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Parallel Transport Frame in 4−dimensional Euclidean Space E^4

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Abstract. In this work, we give parallel transport frame of a curve and we introduce the relations between the frame and Frenet frame of the curve in 4−dimensional Euclidean space. The relation which is well known in Euclidean 3−space is generalized for the first time in 4−dimensional Euclidean space. Also, we obtain the condition for spherical curves using the parallel transport frame of them. The condition in terms of Frenet curvatures is so complicated but in terms of the parallel transport curvatures is simple. So, parallel transport frame is important to make easy some complicated characterizations. Moreover, we characterize curves whose position vectors lie in their normal, rectifying and osculating planes in 4−dimensional Euclidean space $\mathbb{E}^4$

Keywords: Parallel transport frame, Normal curve, Rectifying curve, Osculating curve

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1. Introduction

The Frenet frame is constructed for 3-time continuously differentiable non-degenerate curves but curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. In this situation, we need an alternative frame for these curves in $\mathbb{E}^3$. Therefore

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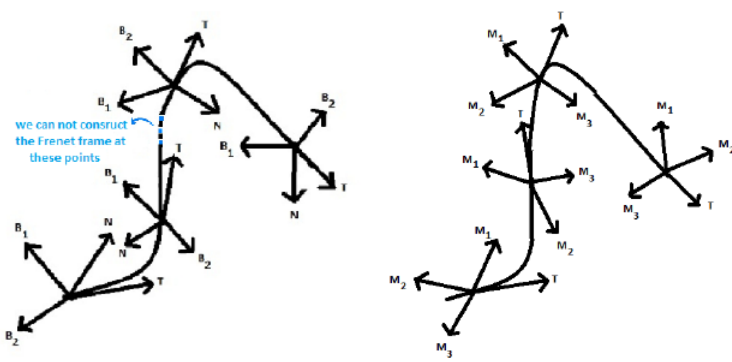
Bishop defined a new frame for a curve and he called it Bishop frame which is well defined even if the curve has vanishing second derivative in 3−dimensional Euclidean space [1]. Also, Bishop and Hanson gave advantages of the Bishop frame and the comparable Bishop frame with the Frenet frame in Euclidean 3−space [1, 5]. In Euclidean 4−space $\mathbb{E}^4$, we have the same problem for a curve like being in Euclidean 3−space. That is, one of the $i-th$ $(1 < i < 4)$ derivative of the curve may be zero. So, using the similar idea we consider to such curves and construct an alternative frame.

Our work is structured as follows. Firstly, we give parallel transport frame of a curve and we introduce the relations between the frame and Frenet frame of the curve in 4−dimensional Euclidean space. Thus, the relation which is well known in Euclidean 3−space is generalized for the first time in 4−dimensional Euclidean space. For construction of parallel transport frame we use the following method.

Let $\alpha(s)$ be a space curve parametrized by arc-length s and its normal vector field be $V(s)$ which is perpendicular to its tangent vector field $T(s)$ said to be relatively parallel vector field if its derivative is tangential along the curve $\alpha(s)$. If the tangent vector field $T(s)$ is given uniquely for the curve, we can choose any convenient arbitrary basis which consist of relatively parallel vector field $\{M_1(s), M_2(s), M_3(s)\}$ of the frame, they are perpendicular to $T(s)$ at each point.



(a) (b)

`Figure 1` : Comparing the Frenet and parallel transport frames. The Frenet frame which is undefined at the some points and the parallel transport frame is smooth along the curve in Figure1-(a) and in Figure1-(b), respectively.

Secondly, we obtain the condition for spherical curves using the parallel transport frame of them. The condition in terms of $\varkappa$ and τ is so complicated in Euclidean 3−space but the condition was given by Bishop [1] in terms of k_1 and k_2 is simple. So, parallel transport frame is important to make easy some complicated characterizations. Therefore, we give a formulae for a curve such as: Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a curve with nonzero curvatures k_i $(i = 1, 2, 3)$ according to parallel transport frame in $\mathbb{E}^4$. "The curve α lies on a sphere if and only if $ak_1 + bk_2 + ck_3 + 1 = 0$ where a, b, and c are non-zero constants."

Finally, we give a necessary and sufficient conditions for curves in Euclidean 4−space $\mathbb{E}^4$ to be normal, rectifying and osculating curves in terms of their parallel transport curvature functions. Normal curves are defined as the space curves whose position vector is always orthogonal to the vector field $T(s)$. Rectifying curves are defined in the Euclidean 4−space as a curve whose position vector always lies in orthogonal complement $M_1^{\perp}(s)$ of its principal normal vector field $M_1(s)$. Analogously, osculating curve as the space curves whose position vector always lies in its osculating space, which represents the orthogonal complement of the second binormal vector field $M_2(s)$ of the parallel transport frame.

