On quasi-catenary modules

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Abstract. We call a module $M$, quasi-catenary if for each pair of quasi-prime submodules $K$ and $L$ of $M$ with $K \subset L$, all saturated chains of quasi-prime submodules of $M$ from $K$ to $L$ have a common finite length. We show that any homomorphic image of a quasi-catenary module is quasi-catenary. We prove that if $M$ is a module with following properties:

(i) Every quasi-prime submodule of $M$ has finite quasi-height;

(ii) For every pair of $K \subset L$ of quasi-prime submodules of $M$, $q\text{-height}(L/K) = q\text{-height}(L) - q\text{-height}(K)$;

then $M$ is quasi-catenary.

Keywords: Catenary module; quasi-prime submodule; quasi-catenary module


1. INTRODUCTION

In this paper all rings are commutative with identity and all modules are unitary. A strictly increasing (or decreasing) chain $K_0 \subset K_1 \subset \ldots \subset K_n$ of (quasi-)prime (ideals) submodules of (a) an (ring) $R$-module $M$ is said to be saturated if there does not exist any (quasi-)prime (ideal) submodule strictly contained between any two consecutive terms. Recall that a ring $R$ is catenary if the following condition is satisfied: for any prime ideals $p$ and $p'$ of $R$ with $p \subset p'$, there exists a saturated chain of prime ideals starting from $p$ and ending with $p'$ and all such chains

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have the same finite length. A proper ideal $I$ of $R$ is called quasi-prime, provided $J \cap L \subseteq I$ for ideals $J, L$ of $R$, implies that $J \subseteq I$ or $L \subseteq I$ (see [2] and [4]). In this work, we call a ring $R$, quasi-catenary, if for each quasi-prime ideals $p$ and $q$ of $R$ with $p \subseteq q$, there exists a saturated chain of quasi-prime ideals starting from $p$ and ending at $q$ and all such chains have the same finite length. Let $R$ be a ring and $M$ an $R$-module. A proper submodule $K$ of $M$ is called prime if $am \in K$ implies $m \in K$ or $aM \subseteq K$, for $a \in R$, $m \in M$. S. Namazi and H. Sharif generalized the concept of catenary rings to catenary modules (see [6] and [7]). A module $M$ is called (quasi-)catenary if for each pair of (quasi-)prime submodules $K$ and $L$ of $M$ with $K \subseteq L$ all saturated chains of (quasi-)prime submodules of $M$ from $K$ to $L$ have a common finite length. They investigated some properties of such modules. We say that a (quasi-)prime submodule $K$ of $M$ has $(q)$-height $n$, if there exists a chain $K_0 \subset K_1 \subset \ldots \subset K_n = K$ of (quasi-)prime submodules $K_i$ ($0 \leq i \leq n$) of $M$, but no such longer chain exists. Otherwise, we say that it has an infinite (quasi-)height. We shall denote the (quasi-height) height of $K$ by $q\text{ht}(K)$ $ht(K)$. It is defined that $h - \dim(M)$ to be the supremum of the heights of all prime submodules of $M$. If $M$ has no prime submodule, it is defined to be $h - \dim(M) = -1$. Based on this definition we use the notion $q\text{h} - \dim(M)$ for the supremum of the $q$-heights of all quasi-prime submodules of $M$ and if $M$ has no quasi-prime submodule we set $q\text{h} - \dim(M) = -1$.

2. QUASI-PRIME IDEALS AND SUBMODULES

In this section we will study quasi-prime ideals and submodules which are a generalization of prime ideals and submodules. A proper ideal $I$ of a ring $R$ is said to be quasi-prime if for each pair of ideals $A$ and $B$ of $R$, $A \cap B \subseteq I$ yields either $A \subseteq I$ or $B \subseteq I$ (see [2] and [4]). Clearly every prime ideal is a quasi-prime ideal.

A ring $R$ is Laskerian, if each ideal has a finite primary decomposition. Let $R$ be a ring with just one prime ideal $m$ such that $m^n = 0$. Then $R$ is Laskerian. In particular, every Noetherian ring is Laskerian.

Some properties of quasi-prime ideals of a ring are listed below.

