Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **2**(2)(2013), 125-136

Common Fixed Point Theorems for Generalized Weakly Contractive Mappings under The Weaker Meir-Keeler Type Function

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Abstract. In this paper, we prove some common fixed point theorems for multivalued mappings and we present some new generalization contractive conditions under the condition of weak compatibility. Our results extends Chang-Chen's results [6] as well as Ćirić results [7]. An example is given to support the usability of our results.

Keywords: Metric space; Common fixed point; Contractive mapping; Weakly compatible.

2000 Mathematics subject classification: 47H10, 47H09, 54H25, 55M20.

1. Introduction

It is common that the contractive-type conditions are very important in the study a fixed point theory. The first important result of fixed points for contractive-type mapping was the well-known Banach-Caccioppoli theorem published for the first time in 1922 in [3] and also found in [5].

In recent years, many authors had proved fixed point theorems for mappings in metric spaces satisfying general contractive integral type

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Received: 22 Oct 2012
Revised: 20 Aug 2013
Accepted: 19 Sep 2013

inequalities for example see [12, 18, 19, 15, 17, 16] and weakly compatible mappings, see for example [1, 2, 4, 7, 9, 10, 13, 20, 21].

The purpose of this paper is to establish the existence of fixed point theorem for generalized contractive multivalued mappings.

2. Preliminaries

At first we recall the notion of the Meir-Keeler and weaker Meir-Keeler type functions as follows.

Definition 2.1 ([14])**.** A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler type function , if for each $\eta \in \mathbb{R}^+$, there exists $\delta = \delta(\eta) > 0$ such that for $t \in \mathbb{R}^+$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$.

Definition 2.2 ([6])**.** The function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a weaker Meir-Keeler type function, if for each $\eta > 0$, there exists $\delta > \eta$ such that for $t \in \mathbb{R}^+$ with $\eta \leq t < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

Throughout this paper, let $\mathcal{B}(X)$ stand for the set of all nonempty bounded subsets of X and two functions $\delta, D : \mathcal{B}(X) \times \mathcal{B}(X) \to [0, +\infty)$ are defined be:

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\},$$

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

The following definition is given by Fisher in [8].

Definition 2.3. Let $\{A_n : n = 1, 2, ...\}$ be a sequence of $2^X - \{\emptyset\}$. We say that the sequence $\{A_n\}$ converges to a subset A of X if

(i) each point a in A is the limit of a convergent sequence $\{a_n\}$ with $\{a_n \in A_n : n = 1, 2, ...\}$;
(ii) For arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A_\epsilon$ for $n > N$, where A_ϵ is a the union of all open spheres with center in A and radius ϵ.

The set A is then said to be the limit of the sequence $\{A_n\}$.

The following lemmas that appear in [8] and [10], are useful for the main results of this paper.

Lemma 2.4 ([8])**.** *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded sets of (X, d) which converge to the bounded sets A and B, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 2.5 ([10])**.** *If $\{A_n\}$ is a sequences of bounded sets in the compelet metric space (X, d) and if $\lim_{n\to\infty} \delta(A_n, \{y\}) = 0$, for some $y \in X$, then $\{A_n\} \to \{y\}$.*

Let $T : X \to \mathcal{B}(X)$ be a multivalued mapping . If U is any nonempty subset of X then we define

$$T(U) = \bigcup_{x\in U} Tx.$$

Also, if f is a self mapping of X, then by $T(X) \subseteq f(X)$, we mean

$$T(U) = \bigcup_{x\in U} Tx \subseteq f(X),$$

that is, for all $x \in U$, we have $Tx \subseteq f(X)$.

The following definitions were given by Jungck and Rhoades [11].

Definition 2.6. Let $f : X \to X$ and $S : X \to \mathcal{B}(X)$ be two mappings. The pair (f, S) is said to be weakly compatible if f and S commute at coincidence; i.e., for each point u in X such that $Su = \{fu\}$, we have $Sfu = fSu$.

