

Some New Results On Semi Fully Fuzzy Linear Programming Problems

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ABSTRACT. There are two interesting methods, is assumed in the literature for solving fuzzy linear programming problems in which the elements of coefficient matrix of the constraints are represented by real numbers and rest of the parameters are represented by symmetric trapezoidal fuzzy numbers. The first method, named as fuzzy primal simplex method, is assumed an initial primal basic feasible solution is at hand. The second method, named as fuzzy dual simplex method, is assumed an initial dual basic feasible solution is at hand. In this paper, the shortcomings of these methods are pointed out and to overcome these shortcomings, a new method is proposed to determine the fuzzy optimal solution of such fuzzy problems. The advantages of the proposed method over existing methods are discussed. To illustrate the proposed method a numerical example is solved by using the proposed method and the obtained results are discussed.

Keywords: Linear programming, Symmetric trapezoidal fuzzy number, Fuzzy primal simplex method, Fuzzy dual simplex method.

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Received: 23 Jan 2013

Revised: 21 July 2013

Accepted: 25 July 2013

1. INTRODUCTION

The fuzzy set theory is currently being applied considerably in many fields these days. One of these is linear programming problems. However, in most practical applications of linear programming the possible values of the parameters required in the modeling of the problem are provided by either a decision maker subjectively or a statistical inference from the past data due to the uncertainty existence. In order to reflect this uncertainty, the model of the problem is often constructed with fuzzy data [8]. Since the fuzziness may appear in a linear programming problem in many ways, the definition of fuzzy linear programming problem is not unique. Kumar et al [4] pointed out that there is no method in literature to find the exact fuzzy optimal solution of Fully Fuzzy Linear Programming (FFLP) problems and proposed a new method to find the fuzzy optimal solution of these problems with equality constraints having nonnegative fuzzy variables and unrestricted fuzzy coefficients. In the literature, there may exist several kinds of FFLP problems but each model and/or the associated approaches have some shortages in the solving process (see in [2-7]). In this paper, we consider a kind of fuzzy linear programming problems in which the elements of coefficient matrix of the constraints are represented by real numbers and rest of the parameters are represented by symmetric trapezoidal fuzzy numbers. We name this kind of fuzzy linear programming problem as Semi Fully Fuzzy Linear Programming (SFFLP) problem.

Ganesan and Veeramani [2] defined SFFLP problems for the first time and then extend the primal simplex method in crisp environment for finding the fuzzy optimal solution. Their method begins with a fuzzy primal basic feasible solution for SFFLP problem and moves to an optimal basis by walking through a sequence of fuzzy primal feasible bases of SFFLP problem. All the bases with the possible exception of the optimal basis found in fuzzy primal simplex method don't satisfy the optimality criteria for SFFLP problem. Also their method is not efficient when a primal basic feasible solution is not at hand.

Nasserri and Mahdavi-Amiri [6] and Nasserri et al. [7] developed the concept of duality for the SFFLP problem proved the duality results in fuzzy sense. Based on these results, Ebrahimnejad and Nasserri [1] generalized the dual simplex method in crisp environment for obtaining the fuzzy optimal solution. Their method begins with a basic dual solution and proceeds by pivoting through a series of dual basic fuzzy solution until the associated complementary primal basic solution is feasible. However, the fuzzy dual simplex method needs to an initial dual basic feasible solution. Here, we develop the fuzzified version of conventional primal-dual method of linear programming problems that any

dual feasible solution, whether basic or not, is adequate to initiate this method.

The rest of this paper is organized as follows: In Section 2, we review the basic definitions and results on fuzzy sets and related topics. Section 3 gives the definition of SFFLP problem proposed by Ganesan and Veeramani. We give a new method for solving SFFLP problem in Section 4 and explain it by a numerical example. The conclusions are discussed in Section 6.

2. PRELIMINARIES

In this section, some basic definitions, arithmetic operations of fuzzy numbers and an existing ranking approach for comparing fuzzy numbers are presented [1,2].