**Proposition 2.1.** ([4] Lemma 2.2] and [1] Remark 2.2]) Let $I$ be an ideal in a ring $R$. Then

1. If $I$ is quasi-prime, then $I$ is irreducible ($I$ is not the intersection of two ideals of $R$ that properly contain it);
2. If $R$ is a Laskerian ring, then every quasi-prime ideal is a primary ideal;
3. If $I$ is a prime ideal, then $I$ is quasi-prime;
4. Every proper ideal of a serial ring is quasi-prime;
(5) If $R$ is an arithmetical ring, $I$ is irreducible if and only if $I$ is quasi-prime;
(6) If $R$ is a Dedekind domain, then $I$ is quasi-prime if and only if $I$ is a primary ideal.
(7) Every primary principal ideal of a UFD, is quasi-prime.

**Definition 2.2.** A proper submodule $N$ of an $R$-module $M$ is called quasi-prime if $(N :_R M)$ is a quasi-prime ideal of $R$. (see [1])

We define the quasi-prime spectrum of an $R$-module $M$ to be the set of all quasi-prime submodules of $M$ and denote it by $q\text{Spec}(M)$. Recall from [5], the set of prime submodules of a module $M$ is called spectrum of $M$ denoted by $\text{Spec}(M)$. Also, the set of maximal submodules of $M$ is denoted by $\text{Max}(M)$.

**Remark 2.3.** Let $M$ be an $R$-module.
(1) By [6, Proposition 4], every maximal submodule of an $R$-module $M$ is prime, so that every prime submodule of $M$ is a quasi-prime submodule. Therefore, $\text{Max}(M) \subseteq \text{Spec}(M) \subseteq q\text{Spec}(M)$.
(2) Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module and $N = (2, 0)\mathbb{Z}$ a submodule of $M$. Then $(N : M) = (0) \in \text{Spec}(\mathbb{Z})$, i.e., $N \in q\text{Spec}(M)$ though $N$ is not a $(0)$-prime submodule of $M$. Thus in general, $\text{Spec}(M) \neq q\text{Spec}(M)$.

We say that $R$ is a uniserial ring if the set of all ideals of $R$ is linearly ordered and a ring $R$ is serial, if it is a direct sum of uniserial rings. Recall that a ring $R$ is said to be arithmetical, if for any maximal ideal $P$ of $R$, $RP$ is a serial ring. Recall that a module $M$ is said to be a Laskerian module, if every proper submodule of $M$ has a primary decomposition.

**Lemma 2.4. ([1] Lemma 2.4)** Let $M$ be an $R$-module and let $S$ be a multiplicatively closed subset of $R$.
(1) If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a family of quasi-prime submodules with $(N_\lambda :_R M) = J$ for each $\lambda$, then $N = \bigcap_{\lambda \in \Lambda} N_\lambda$ is a quasi-prime submodule of $M$ such that $(N :_R M) = J$;
(2) If $M$ is a fully prime module (every proper submodule of $M$ is prime), then every proper submodule of $M$ is quasi-prime. In particular, every proper subspace of a vector space over a field is quasi-prime;
(3) If $R$ is a uniserial ring, then every proper submodule of $M$ is quasi-prime;
(4) Let $N$ be a quasi-prime submodule of the $R_S$-module $M_S$. Then $N \cap M$ is a quasi-prime submodule of $M$;
(5) Let $R$ be an arithmetical ring. Then every primary submodule of $M$ is quasi-prime.

A quasi-prime submodule need not be prime as following example shows.
Example 2.5. (1) Every proper submodule of the \( \mathbb{Z} \)-module \( M = \mathbb{Z}_{(p^{\infty})} \) is a quasi-prime submodule, in which \( p \) is a prime integer. We note that \( \text{Spec}(M) = \emptyset \).
(2) It is known that \( (p^n) \) is a primary ideal of \( \mathbb{Z} \). So by Proposition 2.1(7), \( (p^n) \) is quasi-prime. But it is not prime.

3. QUASI-CATENARY RINGS AND MODULES

In this section we define and study quasi-catenary rings and modules. We investigate some properties of these new classes of rings and modules.

Definition 3.1. We call a ring \( R \), quasi-catenary, if for each quasi-prime ideals \( p \) and \( q \) of \( R \) with \( p \subset q \), there exists a saturated chain of quasi-prime ideals starting from \( p \) and ending at \( q \) and all such chains have the same finite length.

