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Modified Laplace Decomposition Method for Singular IVPs in the Second-Order Ordinary Differential Equations

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Abstract. In this paper, we use modified Laplace decomposition method for solving initial value problems (IVP) of the second order ordinary differential equations. The proposed method can be applied to linear and non - linear problems.

Keywords: Singular initial value problems, Laplace decomposition method, Adomian decomposition method.

1. INTRODUCTION

In recent years, studies of initial value problems in the second order ordinary differential equations (ODEs) have been attracted the attention of many mathematicians and physicists. A large amount of literatures developed concerning Adomian decomposition method [1-4] and the related modification [5-6] to investigate various scientific models. This paper present a Laplace transform numerical scheme, based on the decomposition method, for solving linear and non - linear differential equations. The technique is described and illustrated with some numerical examples.

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The aim of this paper is to introduce a new reliable modification of Laplace decomposition method (MLDM) [17-21] . For convenience, we consider the general form of the second order non homogeneous ordinary differential equations with initial conditions is given below:

$$\mathrm{y}'' + \frac{2\mathrm{n}}{\mathrm{x}}y' + \frac{\mathrm{n}(\mathrm{n}-1)}{\mathrm{x}^2}\mathrm{y} + \mathrm{f}(\mathrm{x},\mathrm{y}) = \mathrm{g}(\mathrm{x}) \qquad \mathrm{n} = 1, 2 \quad (1.1)$$

$$\mathrm{y}\ (0) = \mathrm{A}\ , \quad \mathrm{y}'(0) = \mathrm{B},$$

Where $\mathrm{f}(\mathrm{x},\mathrm{y})$ is a real function, $\mathrm{g}(\mathrm{x})$ is given function, A and B are constants.

2. The Method

The technique consists first of applying Laplace transformation denoted throughout this paper by L to both sides of (1.1), hence when n= 1, we obtain

$$y'' + \frac{2}{x}y' + f(x,y) = g(x) \qquad (1.2)$$

Applying the Laplace transform denoted by L we have

$$L(xy'' + 2y' + xf(x,y) - xg(x)\ ) = 0$$

Using the properties of Laplace transform, we obtain

$$-L(y'')' + 2L\left(y'\right) + L\left(xf(x,y) - xg(x)\right) = 0 \qquad (1.3)$$

$$-(s^2F(s) - sf(0) - f'(0))' + 2\left(sF(s) - f(0)\right) + L\left(xf(x,y) - xg(x)\right) = 0$$

$$-(s^2F(s) - sA - B)' + 2\left(sF(s) - A\right) + L(xf(x,y) - xg(x)) = 0$$

Using the initial conditions, we have

$$-s^2F'(s) - y(0) + L\left(xf(x,y) - xg(x)\right) = 0 \qquad (1.4)$$

We decompose $F(x,y)$ in to two parts:

$$F(x,y) = R(y(x) + N(y(x)) \qquad (1.5)$$

where *R(y(x))* and *N(y(x))* denote the liner term and the nonlinear term respectively.

The Adomian decomposition method (ADM) polynomials can be used to handle Eq. (1.4) and to address the nonlinear term *N(y(x))* MLDM defines a solution $\mathrm{y}(x)$ and the nonlinear function $F(x,y)$ by infinity series

$$y(x) = \sum_{\mathrm{n}=0}^{8} \mathrm{y_n}(\mathrm{x}) \qquad (1.6)$$

$$F(x,y) = \sum_{\mathrm{n}=0}^{8} \mathrm{A_n} \qquad (1.7)$$

Where A_n are the Adomian polynomials and it can be calculated by formula give below.

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{n=0}^{8} \lambda^n u_n\right))_{\lambda=0}\right] \qquad n=0,\ 1,2,\ldots \quad (1.8)$$

Therefore Adomian's polynomials are given by:

$$\begin{aligned} A_0 &= N[u_0], \\ A_1 &= u_1 N'[u_0], \\ A_2 &= u_2 N'[u_0] + \frac{1}{2!}u_1^2 N''[u_0], \\ A_3 &= u_3 N'[u_0] + u_1 u_2 N''[u_0] + \frac{1}{3!}u_1^3 N'''[u_0], \end{aligned} \tag{1.9}$$

...