Definition 2.7. Let Φ denotes all function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions:
(i) $\phi(0) = 0$, and $\phi(t) > 0$ for all $t \in \mathbb{R}^+$,
(ii) ϕ is continuous from the right and
(iii) ϕ is nondecreasing on $\mathbb{R}^+$.

Definition 2.8. Let Ψ denotes all function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy weaker Meir-Keeler type function such that for $t > 0$ with $\psi(t) < t$ and $\{\psi^n(t)\}_{n\in\mathbb{N}}$ is non-increasing.

The following Lemma is useful for the main results of this paper, that appear in [22].

Lemma 2.9. *Let $\phi \in \Phi$. If $\lim_{n\to\infty} \phi(\epsilon_n) = 0$, for $\{\epsilon_n\} \subset \mathbb{R}^+$, then $\lim_{n\to\infty} \epsilon_n = 0$.*

Definition 2.10 ([6])**.** Let (X, d) be a metric space, and let $T, S : E \to \mathcal{B}(E)$. If the following inequality holds:

$$\phi(\delta(Sx, Ty)) \leq \psi(\phi(M(x, y))) \tag{2.1}$$

for all $x, y \in X$, where

$$M(x, y) := \max\left\{d(x, y), \delta(Sx, x), \delta(y, Ty), \frac{1}{2}[D(x, Ty) + D(Sx, y)]\right\}, \tag{2.2}$$

then we call that the pair (T, S) having the (ϕ, ψ)- contraction property.

Using this definition Chang and Chen proved the following theorem and extended the Ćirić results.

Theorem 2.11 (see [6, Theorem 1]). *Let (X,d) be a complete metric space and let $T,S:X\to\mathcal{B}(X)$. If (T,S) have the (ϕ,ψ)-contraction property, where $\varphi\in\Phi$ and $\psi\in\Psi$, then S and T have a unique common fixed point a in X. Moreover, $Sa=Ta=\{a\}$.*

Now, we define a generalized (ϕ,ψ)-contractive for the pair (T,S) that $T,S:E\to\mathcal{B}(E)$ as follows:

Definition 2.12. Two mappings $T,S:X\to\mathcal{B}(X)$ are called generalized (ϕ,ψ)- contractive if there exist two maps $f,g:X\to X$ such that

$$\phi(\delta(Sx,Ty))\le\psi(\phi(M(x,y))) \tag{2.3}$$

for all $x,y\in X$, where

$$M(x,y):=\max\big\{d(fx,gy),\delta(Sx,fx),\delta(gy,Ty),\frac{1}{2}[D(fx,Ty)+D(Sx,gy)]\big\}. \tag{2.4}$$

In next section, we give a new fixed point theorem for (ϕ,ψ)-contractive mappings and extend Chang-Chen's Theorem. After that an example shows that our results extend Chang-Chen's Theorem.

3. Main result

The following theorem extends Chang-Chen's Theorem

Theorem 3.1. *Let (X,d) be a complete metric space, and let E be a nonempty closed subset of X. Let $T,S:E\to\mathcal{B}(E)$ be two generalized (ϕ,ψ)- contractive, where $\varphi\in\Phi$ and $\psi\in\Psi$, and $f,g:E\to X$ verifying the following:*
(A) (f,S) and (g,T) are weakly compatible;
(B) $T(E)\subseteq f(E)$ and $S(E)\subseteq g(E)$.
Assume that $f(E)$ or $g(E)$ is a closed subset of X. Then f, T, g and S have a unique common fixed point, that is, there exist $x\in E$ such that $\{fx\}=\{gx\}=Tx=Sx=\{x\}$.