Clearly every quasi-catenary ring is catenary since every prime ideal is quasi-prime.

Recall from [5, Theorem 11.2], that a ring \( R \) is a discrete valuation ring (DVR for short) if and only if \( R \) is local principal ideal domain which is not a field.

Example 3.2. If the ideals of \( R \) are linearly ordered, then each ideal in \( R \) is quasi-prime. So, for example, if \( R \) is either a DVR or a homomorphic image of a DVR, then each ideal in \( R \) is quasi-prime. In particular, if \( F \) is a field, \( X \) is an indeterminate, and \( n \) is a positive integer, then each ideal in \( R = F[[X]] = (X^n) \) is quasi-prime. Now let \( p \subset q \) be two ideals of a DVR, \( R \). Since each ideal of \( R \), is a power of \( m \), unique maximal ideal of \( R \), there exists a unique saturated chain of ideals of \( R \), \( p = m^r \subset m^{r+1} \subset \ldots \subset m^{s-1} \subset m^s = q \) and since \( R \) is Noetherian, its length is finite. So \( R \) is quasi-catenary. In particular, every field is a quasi-catenary ring.

Definition 3.3. We call a module \( M \), quasi-catenary (q-catenary for short) if for each pair of quasi-prime submodules \( K \) and \( L \) of \( M \) with \( K \subset L \), all saturated chains of quasi-prime submodules of \( M \) from \( K \) to \( L \) have a common finite length.

Since every prime submodule is a quasi-prime submodule, every quasi-catenary module is catenary.

Example 3.4. It is easy to check that any vector space is q-catenary if and only if it is a finite dimension.

Proof. Let \( V \) be a q-catenary vector space over a field \( F \). Then \( V \) is a catenary vector space. By [6] Example 2.1(ii)], \( V \) is a finite dimension. In contrast, let \( V \) be a finite dimensional vector space over \( F \) such that
$K$ and $L$ are (quasi-prime) submodules of $V$ with $K \subset L$. Since $V$ is artinian and noetherian, by Jordan-Holder Theorem, every two saturated chains of (quasi-prime) submodules of $V$ has equal finite lengths.

\[ \square \]

**Remark 3.5.** Let $M$ be an $R$-module and $N \subset K$ be submodules of $M$, then $K$ is a quasi-prime submodule of $M$ if and only if $K/N$ is a quasi-prime submodule of the $R$-module $M/N$.

**Proof.** Let $K \leq M$ be quasi-prime. Since $(K/N : R M/N) = (K : R M)$, then $(K/N : R M/N)$ is a quasi-prime ideal of $R$. The converse is similar. \[ \square \]

We call a module $M$, *homogeneous semisimple* if $M = \bigoplus_{i \in I} M_i$ where each $M_i \cong N$ is a simple $R$-module.

**Proposition 3.6.** Let $M$ be a module such that every submodule of $M$ is quasi-prime. If $M$ is Artinian and Noetherian, Then $M$ is $q$-catenary. In particular every finitely generated homogeneous semisimple (e.g. every finitely generated semisimple module over a local ring) module is $q$-catenary.

**Proof.** Let $M$ be a module such that every submodule of $M$ is quasi-prime. Now let $K \subset L$ be two submodules of $M$. Since every saturated chain of quasi-prime submodules of $M$ between $K$ and $L$ is a saturated chain of quasi-prime submodules of $M/K$ by Remark 3.5 and $M/K$ is Noetherian and Artinian, by Jordan-Holder Theorem, every two saturated chains of quasi-prime submodules has a finite common length. For the second part, let $M = M_1 \oplus \ldots \oplus M_n$, where all $M_1, \ldots, M_n$ are simple and isomorphic to each other. Then $Ann(M) = m$ is a maximal ideal of $R$. By [3, Proposition 1.10], every submodule of $M$ is quasi-prime. The rest is similar. \[ \square \]

In [6, Lemma 2.2], it is shown that any homomorphic image of a catenary module is catenary. We have the similar result for $q$-catenary modules.