After substituting (1.6) and (1.7) into (1.4), we have

$$\begin{aligned} &-s^2 L'\left\{\sum_{n=0}^{8} y_n(x)\right\} - y\ (0) - L\{xg(x)\} \\ &+L\left\{xR\left[\sum_{n=0}^{8} y_n(x)\right] + x\sum_{n=0}^{8} A_n(x)\right\} = 0 \end{aligned} \tag{1.10}$$

Using the linearity of Laplace transform, it follows that

$$\begin{aligned} &-s^2 \sum_{n=0}^{8} L'\{y_n(x)\} - y(0) - L\{xg(x)\} \\ &+\sum_{n=0}^{8} L\{xR[y_n(x)] + xA_n(x)\} = 0 \end{aligned} \tag{1.11}$$

In general, the recursive relation is given by:

$$\begin{aligned} L'\{y_0(x)\} &= -s^{-2}y(0) - s^{-2}L\{xg(x)\}, \\ L'\{y_{n+1}(x)\} &= s^{-2}L\{xR[y_n(x)] + xA_n(x)\}, \end{aligned} \tag{1.12}$$

By integrating both sides of Eq. (1.12), we have

$$\begin{aligned} L\{y_0(x)\} &= \int \left[-s^{-2}y(0) - s^{-2}L\{xg(x)\}\right] ds\ , \\ L\{y_{n+1}(x)\} &= \int s^{-2}L\{xR[y_n(x)] + xA_n(x)\}\, ds \end{aligned} \tag{1.13}$$

Taking the inverse Laplace transform to Eq. (1.13) one obtains

$$y_0(x) = L^{-1}\left\{\int \left[-s^{-2}y(0) - s^{-2}L\{xg(x)\}\right] ds\right\} = H(x). \tag{1.14}$$

$$\mathrm{y_{n+1}}\left(\mathrm{x}\right) = \mathrm{L}^{-1}\left\{\int \mathrm{s}^{-2}\mathrm{L}\left\{\mathrm{xR}\left[\mathrm{y_n}(\mathrm{x})\right] + \mathrm{xA_n}(\mathrm{x})\right\}\mathrm{ds}\right\},$$

where H(x) represents the term arising from source equation and prescribed initial condition. The initial solution is important, and the choice of Eq. (1.14) as the initial solution always leads to noise oscillation during the iteration procedure.

3. Modified Laplace decomposition method

In order to overcome the shortcoming, we assume That H (x) can be divided into the sum of two parts namely $\mathrm{H_0(x)}$ and $\mathrm{H_1(x)}$, therefore we get

$$H(x) = \mathrm{H_0}(x) + \mathrm{H_1}(x). \tag{1.15}$$

Instead of the iteration procedure expresed at eq 14 we suggest the following modification

$$\mathrm{y_0}(x) = \mathrm{H_0}(x), \tag{1.16}$$

$\mathrm{y_1}\ (\mathrm{x}) = \mathrm{H_1}\ (\mathrm{x}) + \mathrm{L}^{-1}\left\{\int \mathrm{s}^{-2}\mathrm{L}\left\{\mathrm{xR}\left[\mathrm{y_n}(\mathrm{x})\right] + \mathrm{xA_n}(\mathrm{x})\right\}\mathrm{ds}\right\}$

$\mathrm{y_{n+1}}(\mathrm{x}) = \mathrm{L}^{-1}\left\{\int \mathrm{s}^{-2}\mathrm{L}\left\{\mathrm{xR}\left[\mathrm{y_n}(\mathrm{x})\right] + \mathrm{xA_n}(\mathrm{x})\right\}\mathrm{ds}\right\},$

The solution through the modified Laplace decomposition method highly depends upon the choice of $\mathrm{H_0(x)}$ and $\mathrm{H_1(x)}$.

4. Numerical Examples

Example 1. Consider the nonlinear singular IVP

$$y'' + \frac{2}{t}y' + y - \left(t^2 + 6\right) = 0, \qquad y\left(0\right) = y'\left(0\right) = 0. \tag{1.17}$$

According to the MLDM and initial conditions we have

$$-s^2 L'\left(y\right) + L\left(ty\right) - L\left(6t + t^3\right) = 0$$

The recursive relation is obtained as

$$L'\left(y_0\right) = -s^{-2}L(6t)$$

$\mathrm{y_0(t)} = \mathrm{L}^{-1}(\int -\mathrm{s}^{-2}\mathrm{L}\ (6\mathrm{t})\ ds)$

$$y_0\left(t\right) = L^{-1}\left(\int\int (-s^{-2}(\frac{6}{s^2}(-s^{-2}(\frac{6}{s^2}\right) ds = L^{-1}\left(\int \frac{-6}{s^4}\right) ds = L^{-1}(\frac{-6s^{-3}}{-3})$$

$\mathrm{L}^{-1}(2\mathrm{s}^{-3}) = \mathrm{L}^{-1}(\frac{2}{\mathrm{s}^3}) = \mathrm{t}^2$ $\qquad \mathrm{y_0}(t) = \mathrm{t}^2$