2.1. Basic definitions.

Definition 2.1. Let \tilde{a} be a fuzzy set in \mathbb{R} . Then, \tilde{a} is a fuzzy number if and only if there exist a closed interval $[m, n] \neq \emptyset$ such that

$$\mu_{\tilde{a}}(x) = \begin{cases} L(x), & \text{for } x \in (-\infty, m], \\ 1, & \text{for } x \in [m, n], \\ R(x), & \text{for } x \in [n, \infty), \end{cases}$$

where $L : (-\infty, m] \rightarrow [0, 1]$ is monotonic increasing, continuous from the right and $L(x) = 0$ for $x \in (-\infty, w_1], w_1 < m$; $R : [n, \infty) \rightarrow [0, 1]$ is monotonic decreasing, continuous from the left and $R(x) = 0$ for $x \in [w_2, \infty), w_2 > n$.

As the set of fuzzy numbers is rather large and their arithmetic is in general computationally expensive based on Zadeh's extension principle, it is imperative to define and select a few special types of fuzzy numbers to be used for real life applications. Some such special types of fuzzy numbers and their arithmetic are being discussed here which will be used extensively in later section on fuzzy linear programming problems.

Definition 2.2. A fuzzy number $\tilde{a} = (a_1, a_2, \alpha_1, \alpha_2)$ is called a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-(a_1-\alpha_1)}{\alpha_1}, & \text{for } a_1 - \alpha_1 \leq x \leq a_1, \\ 1, & \text{for } a_1 \leq x \leq a_2, \\ \frac{(a_2+\alpha_2)-x}{\alpha_2}, & \text{for } a_2 \leq x \leq a_2 + \alpha_2, \\ 0, & \text{else.} \end{cases}$$

Remark 2.3. If $\alpha_1 = \alpha_2 = \alpha$ in the trapezoidal fuzzy number $\tilde{a} = (a^L, a^U, \alpha_1, \alpha_2)$, we obtain a symmetric trapezoidal fuzzy number, and

we denote it as $\tilde{a} = (a^L, a^U, \alpha, \alpha)$.

2.2. Arithmetic on fuzzy numbers.

Let $\tilde{a} = (a_1, a_2, \alpha, \alpha)$ and $\tilde{b} = (b_1, b_2, \beta, \beta)$ be two symmetric trapezoidal fuzzy numbers. Then the arithmetic operations on \tilde{a} and \tilde{b} are given by (taken from [2]):

$$\text{Addition: } \tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, \alpha + \beta, \alpha + \beta)$$

$$\text{Subtraction: } \tilde{a} - \tilde{b} = (a_1 - b_2, a_2 - b_1, \alpha + \beta, \alpha + \beta)$$

$$\text{Multiplication: } \tilde{a}\tilde{b} = ((\frac{a_1+a_2}{2})(\frac{b_1+b_2}{2}) - \omega, (\frac{a_1+a_2}{2})(\frac{b_1+b_2}{2}) + \omega, |a_2\beta + b_2\alpha|, |a_2\beta + b_2\alpha|), \text{ where}$$

$$\omega = \frac{t_2 - t_1}{2}, t_1 = \min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, t_2 = \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}.$$

From the above definition it can be seen that

$$\lambda \geq 0, \lambda \in \mathbb{R}; \quad \lambda \tilde{a} = (\lambda a_1, \lambda a_2, \lambda \alpha, \lambda \alpha)$$

$$\lambda < 0, \lambda \in \mathbb{R}; \quad \lambda \tilde{a} = (\lambda a_2, \lambda a_1, -\lambda \alpha, -\lambda \alpha).$$

Note that depending upon the need, one can also use a smaller ω in the definition of multiplication involving symmetric trapezoidal fuzzy numbers.

2.3. Order on fuzzy numbers.

Ranking of fuzzy numbers is an important issue in the study of fuzzy set theory. Ranking procedures are also useful in various applications and one of them will be in the study of fuzzy mathematical programming in later sections. There are numerous methods proposed in the literature for the ranking of fuzzy numbers, some of them seem to be good in a particular context but not in general. Here, we describe only a simple method for the ordering of fuzzy numbers.