**Lemma 3.7.** Any homomorphic image of a $q$-catenary module is $q$-catenary.

**Proof.** This follows from the fact that for any $R$-module $M$ with $N \subset K \subset M$, $(K : R M) = (K/N : R M/N)$ and Remark 3.5. \[ \square \]

Recall that a module $M$ has a distributive set of submodules or is called a distributive module, in case for every submodules $L, K, N$ of $M$ we have $N + (K \cap L) = (N + K) \cap (N + L)$ or $N \cap (L + K) = (N \cap L) + (N \cap K)$.

The following lemma gives us a sufficient condition for a module to be $q$-catenary over a distributive ring.
Lemma 3.8. Let $R$ be a ring such that $R_R$ has a distributive lattice of ideals and $M$ be a $q$-catenary $R$-module. Then for each quasi-prime submodule $K$ of $M$ with $(K :_RM) = p$, the $R/p$-module $M/K$ is $q$-catenary.

Proof. Let $L/K \subset L_1/K \subset \ldots \subset T/K$ be a saturated chain of quasi-prime submodules of $R/p$-module $M/K$. Since $R_R$ is distributive, each $L_i$ is a quasi-prime submodule of $M$. Hence we have a saturated chain of quasi-prime submodules of $M$ namely, $L \subset L_1 \subset \ldots \subset T$. Since $M$ is $q$-catenary all these chains have common finite length. Therefore, $M/K$ is $q$-catenary as a $R/p$-module. □

Proposition 3.9. Let $M$ be an $R$-module and $0 \leq qh - \dim(M) \leq 2$, then $M$ is $q$-catenary.

Proof. For the case $qh - \dim(M) = 0$ or $1$ the proof is obvious. Let $qh - \dim(M) = 2$ and $K \subset L$ be quasi-prime submodules of $M$. Then there can be just one quasi-prime submodule between $K$ and $L$. Hence all saturated chains of quasi-prime submodules of $M$ from $K$ to $L$ have length 2. So in this case also $M$ is $q$-catenary. □

In the above proposition every simple $R$-module is $q$-catenary.

Proposition 3.10. Let $M$ be an $R$-module such that every quasi-prime submodule of $M$ has a finite $q$-height. If $K \subset L$ of quasi-prime submodules of $M$, we have $qht(L/K) = qht(L) - qht(K)$ for each pair, then $M$ is $q$-catenary.

Proof. Let $K \subset L$ be quasi-prime submodules of $M$. Since $n = qht(L/K) < \infty$, there exists a saturated chain of quasi-prime submodules of $M$ from $K$ to $L$ of length $n$. Now let $K = K_0 \subset K_1 \subset \ldots \subset K_m = L$ be any saturated chain of quasi-prime submodules of $M$. We show that $m = n$. Since there is no quasi-prime submodule of $M$ between $K_i$ and $K_{i+1}$, we have $qht(K_{i+1}/K_i) = 1$ and hence $qht(K_{i+1}) = qht(K_i) + 1$, for $i = 0, 1, \ldots, m - 1$. So $qht(L) = qht(K) + m$. Thus $m = qht(L) - qht(K) = qht(L/K) = n$. □

It is easy to see that if $M$ is an $R$-module with $qh - \dim(M) \geq 0$ and for each pair $K \subset L$ of quasi-prime submodules of $M$ with $qht(K) \leq qht(L) - 2$, then $M$ is $q$-catenary.

Lemma 3.11. Let $\varphi : R \to R'$ be a ring epimorphism. Let $M$ be an $R$-module such that $(\ker \varphi)M = 0$. Then $M$ is an $R'$-module and we have $M$ is a $q$-catenary $R$-module if and only if $M$ is a $q$-catenary $R'$-module.

Proof. It is clear since $K$ is a quasi-prime $R$-submodule of $M$ if and only if $K$ is a quasi-prime $R'$-submodule of $M$. □
Corollary 3.12. Let $M$ be an $R$-module and $I$ be an ideal of $R$ such that $IM = 0$. Then $M$ is a $q$-catenary $R/I$-module if and only if $M$ is a $q$-catenary $R$-module.

Example 3.13. Let $R$ be a Noetherian ring and $m$ be a maximal ideal of $R$. Then $M = m/m^2$ is a $q$-catenary $R/m$-module, hence $M$ is a $q$-catenary $R$-module.

Proof. Since $M$ is a finite dimensional vector space over the field $R/m$, it is $q$-catenary by Example 3.4. So the result supports Lemma 3.11. □

References