$L'(y_1) = s^{-2}\mathrm{L}\ ((\mathrm{t}y_0) - L(t^3))$

$$y_1\left(t\right) = L^{-1}\left(\int s^{-2}(L\left(ty_0\right) - L\left(t^3\right))\right) ds \qquad y_1(t) = 0$$

...

$$L'\left(y_n\right) = s^{-2} L\left(t y_{n-1}\right)$$

$$y_n\left(t\right) = L^{-1}(s^{-2} L\left(t y_{n-1}\right) ds \qquad y_n\left(t\right) = 0, n > 1$$

The solution series in general gives

$$y(t) \ = \mathrm{y}_0\left(\mathrm{t}\right) + \mathrm{y}_1\left(\mathrm{t}\right) + \mathrm{y}_2\left(\mathrm{t}\right) + \ldots$$

The exact solution is $y(t) \ = \ t^2$

Example2.consider the linear singular IVP

$$y'' + \frac{2}{t} \ y' - 10y = 12 - 10t^4 \qquad , \qquad y(0) = y'(0) = 0 \qquad (1.18)$$

According to the MLDM and initial conditions we have

$$-s^2 L'\left(y\right) + L\left(-10yt\right) + L(-12t + 10t^5) = 0$$

The recursive relation is obtained as

$$y_0\left(t\right) = L^{-1}\left(\int -s^{-2} L\left(12t - 10t^5\right)\right) ds$$

y_0(t)=$L^{-1}\left(\frac{4}{s^3} - \frac{1200}{7s^7}\right)$ $\qquad$ $y_0\left(t\right) = 2t^2 - \frac{5}{21}t^6$

$$y_1\left(t\right) = L^{-1}(\int s^{-2} L(-10 y_0 t)) = L^{-1}(s^{-2} L(-20t^3 + \frac{50}{21}t^7))$$

y_1(t)=$t^4 - \frac{25}{756}t^8$

$$y_2\left(t\right) = L^{-1}\left(\int s^{-2} L(-10 y_1 t)\right) ds$$

$$y_2\left(t\right) = \frac{5}{21}t^6 - \frac{25}{8316}t^{10}$$

...

The solution series in general gives

$$y(t) \ = \mathrm{y}_0\left(\mathrm{t}\right) + \mathrm{y}_1\left(\mathrm{t}\right) + \mathrm{y}_2\left(\mathrm{t}\right) + \ldots$$

So the exact solution is obtained as

$$y\left(t\right) = 2t^2 + t^4$$

When n = 2 in Eq.(1.1),we obtain

$$y'' + \frac{4}{\mathrm{x}}\mathrm{y}' + \frac{2}{\mathrm{x}^2}\mathrm{y} + \mathrm{f}\left(\mathrm{x}, \mathrm{y}\right) = \mathrm{g}\left(\mathrm{x}\right) \qquad (1.19)$$

$y\ (0) = \mathrm{A},\ y'\ (0) = \mathrm{B}$

By applying x^2 to both sides of (1.19) we have

$\mathrm{x}^2 y'' + 4\mathrm{x}\ y' + 2\ \mathrm{y}\ + \mathrm{x}^2\mathrm{f}\left(\mathrm{x}, \mathrm{y}\right) = \mathrm{x}^2\mathrm{g}(\mathrm{x})$

Applying the Laplace transform (denoted by L) we have

$$L\left(x^2y^{''}\right)+4L\left(xy^{'}\right)+2L\left(y\right)+L\left(x^2f\left(x,y\right)-x^2g\left(x\right)\right)=0$$

$$L(s^2F\left(S\right)-sf\left(0\right)-f'\left(0\right))^{''}-4L(sF\left(s\right)-f\left(0\right))'+2F\left(s\right)+L\left(x^2f\left(x,y\right)-x^2g\left(x\right)\right)=0$$

Using the initial conditions, we have

$$s^2f^{''}\left(s\right)+L\left(x^2f\left(x,y\right)-x^2g\left(x\right)\right)=0$$

Taking the inverse Laplace transform to Eq. (1.13) we obtains

$$\mathrm{y_0\left(x\right)=L^{-1}\left\{\int\int\left[s^{-2}L\left\{x^2g(x)\right\}\right]ds\ ds\right\}=H\left(x\right).}$$

$$\mathrm{y_{n+1}\left(x\right)=L^{-1}\left\{\int\int s^{-2}-(L\left\{x^2R\left[y_n\left(x\right)\right]+x^2A_n\left(x\right)\right\})ds\ ds\right\}}$$