Let $\tilde{a} = (a_1, a_2, \alpha, \alpha)$ and $\tilde{b} = (b_1, b_2, \beta, \beta)$ be two symmetric trapezoidal fuzzy numbers. Define the relations \preceq and \approx as given below:

$\tilde{a} \preceq \tilde{b}$ (or $\tilde{b} \succeq \tilde{a}$) if and only if

$$(i) \quad \frac{(a_1 - \alpha) + (a_2 + \alpha)}{2} < \frac{(b_1 - \beta) + (b_2 + \beta)}{2},$$

that is $\frac{a_1 + a_2}{2} < \frac{b_1 + b_2}{2}$ (in this case, we may write $\tilde{a} \prec \tilde{b}$),

$$(ii) \quad \text{or } \frac{a_1 + a_2}{2} = \frac{b_1 + b_2}{2}, b_1 < a_1 \text{ and } a_2 < b_2,$$

$$(iii) \quad \text{or } \frac{a_1 + a_2}{2} = \frac{b_1 + b_2}{2}, b_1 = a_1, a_2 = b_2 \text{ and } \alpha \leq \beta.$$

3. SEMI FULLY FUZZY LINEAR PROGRAMMING PROBLEM

Ganesan and Veeramani [2] introduced a new type fuzzy linear programming problems in which the elements of coefficient matrix of the constraints are represented by real numbers and rest of the parameters are represented by symmetric trapezoidal fuzzy numbers. We named these kinds of problems as Semi Fully Fuzzy Linear Programming (SF-FLP) problems. Here we first review two existing methods for solving SFFLP problem and then point out the shortcomings of these methods.

A SFFLP problem is defined as follows:

$$\begin{aligned} \min \quad & \tilde{z} \simeq \tilde{c}\tilde{x} \\ \text{s.t.} \quad & A\tilde{x} \succeq \tilde{b} \\ & \tilde{x} \succeq \tilde{0} \end{aligned} \quad (3.1)$$

where $\tilde{b} \in (F(\mathbb{R}))^m$, $\tilde{c}^T \in (F(\mathbb{R}))^n$, $A \in \mathbb{R}^{m \times n}$ are given and $\tilde{x} \in (F(\mathbb{R}))^n$ is to be determined.

Definition 3.1. Let $F(X)$ be the set of all fuzzy feasible solutions of (3.1). Any vector $\tilde{x}^* \in F(X)$ is said to be an optimum solution to (3.1) if $c\tilde{x}^* \preceq c\tilde{x}$ for all $\tilde{x} \in F(X)$.

Definition 3.2. Suppose $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ solves $A\tilde{x} \simeq \tilde{b}$. If all $\tilde{x}_j \simeq (-\bar{x}_j, \bar{x}_j, \alpha_j, \alpha_j)$ for some $\bar{x}_j \geq 0$ and $\alpha_j \geq 0$, then \tilde{x} is said to be a fuzzy basic feasible solution. If $\tilde{x}_j \not\simeq (-\bar{x}_j, \bar{x}_j, \alpha_j, \alpha_j)$ for some $\bar{x}_j \geq 0$ and $\alpha_j \geq 0$, then \tilde{x} has some non-zero components, say $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_k$, $1 \leq k \leq m$. Then $A\tilde{x} \simeq \tilde{b}$ can be written as:

$$a_1\tilde{x}_1 + a_2\tilde{x}_2 + \dots + a_k\tilde{x}_k + a_{k+1}(-\bar{x}_{k+1}, \bar{x}_{k+1}, \alpha_{k+1}, \alpha_{k+1}) + \dots + a_n(-\bar{x}_n, \bar{x}_n, \alpha_n, \alpha_n) \simeq \tilde{b}.$$

If the columns a_1, a_2, \dots, a_k corresponding to these non-zero components $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_k$ are linear independent, then \tilde{x} is said to be fuzzy basic feasible solution.

Remark 3.3. Given a system of m simultaneous fuzzy linear equations involving symmetric trapezoidal fuzzy numbers in n unknowns $A\tilde{x} \simeq \tilde{b}$, where A is an $(m \times n)$ real matrix and rank of A is m . Let B be any $(m \times m)$ matrix formed by m linearly independent of A . In this case $\tilde{x} = (\tilde{x}_B, \tilde{x}_N) = (B^{-1}\tilde{b}, \tilde{0})$ is a fuzzy basic feasible solution.

