respect to K. Hence, $A|_K$ is dense in $(\text{lip}(K, d), ||\cdot||_{\text{Lip}(K,d)})$ by Theorem 1.5. Therefore, A is dense in $(\text{lip}(X, K, d), || \cdot ||_{\text{Lip}(X,K,d)})$ by Theorem 2.1. □

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