Proof. Lex $x_o\in E$ be arbitrary. Using (B), we choose $x_1\in E$ such that $gx_1\in Sx_0=A_0$. There exists $x_2\in E$ such that $fx_2\in Tx_1=A_1$, and so on. Using induction, we can define a sequence $\{x_n\}$ in E as follows:

$$gx_{2n+1}\in Sx_{2n}=A_{2n},\ fx_{2n+2}\in Tx_{2n+1}=A_{2n+1}, \tag{3.1}$$

for $n=0,1,....$ We break the argument into three steps.

Step 1. $\lim\limits_{n\to\infty}\delta(A_n,A_{n+1})=0.$

Proof. Using (2.3), have

$$\begin{aligned} \phi(\delta(Sx_{2n}, Tx_{2n+1})) &= \phi(\delta(A_{2n}, A_{2n+1})) \\ &\leq \psi(\phi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} & M(x_{2n}, x_{2n+1}) \\ = \;& \max\big\{d(fx_{2n}, gx_{2n+1}), \delta(Sx_{2n}, fx_{2n}), \delta(gx_{2n+1}, Tx_{2n+1}), \\ & \frac{1}{2}[D(fx_{2n}, Tx_{2n+1}) + D(Sx_{2n}, gx_{2n+1})]\big\} \\ = \;& \max\big\{d(fx_{2n}, gx_{2n+1}), \delta(A_{2n}, fx_{2n}), \delta(gx_{2n+1}, A_{2n+1}), \\ & \frac{1}{2}[D(fx_{2n}, A_{2n+1}) + 0]\big\} \qquad (3.3) \\ \leq \;& \max\big\{\delta(A_{2n-1}, A_{2n}), \delta(A_{2n}, A_{2n+1}), \frac{1}{2}\delta(A_{2n-1}, A_{2n+1})\big\} \\ \leq \;& \max\big\{\delta(A_{2n-1}, A_{2n}), \delta(A_{2n}, A_{2n+1}), \\ & \frac{1}{2}[\delta(A_{2n-1}, A_{2n}) + \delta(A_{2n}, A_{2n+1})]\big\} \\ = \;& \max\big\{\delta(A_{2n-1}, A_{2n}), \delta(A_{2n}, A_{2n+1})\big\} = \delta(A_{2n-1}, A_{2n}). \end{aligned}$$

If $\max\big\{\delta(A_{2n-1}, A_{2n}), \delta(A_{2n}, A_{2n+1})\big\} = \delta(A_{2n-1}, A_{2n})$, then By (3.2), we have

$$\begin{aligned} \phi(\delta(A_{2n}, A_{2n+1})) &= \phi(\delta(Sx_{2n}, Tx_{2n+1})) \\ &\leq \psi(\phi(M(x_{2n}, x_{2n+1})) \\ &= \psi(\phi(\delta(A_{2n}, A_{2n+1})) \qquad (3.4) \\ &< \phi(\delta(A_{2n}, A_{2n+1}), \end{aligned}$$

where that is a contradiction. Hence

$$\delta(A_{2n}, A_{2n+1}) \leq \delta(A_{2n-1}, A_{2n}). \tag{3.5}$$

Similarly,

$$\delta(A_{2n+1}, A_{2n+2}) \leq \delta(A_{2n+1}, A_{2n}). \tag{3.6}$$

Consequently for each $n \in \mathbb{N}$, we have

$$\delta(A_n, A_{n+1}) \leq \delta(A_n, A_{n-1}), \tag{3.7}$$

and hence,

$$\begin{aligned} \psi(\delta(A_n, A_{n+1})) &\leq \psi(\phi(\delta(A_{n-1}, A_n))) \\ &\leq \ldots \leq \psi^n(\phi(\delta(A_0, A_1))). \end{aligned} \tag{3.8}$$