Suppose a fuzzy basic feasible solution of (3.1) with basis B is at hand. Let y_k and \tilde{w} be the solutions to $By_k = a_k$ and $\tilde{w}B = \tilde{c}_B$, respectively. Define $\tilde{z}_j = \tilde{w}a_j = \tilde{c}_B B^{-1}a_j$, where $\tilde{c}_B = (\tilde{c}_{B_1}, \dots, \tilde{c}_{B_m})$. Now, we are in a position to state some important theorems of fully fuzzy linear programming problems concerning to improving a fuzzy feasible

solution, unbounded criteria and the optimality conditions which proved by Ganesan and Veeramani [2].

Theorem 3.4. *If we have a fuzzy basic feasible solution with fuzzy objective value \tilde{z} such that $\tilde{z}_k \succ \tilde{c}_k$ for some nonbasic variable \tilde{x}_k , and $y_k \not\leq 0$, then it is possible to obtain a new basic feasible solution with new fuzzy objective value $\tilde{\tilde{z}}$, that satisfies $\tilde{\tilde{z}} \preceq \tilde{z}$.*

Theorem 3.5. *If we have a fuzzy basic feasible solution with $\tilde{z}_k \succ \tilde{c}_k$ for some nonbasic variable \tilde{x}_k , and $y_k \leq 0$, then the problem (3.1) has an unbounded optimal solution.*

Theorem 3.6. *If a fuzzy basic solution $\tilde{x}_B = B^{-1}\tilde{b}$, $\tilde{x}_N \simeq \tilde{0}$ is feasible to (3.1) and $\tilde{z}_j \preceq \tilde{c}_j$ for all $j, 1 \leq j \leq n$, then the fuzzy basic feasible solution is a fuzzy optimal feasible solution to (3.1).*

3.1. Primal simplex method for SFFLP problem.

Ganesan and Veeramani [2] based on these theorems proposed a new algorithm for solving FFLP problems. Here, we give a summary of their method in tableau format.

Algorithm 1. A fuzzy primal simplex method for the SFFLP problem

Initialization Step

Suppose an initial fuzzy basic feasible solution with basis B is at hand. Form the following initial tableau.

Table 1. The initial SFFLP primal simplex tableau

Basis	\tilde{x}_B	\tilde{x}_N	R.H.S.
\tilde{z}	$\tilde{0}$	$\tilde{z}_N - \tilde{c}_N = \tilde{c}_B Y_N - \tilde{c}_N$	$\tilde{\tilde{z}} = c_B B^{-1} \tilde{b}$
\tilde{x}_B	I	Y_N	$\tilde{\tilde{b}} = B^{-1} \tilde{b}$

Main steps

- (1) Calculate $\tilde{z}_j - \tilde{c}_j$ for all nonbasic variables. Suppose $\tilde{z}_j - \tilde{c}_j = (h_j^L, h_j^U, h_j, h_j)$. Let

$$h_k^U + h_k^L = \underset{j \in R}{Max} \{h_j^U + h_j^L\}$$

where T is the index set of the current nonbasic variables.

If $h_k^U + h_k^L \leq 0$, then stop; the current solution is optimal. Otherwise, go to (2).

- (2) Let $y_k = B^{-1}a_k$. If $y_k \leq 0$, then stop; the problem is unbounded. Otherwise, suppose $\tilde{b}_i = (\bar{b}_i^U, \bar{b}_i^L, \alpha_i, \alpha_i)$ and determine the index

r as follows:

$$\frac{\bar{b}_r^U + \bar{b}_r^L}{y_{rk}} = \min_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i^U + \bar{b}_i^L}{y_{ik}} \mid y_{ik} > 0 \right\}.$$

- (3) Update the tableau by pivoting at y_{rk} . Update the fuzzy basic and nonbasic variables where \tilde{x}_k enters the basis and \tilde{x}_{B_r} leaves the basis, and go to (1).

3.2. Dual simplex method for SFFLP problem.

It is common that in linear programming, duality theory takes a central place. In this theory, for a given problem, named primal, one analyzes another problem called the dual and the relationship between these two problems is used to highlight the properties of the optimal solutions to the both problems. It is imperative, therefore, to study duality in fuzzy linear programming. Here, we review the duality results in SFFLP problems leading to a new method for solving SFFLP problems (taken from Nasser and Mahdavi-Amiri [6] and Nasser et al. [7]).

