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On quasi-catenary modules

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Abstract. We call a module M, quasi-catenary if for each pair of quasi-prime submodules K and L of M with $K \subset L$ all saturated chains of quasi-prime submodules of M from K to L have a common finite length. We show that any homomorphic image of a quasi-catenary module is quasi-catenary. We prove that if M is a module with following properties:
(i) Every quasi-prime submodule of M has finite quasi-height;
(ii) For every pair of $K \subset L$ of quasi-prime submodules of M, $q - height(L/K) = q - height(L) - q - height(K)$;
then M is quasi-catenary.

Keywords: Catenary module; quasi-prime submodule; quasi-catenary module

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1. INTRODUCTION

In this paper all rings are commutative with identity and all modules are unitary. A strictly increasing (or decreasing) chain $K_0 \subset K_1 \subset \ldots \subset K_n$ of (quasi-)prime (ideals) submodules of (a) an (ring) R-module M is said to be saturated if there does not exist any (quasi-)prime (ideal) submodule strictly contained between any two consecutive terms. Recall that a ring R is *catenary* if the following condition is satisfied: for any prime ideals p and p' of R with $p \subset p'$, there exists a saturated chain of prime ideals starting from p and ending with p' and all such chains

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have the same finite length. A proper ideal I of R is called *quasi-prime*, provided $J \cap L \subseteq I$ for ideals J, L of R, implies that $J \subseteq I$ or $L \subseteq I$(see [2] and [4]). In this work, we call a ring R, *quasi-catenary*, if for each quasi-prime ideals p and q of R with $p \subset q$, there exists a saturated chain of quasi-prime ideals starting from p and ending at q and all such chains have the same finite length. Let R be a ring and M an R-module. A proper submodule K of M is called *prime* if $am \in K$ implies $m \in K$ or $aM \subset K$, for $a \in R$, $m \in M$. S. Namazi and H. Sharif generalized the concept of catenary rings to catenary modules (see [6] and [7]). A module M is called *(quasi-)catenary* if for each pair of (quasi-)prime submodules K and L of M with $K \subset L$ all saturated chains of (quasi-)prime submodules of M from K to L have a common finite length. They investigated some properties of such modules. We say that a (quasi-)prime submodule K of M has (q-)height n, if there exists a chain $K_0 \subset K_1 \subset \ldots \subset K_n = K$ of (quasi-)prime submodules K_i $(0 \leq i \leq n)$ of M, but no such longer chain exists. Otherwise, we say that it has an infinite (quasi-)height. We shall denote the (quasi-height) height of K by $(qht(K))$ $ht(K)$. It is defined that $h - dim(M)$ to be the supremum of the heights of all prime submodules of M. If M has no prime submodule, it is defined to be $h - dim(M) = -1$. Based on this definition we use the notion $qh - dim(M)$ for the supremum of the q-heights of all quasi-prime submodules of M and if M has no quasi-prime submodule we set $qh - dim(M) = -1$.

2. QUASI-PRIME IDEALS AND SUBMODULES

In this section we will study quasi-prime ideals and submodules which are a generalization of prime ideals and submodules. A proper ideal I of a ring R is said to be *quasi-prime* if for each pair of ideals A and B of R, $A \cap B \subseteq I$ yields either $A \subseteq I$ or $B \subseteq I$ (see [2] and [4]). Clearly every prime ideal is a quasi-prime ideal.

A ring R is *Laskerian*, if each ideal has a finite primary decomposition. Let R be a ring with just one prime ideal m such that $m^n = 0$. Then R is Laskerian. In particular, every Noetherian ring is Laskerian.

Some properties of quasi-prime ideals of a ring are listed below.

Proposition 2.1. *(*[4, Lemma 2.2] *and* [1, Remark 2.2]*) Let I be an ideal in a ring R. Then*
(1) If I is quasi-prime, then I is irreducible (I is not the intersection of two ideals of R that properly contain it);
(2) If R is a Laskerian ring, then every quasi-prime ideal is a primary ideal;
(3) If I is a prime ideal, then I is quasi-prime;
(4) Every proper ideal of a serial ring is quasi-prime;

(5) If R is an arithmetical ring, I is irreducible if and only if I is quasi-prime;
(6) If R is a Dedekind domain, then I is quasi-prime if and only if I is a primary ideal.
(7) Every primary principal ideal of a UFD, is quasi-prime.

Definition 2.2. A proper submodule N of an R-module M is called quasi-prime if $(N :_R M)$ is a quasi-prime ideal of R. (see [1])

We define the quasi-prime spectrum of an R-module M to be the set of all quasi-prime submodules of M and denote it by $qSpec(M)$. Recall from [5], the set of prime submodules of a module M is called *spectrom of* M denoted by $Spec(M)$. Also, the set of maximal submodules of M is denoted by $Max(M)$.