Since $\{\psi^n(\phi(\delta(A_0, A_1)))\}_{n\in\mathbb{N}}$ is nonincreasing and so it must converges to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. From $\psi(t) < t$ we get $\phi(\delta(A_0, A_1)) \geq \eta$, then by the definition of the weaker Meir-Keeler type function, there exists $\delta > \eta$ such that for $\phi(\delta(A_0, A_1)) > 0$ with $\eta \leq \phi(\delta(A_0, A_1)) < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\phi(\delta(A_0, A_1))) < \delta$. Since

$$\lim_{n\to\infty} \psi^n(\phi(\delta(A_0, A_1))) = 0, \tag{3.9}$$

and this is a contradiction. Consequently, $\eta = 0$. Thus By (3.8), we have

$$\lim_{n\to\infty} \psi(\delta(A_n, A_{n+1}))) = 0, \tag{3.10}$$

so $\lim\limits_{n\to\infty} \delta(A_n, A_{n+1}) = 0$. □

Step 2. $\{A_n\}$ is Cauchy.

Proof. For each $m \in \mathbb{N}$, we suppose $C_m = \delta(A_m, A_{m+1})$ and we claim that following result holds:

$$\forall\ \gamma > 0\ \exists\ n_0(\gamma) \in \mathbb{N}\ s.t\ \forall\ m, n > n_0(\gamma),\ \ \delta(A_m, A_n) < \gamma. \tag{3.11}$$

Suppose (3.11), is not held. Then using **Step 1** there exists some $\gamma > 0$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ that $m_k - n_k > 3$ and m_k is even, n_k is odd and

$$\delta(A_{m_k}, A_{n_k}) \geq \gamma. \tag{3.12}$$

For every $k \in \mathbb{N}$, let m_k be the smallest even number satisfying (3.12).

Since $\lim\limits_{m\to\infty} C_m = 0$, there exist $k_0 \in \mathbb{N}$ such that for $m \geq k_0$, $\delta(A_m, A_{m+1}) < \gamma$. Thus By (3.12), we have

$$\begin{aligned} \gamma &\leq \delta(A_{m_k}, A_{n_k}) \\ &\leq \delta(A_{m_k}, A_{m_k-1}) + \delta(A_{m_k-1}, A_{m_k-2}) + \delta(A_{m_k-2}, A_{n_k}) \\ &\leq \delta(A_{m_k}, A_{m_k-1}) + \delta(A_{m_k-1}, A_{m_k-2}) + \gamma \end{aligned} \tag{3.13}$$

letting $k \to \infty$, we get

$$\lim_{k\to\infty} \delta(A_{m_k}, A_{n_k}) = \gamma. \tag{3.14}$$

So with using (2.3),

$$\begin{aligned} \phi(\delta(A_{m_k}, A_{n_k})) &= \phi(\delta(Sx_{m_k}, Tx_{n_k})) \\ &\leq \psi(\phi(M(x_{m_k}, x_{n_k})), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
& M(x_{m_k}, x_{n_k}) \\
= \ & \max\big\{d(fx_{m_k}, gx_{n_k}), \delta(Sx_{m_k}, fx_{m_k}), \delta(gx_{n_k}, Tx_{n_k}), \\
& \frac{1}{2}[D(fx_{m_k}, Tx_{n_k}) + D(S_{m_k}, gx_{n_k})]\big\} \\
\leq \ & \max\big\{\delta(A_{m_k-1}, A_{n_k-1}), \delta(A_{m_k}, A_{m_k-1}), \delta(A_{n_k-1}, A_{n_k}), \\
& \frac{1}{2}[\delta(A_{m_k-1}, A_{n_k}) + \delta(A_{m_k}, A_{n_k-1})]\big\} \\
\leq \ & \max\big\{\delta(A_{m_k-1}, A_{n_k}) + \delta(A_{m_k}, A_{n_k}) + \delta(A_{n_k}, A_{n_k-1}), \\
& \delta(A_{m_k}, A_{m_k-1}), \delta(A_{n_k-1}, A_{n_k}), \\
& \frac{1}{2}[\delta(A_{m_k-1}, A_{m_k}) + \delta(A_{m_k}, A_{n_k}) + \delta(A_{n_k-1}, A_{n_k}) + \delta(A_{n_k}, A_{m_k})]\big\} \\
= \ & \max\big\{C_{m_k-1} + \delta(A_{m_k}, A_{n_k}) + C_{n_k-1}, C_{m_k-1}, C_{n_k-1}, \\
& \frac{1}{2}[C_{m_k-1} + 2\delta(A_{m_k}, A_{n_k}) + C_{n_k-1}]\big\} \qquad (3.16) \\
\leq \ & C_{m_k-1} + C_{n_k-1} + \delta(A_{m_k}, A_{n_k}).
\end{aligned}$$