Definition 3.7. Dual of the SFFLP problem (3.1) is defined as follows:

$$\begin{aligned} \max \quad & \tilde{u} \approx \tilde{w} \tilde{b} \\ \text{s.t.} \quad & \tilde{w} A \preceq \tilde{c} \\ & \tilde{w} \succeq \tilde{0} \end{aligned} \quad (3.2)$$

where $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_m) \in F(\mathbb{R})^m$ is including the fuzzy variables corresponding to constraints of problem (3.1). In fact, \tilde{w}_i , $i = 1, \dots, m$ is defined for the i th constraint of the problem (3.1). We name this problem as the DSFFLP problem.

Nasser and Mahdavi-Amiri [6] and Nasser et al. [7] proved the relationships between the SFFLP problem and its corresponding dual DSFFLP problem such as the weak duality property, the Strong duality and complementary. Then based on these properties Ebrahimnejad and Nasser [1] proposed the dual simplex method for solving SFFLP problem.

Algorithm 2. A fuzzy dual simplex algorithm

Initialization Step:

Suppose that basis B be dual feasible for the problem (3.2). Form the Tableau 1 as an initial dual feasible simplex tableau. Suppose $\tilde{z}_j - \tilde{c}_j = (h_j^U, h_j^L, \alpha_j, \alpha_j)$, so $h_j^U + h_j^L \leq 0$ for all j .

Main steps

- (1) Suppose $\tilde{b} = B^{-1}\tilde{b}$. If $\tilde{b} \succeq \tilde{0}$, then Stop; the current fuzzy solution is optimal.
 Else suppose $\tilde{b}_i = (\bar{b}_i^U, \bar{b}_i^L, \alpha_i, \alpha_i)$ and let $\bar{b}_r^U + \bar{b}_r^L = \min_{1 \leq i \leq m} \{\bar{b}_i^U + \bar{b}_i^L\}$.
- (2) If $y_{rj} \geq 0$ for all j , then Stop; the problem (3.2) is infeasible.
 Else select the pivot column k by the following test:

$$\frac{h_k^U + h_k^L}{y_{rk}} = \min_{1 \leq j \leq n} \left\{ \frac{h_j^U + h_j^L}{y_{rj}} \mid y_{rj} < 0 \right\}.$$

- (3) Update the tableau by pivoting at y_{rk} . Update the fuzzy basic and nonbasic variables where x_k enters the basis and x_{B_r} leaves the basis, and go to (1).

4. PRIMAL-DUAL SIMPLEX METHOD FOR SFFLP PROBLEM

It needs to point out that the method which is proposed by Ganesan and Veeramani in [2], starts with a fuzzy basic feasible solution for FLP and moves to an optimal basis by walking through a sequence of fuzzy feasible bases of SFFLP. All the bases with the possible exception of the optimal basis obtained in fuzzy primal simplex algorithm don't satisfy the optimality criteria for SFFLP or feasibility condition for DSFFLP. But their method is not efficient when a primal basic feasible solution is not at hand. So, Ebrahimnejad and Nasseri [1] developed a new dual simplex algorithm to overcome this shortcoming by using the duality results which have been proposed by Nasseri and Mahdavi-Amiri [6] and Nasseri et al. [7]. This algorithm starts with a dual basic feasible solution, but primal basic infeasible solution and walks to an optimal solution by moving among adjacent dual basic feasible solutions. However, the dual simplex method for solving FLP problem needs to an initial dual basic feasible solution. Here, we develop the fuzzified version of conventional primal-dual method of linear programming problems that any dual feasible solution, whether basic or not, is adequate to initiate this method.