Remark 2.3. Let M be an R-module.
(1) By [6, Proposition 4], every maximal submodule of an R-module M is prime, so that every prime submodule of M is a quasi-prime submodule. Therefore, $Max(M) \subseteq Spec(M) \subseteq qSpec(M)$.
(2) Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module and $N = (2, 0)\mathbb{Z}$ a submodule of M. Then $(N : M) = (0) \in Spec(\mathbb{Z})$, i.e., $N \in qSpec(M)$ though N is not a (0)-prime submodule of M. Thus in general, $Spec(M) \neq qSpec(M)$.

We say that R is a *uniserial ring* if the set of all ideals of R is linearly ordered and a ring R is *serial*, if it is a direct sum of uniserial rings. Recall that a ring R is said to be *arithmetical*, if for any maximal ideal P of R, R_P is a serial ring. Recall that a module M is said to be a *Laskerian* module, if every proper submodule of M has a primary decomposition.

Lemma 2.4. ([1, Lemma 2.4]) *Let M be an R-module and let S be a multiplicatively closed subset of R.*
(1) If $\{N_\lambda\}_{\lambda\in\Lambda}$ is a family of quasi-prime submodules with $(N_\lambda :_R M) = J$ for each λ, then $N = \bigcap_{\lambda\in\Lambda} N_\lambda$ is a quasi-prime submodule of M such that $(N :_R M) = J$;
(2) If M is a fully prime module (every proper submodule of M is prime), then every proper submodule of M is quasi-prime. In particular, every proper subspace of a vector space over a field is quasi-prime;
(3) If R is a uniserial ring, then every proper submodule of M is quasi-prime;
(4) Let N be a quasi-prime submodule of the R_S-module M_S. Then $N \cap M$ is a quasi-prime submodule of M;
(5) Let R be an arithmetical ring. Then every primary submodule of M is quasi-prime.

A quasi-prime submodule need not be prime as following example shows.

Example 2.5. (1) Every proper submodule of the $\mathbb{Z}$-module $M = Z_{(p^\infty)}$ is a quasi-prime submodule, in which p is a prime integer. We note that $Spec(M) = \emptyset$.

(2) It is known that (p^n) is a primary ideal of $\mathbb{Z}$. So by Proposition 2.1(7), (p^n) is quasi-prime. But it is not prime.

3. QUASI-CATENARY RINGS AND MODULES

In this section we define and study quasi-catenary rings and modules. We investigate some properties of these new classes of rings and modules.

Definition 3.1. We call a ring R, *quasi-catenary*, if for each quasi-prime ideals p and q of R with $p \subset q$, there exists a saturated chain of quasi-prime ideals starting from p and ending at q and all such chains have the same finite length.

Clearly every quasi-catenary ring is catenary since every prime ideal is quasi-prime.

Recall from [5, Theorem 11.2], that a ring R is a *discrete valuation ring* (DVR for short) if and only if R is local principal ideal domain which is not a field.

Example 3.2. If the ideals of R are linearly ordered, then each ideal in R is quasi-prime. So, for example, if R is either a DVR or a homomorphic image of a DVR, then each ideal in R is quasi-prime. In particular, if F is a field, X is an indeterminate, and n is a positive integer, then each ideal in $R = F[[X]] = (X^n)$ is quasi-prime. Now let $p \subset q$ be two ideals of a DVR, R. Since each ideal of R, is a power of m, unique maximal ideal of R, there exists a unique saturated chain of ideals of R, $p = m^r \subset m^{r+1} \subset \ldots \subset m^{s-1} \subset m^s = q$ and since R is Noetherian, its length is finite. So R is quasi-catenary. In particular, every field is a quasi-catenary ring.

Definition 3.3. We call a module M, quasi-catenary (q-catenary for short) if for each pair of quasi-prime submodules K and L of M with $K \subset L$, all saturated chains of quasi-prime submodules of M from K to L have a common finite length.

Since every prime submodule is a quasi-prime submodule, every quasi-catenary module is catenary.

Example 3.4. It is easy to check that any vector space is q-catenary if and only if it is a finite dimension.

Proof. Let V be a q-catenary vector space over a field F. Then V is a catenary vector space. By [6, Example 2.1(ii)], V is a finite dimension. In contrast, let V be a finite dimensional vector space over F such that

K and L are (quasi-prime) submodules of V with $K \subset L$. Since V is artinian and noetherian, by Jordan-Holder Theorem, every two saturated chains of (quasi-prime) submodules of V has equal finite lengths. □

Remark 3.5. Let M be an R-module and $N \subset K$ be submodules of M, then K is a quasi-prime submodule of M if and only if K/N is a quasi-prime submodule of the R-module M/N.

Proof. Let $K \leq M$ be quasi-prime. Since $(K/N :_R M/N) = (K :_R M)$, then $(K/N :_R M/N)$ is a quasi-prime ideal of R. The converse is similar. □

We call a module M, *homogeneous semisimple* if $M = \bigoplus_{i \in I} M_i$ where each $M_i \cong N$ is a simple R-module.