Now with combining (3.15), (3.16), and letting $k \to \infty$, we have

$$\phi(\gamma) \leq \psi(\phi(\gamma)), \qquad (3.17)$$

and this is a contraction.So $\{A_n\}$ is Cauchy. □

Step 3. T, S, g and f have a common fixed point.

Proof. If a_n be an arbitrary point in A_n for $n = 0, 1, ...$, it follows that

$$\lim_{n,m\to\infty} d(a_n, a_m) \leq \lim_{n,m\to\infty} \delta(A_n, A_m) = 0. \qquad (3.18)$$

Therefore, the sequence $\{a_n\}$ and hence any subsequence thereof is a Cauchy sequence in X. Since $gx_{2n+1} \in Sx_{2n} = A_{2n}$ for $n = 0, 1, ...$, we have

$$\lim_{n,m\to\infty} d(gx_{2n+1}, gx_{2m+1}) \leq \lim_{n,m\to\infty} \delta(A_{2n}, A_{2m}) = 0, \qquad (3.19)$$

Therefore, the sequence $\{gx_{2n+1}\}$ is Cauchy. So there exists $z \in X$ such that $\lim_{n\to\infty} gx_{2n+1} = z$. Since E is closed and $\{gx_{2n+1}\} \subseteq X$, we have $z \in E$. Since $g(E)$ is closed, then there exists $u \in E$ such that $z = gu$. But, $fx_{2n} \in Tx_{2n-1} = A_{2n-1}$, so that we have

$$\lim_{n\to\infty} d(fx_{2n}, gx_{2n+1}) \leq \lim_{n\to\infty} \delta(A_{2n-1}, A_{2n}) = 0. \qquad (3.20)$$

Consequently, $\lim_{n\to\infty} fx_{2n} = z$. Now we prove $Tu = \{z\}$. By using (2.3), we have

$$\phi(\delta(Sx_{2n}, Tu)) \quad \leq \quad \psi(\phi(M(x_{2n}, u)), \qquad (3.21)$$

where

$$\begin{aligned}\delta(gu,Tu) &\leq M(x_{2n},u)\\ &= \max\big\{d(fx_{2n},gu),\delta(Sx_{2n},fx_{2n}),\delta(gu,Tu),\\ &\quad \frac{1}{2}[D(fx_{2n},Tu)+D(Sx_{2n},gu)]\big\} \qquad (3.22)\\ &\leq \max\big\{\delta(fx_{2n},gu),\delta(Sx_{2n},fx_{2n}),\delta(gu,Tu),\\ &\quad \frac{1}{2}[\delta(fx_{2n},Tu)+\delta(Sx_{2n},gu)]\big\}.\end{aligned}$$

where

$$\begin{aligned}\delta(Sx_{2n},gu) &= \delta(Sx_{2n},z)\leq \delta(Sx_{2n},fx_{2n})+\delta(fx_{2n},z)\\ &\leq \delta(A_{2n},A_{2n-1})+\delta(fx_{2n},z). \qquad (3.23)\end{aligned}$$

letting $n\to\infty$ in above inequality we noted

$$\lim_{n\to\infty}\delta(Sx_{2n},z)=0. \qquad (3.24)$$

Consequently, by combining (3.22), (3.24) and letting $n\to\infty$, we have got

$$\lim_{n\to\infty}M(x_{2n},u)=\delta(z,Tu). \qquad (3.25)$$

From (3.21) and (3.25) by letting $n\to\infty$ we have

$$\phi(\delta(z,Tu)) \leq \psi(\phi(\delta(z,Tu)). \qquad (3.26)$$

Hence $\delta(z,Tu)=0$. So $Tu=\{z\}$. Consequently, $\{gu\}=Tu=\{z\}$.