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be arbitrary curve in the Euclidean 4−space $\mathbb{E}^4$. Recall that the curve α is parameterized by arclength s if $\langle \alpha'(s), \alpha'(s) \rangle = 1$ where $\langle , \rangle$ is the inner product of $\mathbb{E}^4$ given by

$$\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

for each $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4) \in \mathbb{E}^4$. The norm of a vector $X \in \mathbb{E}^4$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$.

Let $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame along the curve α. Then T, N, B_1 and B_2 are the tangent, the principal normal, first and second binormal vectors of the curve α, respectively. If α is a space curve, then this set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties

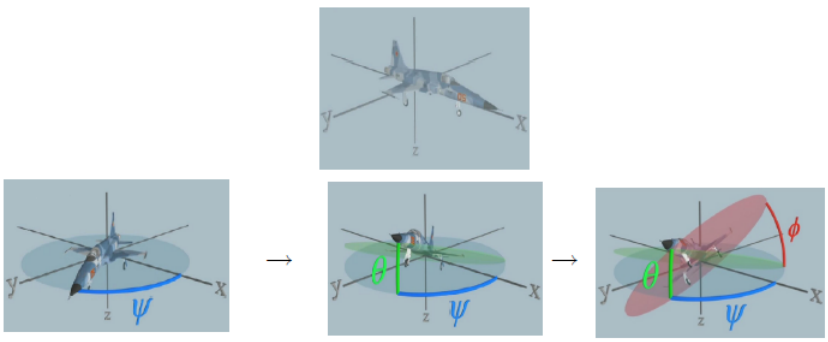
$$\begin{bmatrix} T'(s) \\ N'(s) \\ B_1'(s) \\ B_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & \varkappa(s) & 0 & 0 \\ -\varkappa(s) & 0 & \tau(s) & 0 \\ 0 & -\tau(s) & 0 & \sigma(s) \\ 0 & 0 & -\sigma(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B_1(s) \\ B_2(s) \end{bmatrix}$$

where $\varkappa$, τ and σ denote the curvature functions according to Serret-Frenet frame of the curve α ,respectively.

We use the tangent vector field $T(s)$ and three relatively parallel vector fields $M_1(s)$, $M_2(s)$ and $M_3(s)$ to construct an alternative frame. Since the normal component of the derivatives of the normal vector field is zero then the frame $\{T(s), M_1(s), M_2(s), M_3(s)\}$ is called as parallel transport frame and the curvature functions $\{k_1(s), k_2(s), k_3(s)\}$ according the frame is defined by

$$k_1(s) = \left\langle T'(s), M_1(s) \right\rangle, k_2(s) = \left\langle T'(s), M_2(s) \right\rangle, k_3(s) = \left\langle T'(s), B_2(s) \right\rangle.$$

The Euler angles are introduced by Leonhard Euler to describe the orientation of a rigid body. Also, the Euler angles are used in robotics for speaking about the degrees of freedom of a wrist, electronic stability control in a similar way, gun fire control systems require corrections to gun order angles to compensate for dect tilt, crystallographic texture can be described using Euler angles and texture analysis, the Euler angles provide the necessary mathematical description of the orientation of individual crystallites within a polycrystalline material, allowing for the quantitative description of the macroscopic material. [see more details in [10]], the most application is to describe aircraft attitudes see in Figure 2.



`Figure 2`: Euler angles of the aircraft movements.

Using Euler angles an arbitrary rotation matrix is given by

$$\begin{bmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

where θ, ϕ, ψ are Euler angles [10].

3. Parallel Transport Frame in 4−dimensional Euclidean Space $\mathbb{E}^4$

In this section, we give parallel transport frame of a curve and we introduce the relations between the frame and Frenet frame of the curve in 4−dimensional Euclidean space by using the Euler angles.

