

Exact Solutions of the Generalized Kuramoto-Sivashinsky Equation

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ABSTRACT. In this paper we obtain exact solutions of the generalized Kuramoto-Sivashinsky equation, which describes many physical processes in motion of turbulence and other unstable process systems. The methods used to determine the exact solutions of the underlying equation are the Lie group analysis and the simplest equation method. The solutions obtained are then plotted.

Keywords: Generalized Kuramoto-Sivashinsky equation, Integrability, Lie symmetry methods, Simplest equation method.

1. INTRODUCTION

In this paper, we consider the generalized Kuramoto-Sivashinsky equation [1]

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \quad (1.1)$$

where α , β and γ are real constants. This equation is also called KdV-Burgers-Kuramoto equation. The generalized Kuramoto-Sivashinsky equation (1.1) describes physical processes in motion of turbulence and other unstable process systems. It has been applied to stationary solitary pulses in a falling film [2]. Also, this equation can be used to describe long waves on a viscous fluid flowing down along an inclined plane [3], unstable drift waves in plasma [4] and stress waves in fragmented porous

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media. For more applications of the generalized Kuramoto-Sivashinsky equation (1.1), see [1, 2, 3, 4, 5] and references therein.

In this paper we use Lie group analysis along with the simplest equation method to obtain exact solutions of the generalized Kuramoto-Sivashinsky equation (1.1).

2. LIE POINT SYMMETRIES OF (1.1)

A Lie point symmetry of a differential equation is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. Finding all the symmetries of a differential equation is an alarming exercise. However, in the middle of the nineteenth, Sophus Lie (1842-1899) realized that if we restrict ourself to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries. Lie's continuous symmetry groups have applications in such diverse fields as invariant theory, control theory, classical mechanics, relatively etc. For the theory and application of the Lie group analysis methods, see for example, the Refs. [6, 8, 7, 9].

The symmetry group of the generalized Kuramoto-Sivashinsky equation (1.1) will be generated by the vector field of the form

$$X = \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta^1(t, x, u) \frac{\partial}{\partial u}. \quad (2.1)$$

Applying the third prolongation $\text{pr}^{(4)}X$ (see [8] for details) to (1.1) and solving the resultant overdetermined system of linear partial differential equations one obtains the following two Lie point symmetries:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}.$$

3. SYMMETRY REDUCTION OF (1.1)

In this section we consider the symmetry $X_1 + \nu X_2$ and use it to reduce the generalized Kuramoto-Sivashinsky equation (1.1) into an ordinary differential equation. Later in the next section we will look for exact solutions of (1.1).

The symmetry generator $X_1 + \nu X_2$ gives rise to the group-invariant solution

$$u = F(z), \quad (3.1)$$

where $z = x - \nu t$ is an invariant of the symmetry $X_1 + \nu X_2$. Substitution of (3.1) into (1.1) results in the fourth-order nonlinear ordinary differential equation

$$\gamma F''''(z) + \beta F'''(z) + \alpha F''(z) - \nu F'(z) + F(z)F'(z) = 0. \quad (3.2)$$

4. EXACT SOLUTIONS OF (1.1) USING SIMPLEST EQUATION METHOD

We now employ the simplest equation method to solve the nonlinear ordinary differential equation (3.2). This will result in giving us the solution to the generalized Kuramoto-Sivashinsky equation (1.1) via the equation (3.1). The simplest equation method was introduced by Kudryashov [10, 11] and modified by Vitanov [12] (see also [13]). The simplest equations that will be used here are the Bernoulli and Riccati equations.

Let us consider the solutions of (3.2) in the form

$$F(z) = \sum_{i=0}^M \mathcal{A}_i (H(z))^i, \quad (4.1)$$

where $H(z)$ satisfies the Bernoulli and Riccati equations, M is a positive integer that can be determined by balancing procedure as in [12] and \mathcal{A}_i are parameters to be determined. The solutions of the Bernoulli and Riccati equations can be expressed in terms of elementary functions.

The Bernoulli equation

$$H'(z) = aH(z) + bH^2(z), \quad (4.2)$$

where a and b are constants, has the solution

$$H(z) = a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\},$$

where C is an arbitrary constant of integration.

For the Riccati equation

$$H'(z) = aH^2(z) + bH(z) + c \quad (4.3)$$

where a , b and c are constants, we shall use the solutions

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z + C) \right]$$

and

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)},$$

where $\theta^2 = b^2 - 4ac > 0$ and C is a constant of integration.

4.1. Solutions of (1.1) Using the Bernoulli Equation as the Simplest Equation. The balancing procedure [12] yields $M = 3$ so the solutions of (3.2) are of the form

$$F(z) = \mathcal{A}_0 + \mathcal{A}_1 H(z) + \mathcal{A}_2 H^2(z) + \mathcal{A}_3 H^3(z). \quad (4.4)$$

Substituting (4.4) into (3.2) and making use of the Bernoulli (4.2) and then equating all coefficients of the functions $H^i(z)$ to zero, we obtain

an algebraic system of equations in terms of $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ and \mathcal{A}_3 . Solving the resultant system, one possible set of values are

$$\begin{aligned}\alpha &= a^2\gamma, \\ \beta &= 4a\gamma, \\ \mathcal{A}_0 &= \nu - 6a^3\gamma, \\ \mathcal{A}_1 &= -120a^2b\gamma, \\ \mathcal{A}_2 &= -240ab^2\gamma, \\ \mathcal{A}_3 &= -120b^3\gamma.\end{aligned}$$