Example3. Consider the linear singular IVP

$$y^{''}+\frac{4}{\mathrm{t}}\mathrm{y}^{'}+\frac{2}{\mathrm{t}^2}\mathrm{y}=12\quad,\qquad\mathrm{y\left(0\right)=y^{'}\left(0\right)=0}\tag{1.20}$$

According to the MLDM , we have

$$\mathrm{s^2L^{''}\left(y\right)+L\left(t^2\ 12\right)=0}$$

$$y_0\left(t\right)=L^{-1}\left(\int\int\mathrm{s^{-2}L(12t^2)}\right)\mathrm{ds\ ds)}$$

$\mathrm{y_0\left(t\right)=L^{-1}(\int\int s^{-2}(\frac{24}{s^3})\ ds)}$ $\mathrm{y_0\left(t\right)=t^2}$

The exact solution is $y(t)=\ \mathrm{t}^2$.

Example 4. consider the nonlinear singular IVP

$$y''+\frac{4}{\mathrm{t}}\mathrm{y}^{'}+\frac{2}{\mathrm{t}^2}\mathrm{y}+\mathrm{y}^2=\mathrm{t}^4+12,\tag{1.21}$$

$$y(0)=y'(0)=0.$$

According to the MLDM, we have

$$s^2L^{''}\left(y\right)+L(t^2y^2)-L(t^6+12t^2)=0$$

The recursive relation is obtained as

$$\mathrm{y_0\left(t\right)=L^{-1}\left(\int\int s^{-2}L(12t^2)\right)dsds\qquad y_0(t)=t^2}$$

$$\mathrm{y_1\left(t\right)=L^{-1}(\int\int s^{-2}(-(L(t^2y_0^2))+L(t^6))ds\ ds}$$

$$\mathrm{y_1\left(t\right)=0}$$

$$\mathrm{y_n\left(t\right)=L^{-1}\left(\int\int s^{-2}(-L(t^2y_{n-1}^2)\right)dsds}$$

$$\mathrm{y_n\left(t\right)=0\quad,\ \forall n>1}$$

The solution series in general gives

$$y(t) = \mathrm{y}_0(\mathrm{t}) + \mathrm{y}_1(\mathrm{t}) + \mathrm{y}_2(\mathrm{t}) + \ldots$$

The exact solution is $y(t) = t^2$

Example 5. Consider the singular IVP

$$y'' + \frac{4}{t}y' + \frac{2}{t^2}y + ty = 20t + t^4 \quad , \quad y(o) = y'(0) = 0 \quad (1.22)$$

According to the MLDM, we have

$$s^2 L''(y) - L\left(20t^3 + t^6\right) + L\left(t^3 y\right) = 0.$$

The recursive relation is obtained as

$$\mathrm{y}_0(\mathrm{t}) = \mathrm{L}^{-1}\left(\int\int \mathrm{s}^{-2}\mathrm{L}(20\mathrm{t}^3 + \mathrm{t}^6)\right)\mathrm{dsds}$$

$$\mathrm{y}_0(\mathrm{t}) = \mathrm{t}^3 + \frac{\mathrm{t}^6}{56}$$

$$\mathrm{y}_1(\mathrm{t}) = \mathrm{L}^{-1}(\int\int \mathrm{s}^{-2}(-(\mathrm{L}(\mathrm{t}^3\mathrm{y}_0))\mathrm{ds\ ds})$$

$$\mathrm{y}_1(\mathrm{t}) = -\frac{\mathrm{t}^6}{56} - \frac{\mathrm{t}^9}{6160}$$

$$\mathrm{y}_2(\mathrm{t}) = \mathrm{L}^{-1}(\int\int \mathrm{s}^{-2}(-(\mathrm{L}(\mathrm{t}^3\mathrm{y}_1))\mathrm{ds\ ds})$$

$$\mathrm{y}_2(\mathrm{t}) = \frac{\mathrm{t}^9}{6160} + \frac{\mathrm{t}^{12}}{1121120}$$

...

The solution series in general gives

$$y(t) = \mathrm{y}_0(\mathrm{t}) + \mathrm{y}_1(\mathrm{t}) + \mathrm{y}_2(\mathrm{t}) + \ldots$$

The exact solution is $y(t) = \mathrm{t}^3$

5. Discussion and Conclusion

In the paper, modified Laplace decomposition method (MLDM) is applied to linear and nonlinear differential equation with initial conditions. The MLDM proposed in this investigation is simple and effective for solving in the second order of IVP and can provide an accuracy approximate solution or exact solution. Mathematica has been used for computations in this paper.

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