Algorithm 3. A Fuzzy Primal-Dual Simplex Algorithm

- (1) {dual feasibility} Choose a fuzzy vector \tilde{w} such that $\tilde{z}_j - \tilde{c}_j \preceq \tilde{0}$ for all j .
- (2) $Q = \{j : \tilde{w}a_j - \tilde{c}_j \approx \tilde{0}\}$ and solve the following restricted fuzzy primal problem:

$$\begin{aligned}
& \min \sum_{i \in \Omega} \tilde{0} \tilde{x}_j + \tilde{\mathbf{1}} \tilde{x}_a \\
\text{s.t. } & \sum_{i \in \Omega} a_j \tilde{x}_j + I \tilde{x}_a \approx \tilde{b} \\
& \tilde{x}_j \succeq \tilde{0}, \quad \text{for } j \in \Omega, \\
& \tilde{x}_a \succeq \tilde{0},
\end{aligned} \tag{4.1}$$

where $\tilde{x}_a = (\tilde{x}_{1a}, \dots, \tilde{x}_{ma}) \in F(\mathbb{R})^m$ and $\tilde{\mathbf{1}} = ((1, 1, 0, 0), \dots, (1, 1, 0, 0)) \in (F(\mathbb{R}))^m$.

If $\tilde{x}_0 \approx \tilde{0}$ **then Stop** (the current solution is optimal)

else let \tilde{v} be the optimal dual fuzzy solution to the restricted fuzzy primal problem.

- (3) **If** $\tilde{v} a_j \preceq \tilde{0}$, for all j **then stop** (the FLP problem is infeasible)
else let

$$\alpha = \alpha_k = -\frac{\hat{w}_{1k} + \hat{w}_{2k}}{\hat{v}_{1k} + \hat{v}_{2k}} = \min_j \left\{ -\frac{\hat{w}_{1j} + \hat{w}_{2j}}{\hat{v}_{1j} + \hat{v}_{2j}} \mid \tilde{v} a_j \succ \tilde{0} \right\}.$$

and replace \tilde{w} by $\tilde{w} + \alpha \tilde{v}$ and **go to** (2).

5. CONCLUSIONS

In this paper, we review two interesting methods for solving fuzzy linear programming problems in which the elements of coefficient matrix of the constraints are represented by real numbers and rest of the parameters are represented by symmetric trapezoidal fuzzy numbers. Then, we introduced a fuzzy primal-dual algorithm for solving the SFFLP problems directly without converting them to crisp linear programming problems, based on the interesting results which have been established by Ganesan and Veeramani [2]. This approach can be expected to be efficient if an initial dual fuzzy solution can be computed readily.

ACKNOWLEDGMANT

The authors would like to thank the anonymous reviewers for the valuable suggestions They also led to an improvement in the earlier version of our paper. they thank Dr. Ali Ebrahimejad for his valuable comments.

REFERENCES

- [1] A. Ebrahimnejad and S.H. Nasseri, Linear programs with trapezoidal fuzzy numbers: a duality approach, *International Journal of Operations Research*, **13** (1)(2012) 67-89.
- [2] K. Ganesan and P. Veeramani, Fuzzy linear programming with trapezoidal fuzzy numbers, *Annals of Operations Research*, **143** (2006) 305-315.
- [3] S.M. Hashemi, M. Modarres, E. Nasrabadi and M.M. Nasrabadi, Fully fuzzified linear programming solution and duality, *International Journal of Fuzzy Systems*, **17** (2006) 253-261.
- [4] A. Kumar, J. Kaur and P. Singh, A new method for solving fully fuzzy linear programming problems, *Applied Mathematical Modelling*, **35** (2011) 817-823.
- [5] F.H. Lotfi, T. Allahviranloo, M.A. Jondabeha and L. Alizadeh, Solving a fully fuzzy linear programming using lexicography method and fuzzy approximate solution, *Applied Mathematical Modelling*, **33** (2009) 3151-3156.
- [6] S.H. Nasseri and N. Mahdavi-Amiri, Some duality results on linear programming problems with symmetric fuzzy numbers, *Fuzzy Information and Engineering*, **1** (2009) 59-66.
- [7] S.H. Nasseri, A. Ebrahimnejad and S. Mizuno, Duality in fuzzy linear programming with symmetric trapezoidal numbers, *Applications and Applied Mathematics*, **5** (10) (2010) 1467-1482.
- [8] A. Zadeh, Fuzzy sets, *Inform. and Control*, **8** (1965) 338-353.