Thus a solution of the generalized Kuramoto-Sivashinsky equation (1.1) is

$$\begin{aligned}u(t, x) &= \mathcal{A}_0 + \mathcal{A}_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\} \\ &+ \mathcal{A}_2 a^2 \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\}^2 \\ &+ \mathcal{A}_3 a^2 \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\}^3.\end{aligned}\quad (4.5)$$

4.2. Solutions of (1.1) Using Riccati Equation as the Simplest Equation. The balancing procedure yields $M = 3$ so the solutions of (3.2) are of the form

$$F(z) = \mathcal{A}_0 + \mathcal{A}_1 H(z) + \mathcal{A}_2 H^2(z) + \mathcal{A}_3 H^3(z).\quad (4.6)$$

Substituting (4.6) into (3.2) and making use of the Riccati equation (4.3) and then equating all coefficients of the functions $H^i(z)$ to zero, we obtain an algebraic system of equations in terms of $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ and \mathcal{A}_3 . Solving this algebraic system, one possible set of values are

$$\begin{aligned}a &= 1, \\ b &= 3, \\ c &= 1, \\ \alpha &= 365\gamma, \\ \beta &= -36\gamma - 4\gamma k, \\ \mathcal{A}_0 &= -990\gamma + 60\gamma k + \nu, \\ \mathcal{A}_1 &= 60\gamma + 180\gamma k, \\ \mathcal{A}_2 &= 60\gamma k, \\ \mathcal{A}_3 &= -120\gamma,\end{aligned}$$

where k is any root of $k^2 + 18k + 1 = 0$. Hence solutions of the generalized Kuramoto-Sivashinsky equation (1.1) are

$$\begin{aligned} u(t, x) = & \mathcal{A}_0 + \mathcal{A}_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z + C) \right] \right\} \\ & + \mathcal{A}_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z + C) \right] \right\}^2 \\ & + \mathcal{A}_3 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z + C) \right] \right\}^3 \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} u(t, x) = & A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)} \right\} \\ & + A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)} \right\}^2 \\ & + A_3 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)} \right\}^3, \end{aligned} \quad (4.8)$$

where $z = x - \nu t$ and C is a constant of integration.

5. CONCLUDING REMARKS

In this paper we obtained the exact solutions of the generalized Kuramoto-Sivashinsky equation (1.1) by using the Lie symmetry method along with the simplest equation method. The profiles of the obtained solutions were plotted. Numerical solutions of the generalized Kuramoto-Sivashinsky equation (1.1) were presented in [1] using B-spline functions. In future conservation laws of the generalized Kuramoto-Sivashinsky equation (1.1) will be derived using several different approaches.

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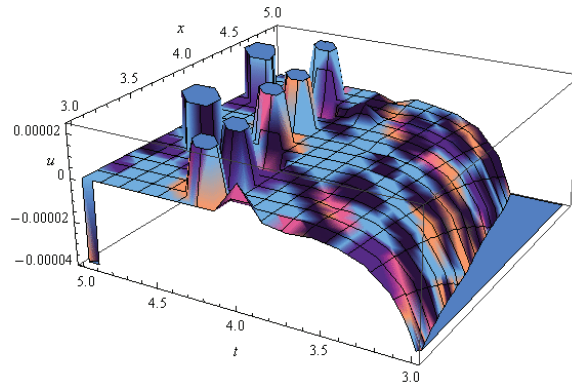


FIGURE 1. Profile of solution (4.5) with $C = 0, a = 1, b = 1, \gamma = 1, \nu = 6$.

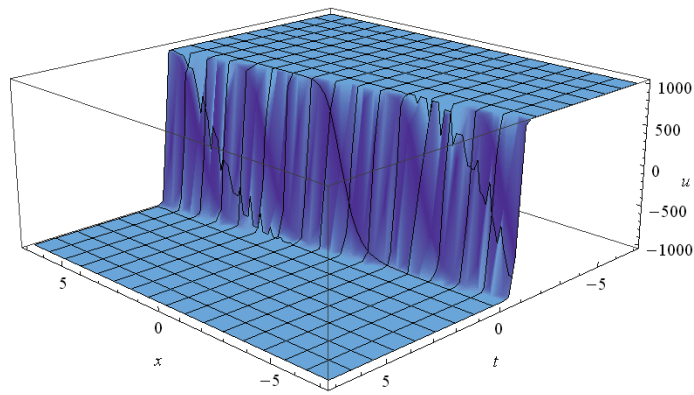


FIGURE 2. Profile of the solution (4.7) with $C = 0, \gamma = 1, \nu = 6$.

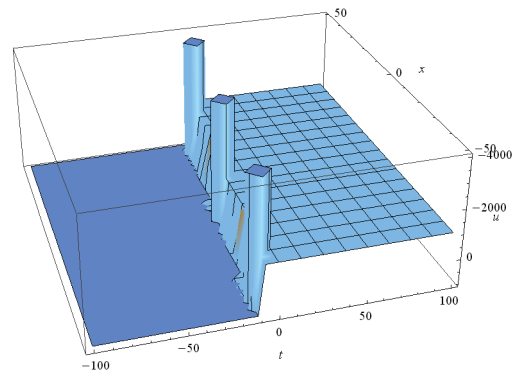


FIGURE 3. Profile of the solution (4.8) with $C = 0, \gamma = 1, \nu = 6$.