Since $T(E)\subseteq f(E)$ and $Tu\in T(E)$, so there exists $w\in E$ exists such that $Tu=\{fw\}=\{gu\}$. Now we prove $Sw=\{z\}$. By using (2.3), we have

$$\phi(\delta(Sw,Tu)) \leq \psi(\phi(M(w,u)), \qquad (3.27)$$

where

$$\begin{aligned}M(w,u) &= \max\big\{d(fw,gu),\delta(Sw,fw),\delta(gu,Tu),\\ &\quad \frac{1}{2}[D(fw,Tu)+D(Sw,gu)]\big\}\\ &= \max\big\{d(fw,gu),\delta(Sw,fw),\delta(fw,Tu),\\ &\quad \frac{1}{2}[0+D(Sw,fw)]\big\} \qquad (3.28)\\ &= \delta(Sw,fw)=\delta(Sw,z).\end{aligned}$$

Now by combining (3.27), (3.28) and $Tu=\{z\}$ we get

$$\phi(\delta(Sw,z)) \leq \psi(\phi(\delta(Sw,z)). \qquad (3.29)$$

From $\psi(t)<t$ for all $t>0$. We conclude that $\delta(Sw,z)=0$, so $Sw=\{z\}$. It follows that $\{gu\}=\{fw\}=Tu=Sw=\{z\}$.

Since the pair (T, g) is weakly compatible, then $gz = gTu = Tgu = Tz$. Now we prove that $Tz = \{z\}$. Using (2.3), we have

$$\phi(\delta(Sw, Tz)) \leq \psi(\phi(M(w, z)), \tag{3.30}$$

where

$$\begin{aligned} M(w, z) &= \max\{d(fw, gz), \delta(Sw, fw), \delta(gz, Tz), \\ &\quad \frac{1}{2}[D(fw, Tz) + D(Sw, gz)]\} \\ &= \delta(Tz, z). \end{aligned} \tag{3.31}$$

Now by combining (3.30), (3.31), we get

$$\phi(\delta(z, Tz)) \leq \psi(\phi(\delta(Tz, z)). \tag{3.32}$$

Consequently, $\delta(Tz, z) = 0$, so $Tz = \{z\}$. Hence $\{gz\} = Tz = \{z\}$.

Similarly, $\{fz\} = Sz = \{z\}$. Therefore, we obtain $\{fz\} = Sz = \{z\} = \{gz\} = Tz$. □

Uniqueness of the common fixed point follows from (2.3). Similarly, if $f(E)$ is closed, we can conclude by a similar argument as noted above that theorem is holds. This completes the proof. □

Remark 3.2. By taking $f = g = I_X$ and $E = X$ in Theorem 3.1, we conclude Theorem 2.11.

The following corollaries are direct results of Theorem 3.1, that extends Ćirić's result [7].

Corollary 3.3. *Let (X, d) be a complete metric space, and let E be a nonempty closed subset of X. Let $T, S, f, g : E \to E$ be four mappings verifying the following conditions:*
(A) (f, S) and (g, T) are weakly compatible,
(B) $T(E) \subseteq f(E)$ and $S(E) \subseteq g(E)$;
(C) $\phi(d(Sx, Ty)) \leq \psi(\phi(M(x, y)))$, for all $x, y \in E$,
where $\phi \in \Phi$, $\psi \in \Psi$ and where

$$M(x, y) := \max\{d(fx, gy), d(Sx, fx), d(gy, Ty), \frac{1}{2}[d(fx, Ty)+d(Sx, gy)]\}.$$

Assume that $f(E)$ or $g(E)$ is a closed subset of X. Then f, T, g and S have a unique common fixed point, that is, there exists $x \in X$ such that $fx = gx = Tx = Sx = x$.