Proposition 3.6. *Let M be a module such that every submodule of M is quasi-prime. If M is Artinian and Noetherian, Then M is q-catenary. In particular every finitely generated homogeneous semisimple (e.g. every finitely generated semisimple module over a local ring) module is q-catenary.*

Proof. Let M be a module such that every submodule of M is quasi-prime. Now let $K \subset L$ be two submodules of M. Since every saturated chain of quasi-prime submodules of M between K and L is a saturated chain of quasi-prime submodules of M/K by Remark 3.5 and M/K is Noetherian and Artinian, by Jordan-Holder Theorem, every two saturated chains of quasi-prime submodules has a finite common length. For the second part, let $M = M_1 \oplus \ldots \oplus M_n$, where all $M_1, \ldots, M_n$ are simple and isomorphic to each other. Then $Ann(M) = m$ is a maximal ideal of R. By [3, Porposition 1.10], every submodule of M is quasi-prime. The rest is similar. □

In [6, Lemma 2.2], it is shown that any homomorphic image of a catenary module is catenary. We have the similar result for q-catenary modules.

Lemma 3.7. *Any homomorphic image of a q-catenary module is q-catenary.*

Proof. This follows from the fact that for any R-module M with $N \subset K \subset M$, $(K :_R M) = (K/N :_R M/N)$ and Remark 3.5. □

Recall that a module M *has a distributive set of submodules* or is called a *distributive* module, in case for every submodules L, K, N of M we have $N + (K \cap L) = (N + K) \cap (N + L)$ or $N \cap (L + K) = (N \cap L) + (N \cap K)$.

The following lemma gives us a sufficient condition for a module to be q-catenary over a distributive ring.

Lemma 3.8. *Let R be a ring such that R_R has a distributive lattice of ideals and M be a q-catenary R-module. Then for each quasi-prime submodule K of M with $(K :_R M) = p$, the R/p-module M/K is q-catenary.*

Proof. Let $L/K \subset L_1/K \subset \ldots \subset T/K$ be a saturated chain of quasi-prime submodules of R/p-module M/K. Since R_R is distributive, each L_i is a quasi-prime submodule of M. Hence we have a saturated chain of quasi-prime submodules of M namely, $L \subset L_1 \subset \ldots \subset T$. Since M is q-catenary all these chains have common finite length. Therefore, M/K is q-catenary as a R/p-module. □

Proposition 3.9. *Let M be an R-module and $0 \leq qh - dim(M) \leq 2$, then M is q-catenary.*

Proof. For the case $qh - dim(M) = 0$ or 1 the proof is obvious. Let $qh - dim(M) = 2$ and $K \subset L$ be quasi-prime submodules of M. Then there can be just one quasi-prime submodule between K and L. Hence all saturated chains of quasi-prime submodules of M from K to L have length 2. So in this case also M is q-catenary. □

In the above proposition every simple R-module is q-catenary.

Proposition 3.10. *Let M be an R-module such that every quasi-prime submodule of M has a finite q-height. If $K \subset L$ of quasi-prime submodules of M, we have $qht(L/K) = qht(L) - qht(K)$ for each pair, then M is q-catenary.*

Proof. Let $K \subset L$ be quasi-prime submodules of M. Since $n = qht(L/K) < \infty$, there exists a saturated chain of quasi-prime submodules of M from K to L of length n. Now let $K = K_0 \subset K_1 \subset \ldots \subset K_m = L$ be any saturated chain of quasi-prime submodules of M. We show that $m = n$. Since there is no quasi-prime submodule of M between K_i and K_{i+1}, we have $qht(K_{i+1}/K_i) = 1$ and hence $qht(K_{i+1}) = qht(K_i) + 1$, for $i = 0, 1, \ldots, m - 1$. So $qht(L) = qht(K) + m$. Thus $m = qht(L) - qht(K) = qht(L/K) = n$. □

It is easy to see that if M is an R-module with $qh - dim(M) \geq 0$ and for each pair $K \subset L$ of quasi-prime submodules of M with $qht(K) \leq qht(L) - 2$, there exists a quasi-prime submodule N such that $K \subset N \subset L$, then M is q-catenary.

Lemma 3.11. *Let $\varphi : R \to R'$ be a ring epimorphism. Let M be an R-module such that $(ker\varphi)M = 0$. Then M is an R'-module and we have M is a q-catenary R-module if and only if M is a q-catenary R'-module.*

Proof. It is clear since K is a quasi-prime R-submodule of M if and only if K is a quasi-prime R'-submodule of M. □

Corollary 3.12. *Let M be an R-module and I be an ideal of R such that $IM = 0$. Then M is a q-catenary R/I-module if and only if M is a q-catenary R-module.*

Example 3.13. Let R be a Noetherian ring and m be a maximal ideal of R. Then $M = m/m^2$ is a q-catenary R/m-module, hence M is a q-catenary R-module.

Proof. Since M is a finite dimensional vector space over the field R/m, it is q-catenary by Example 3.4. So the result supports Lemma 3.11. □

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