Corollary 3.4. *Let (X, d) be a complete metric space, and let two mappings verifying the following conditions:*
(A) (g, T) are weakly compatible;
(B) $T(E) \subseteq g(E)$;

(C) $\phi(d(Tx, Ty)) \leq \psi(\phi(M(x, y)))$, *for all* $x, y \in E$,
where $\phi \in \Phi$, $\psi \in \Psi$ *and where*

$$M(x, y) := \max\left\{d(gx, gy), d(gx, Tx), d(gy, Ty), \frac{1}{2}[d(gx, Ty)+d(gy, Tx)]\right\}.$$

Assume that $g(E)$ *is a closed subset of* X*. Then* T *and* g *have a unique common fixed point, that is, there exists* $x \in X$ *such that* $Tx = gx = x$.

Proof. Let $T = S$ and $f = g$ and apply Corollary 3.3. □

The following example shows that Theorem 3.1 is a real extension of Theorem 2.11.

Example 3.5. Let $X = [0, +\infty)$ endowed with the Euclidean metric and let $E = [0, 1]$. Let $f, g : E \to X$ and $T, S : E \to B(E)$ defined by $fx = 2x$, $gx = 4x^2$, $Tx = [0, \frac{x^6}{6}]$ and $Sx = [0, \frac{x^3}{6}]$ for all $x \in X$.

Also, we define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$\phi(t) = t \ \forall\ t \in \mathbb{R}^+, \ \textit{and} \ \psi(t) = \frac{1}{7}t \ \forall\ t \in \mathbb{R}^+.$$

Obviously E and $f(E)$ and $g(E)$ are nonempty closed subsets of X. Also $\varphi \in \Phi$ and $\psi \in \Psi$ and (f, S) and (g, T) are weakly compatible in $x = 0$ and $T(E) \subseteq f(E)$ and $S(E) \subseteq g(E)$.

We next verify inequality (2.3) of Theorem 3.1. For all $x, y \in E$ where $x \neq y$

$$\begin{aligned}
\delta(Sx, Ty) &= \max\{\frac{x^3}{6}, \frac{y^6}{6}\} = \max\{\frac{1}{7}\frac{7x^3}{6}, \frac{1}{7}\frac{7y^6}{6}\} \\
&\leq \max\{\frac{1}{7}2x, \frac{1}{7}4y^2\} \\
&\leq \frac{1}{7}\max\{2x, 4y^2, |4y^2 - 2x|\} \\
&\leq \frac{1}{7}\max\{2x, 4y^2, |4y^2 - 2x|, \frac{1}{2}[D(x, Ty) + D(Sx, y)]\} \\
&\leq \frac{1}{7}M(x, y) = \psi(M(x, y))
\end{aligned}$$

Hence all conditions of Theorem 3.1 are satisfied. So f, T, g and S have a unique common fixed point in $x = 0$.

The condition (2.1) of Theorem 2.11 is not satisfied for $x = 0$ and $y = 1$. Because

$$\begin{aligned}
\frac{1}{6} &= \delta(S0, T1) \geq \psi(M(0, 1)) \\
&= \psi(\max\{d(0, 1), \delta(S0, 0), \delta(1, T1), \frac{1}{2}[D(0, T1) + D(1, S0)]\}) \\
&= \psi(1) = \frac{1}{7}
\end{aligned}$$

Therefore, our example does not satisfy the condition of Theorem 2.11. Hence Theorem 3.1 is a real extension of Theorem 2.11.

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