Theorem 3.1. *Let* $\{T(s), N(s), B_1(s), B_2(s)\}$ *be a Frenet frame along a unit speed curve* $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ *and* $\{T(s), M_1(s), M_2(s), M_3(s)\}$ *denotes the parallel transport frame of the curve* α*. The relation may be expressed as*

$$\begin{aligned}
&T(s) = T(s)\\
&N(s) = \cos\theta(s)\cos\psi(s)M_1(s) + (-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s))M_2(s)\\
&\qquad + (\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s))M_3(s)\\
&B_1(s) = \cos\theta(s)\sin\psi(s)M_1(s) + (\cos\phi(s)\cos\psi(s) + \sin\phi(s)\sin\theta(s)\sin\psi(s))M_2(s)\\
&\qquad + (-\sin\phi(s)\cos\psi(s) + \cos\phi(s)\sin\theta(s)\sin\psi(s))M_3(s)\\
&B_2(s) = -\sin\theta(s)M_1(s) + \sin\phi(s)\cos\theta(s)M_2(s) + \cos\phi(s)\cos\theta(s)M_3(s)
\end{aligned}$$

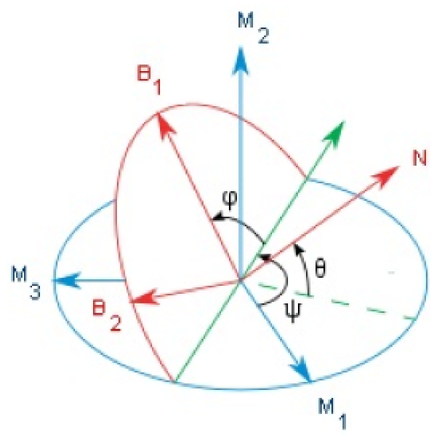


Figure 3: *The relation between the Frenet and the parallel transport frame by means of the rotation matrix.*

The alternative parallel transport frame equations are

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} \tag{3.1}$$

where k_1, k_2, k_3 *are curvature functions according to parallel transport frame of the curve* α *and their expression as follows*

$$\begin{aligned}
k_1(s) &= \varkappa(s)\cos\theta(s)\cos\psi(s),\\
k_2(s) &= \varkappa(s)(-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s)),\\
k_3(s) &= \varkappa(s)(\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s)),
\end{aligned}$$

where $\theta' = \dfrac{\sigma}{\sqrt{\varkappa^2+\tau^2}}$, $\psi' = -\tau - \sigma\dfrac{\sqrt{\sigma^2-\theta'^2}}{\sqrt{\varkappa^2+\tau^2}}$, $\phi' = -\dfrac{\sqrt{\sigma^2-\theta'^2}}{\cos\theta}$ *and the following equalities hold.*

$$\begin{aligned}
\varkappa(s) &= \sqrt{k_1^2+k_2^2+k_3^2},\\
\tau(s) &= -\psi' + \phi'\sin\theta,\\
\sigma(s) &= \frac{\theta'}{\sin\psi},
\end{aligned}$$

$$\phi'\cos\theta + \theta'\cot\psi = 0.$$

Proof. Let $\{T, N, B_1, B_2\}$ be a Frenet frame and $\{T, M_1, M_2, M_3\}$ denotes the parallel transport frame along a unit speed curve α in $\mathbb{E}^4$. The relation between Frenet frame and parallel transport frame as follows

$$\begin{aligned}
T(s) &= T(s),\\
M_1(s) &= (\cos\theta\cos\psi)N(s) + (\cos\theta\sin\psi)B_1(s) - \sin\theta B_2(s),\\
M_2(s) &= (-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi)N(s) +\\
&\quad (\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)B_1(s) + (\sin\phi\cos\theta)B_2(s),\\
M_3(s) &= (\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi)N(s) +\\
&\quad (-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi)B_1(s) + (\cos\phi\cos\theta)B_2(s).
\end{aligned}$$

Differentiating the M_1, M_2, M_3 with respect to s, we get

$$\begin{aligned}
M_1'(s) &= (-\varkappa\cos\theta\cos\psi)T(s)\\
&\quad +(\theta'\sin\theta\cos\psi - \psi'\cos\theta\sin\psi - \tau\cos\theta\sin\psi)N(s)\\
&\quad +(-\theta'\sin\theta\sin\psi + \psi'\cos\theta\cos\psi + \sigma\sin\theta)B_1(s)\\
&\quad +(-\theta'\cos\theta + \sigma\cos\theta\sin\psi)B_2(s),
\end{aligned}$$

$$\begin{aligned}
M_2'(s) &= -\varkappa(-\cos\phi\sin\psi + \sin\phi\sin\theta\sin\psi)T(s)\\
&\quad +(\phi'\sin\phi\sin\psi - \psi'\cos\phi\cos\psi + \phi'\cos\phi\sin\theta\cos\psi\\
&\quad +\theta'\sin\phi\cos\theta\cos\psi - \psi'\sin\phi\sin\theta\sin\psi - k_2(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi))N(s)\\
&\quad +(-\phi'\sin\phi\cos\psi - \psi'\cos\phi\sin\psi + \phi'\cos\phi\sin\theta\sin\psi\\
&\quad +\theta'\sin\phi\cos\theta\sin\psi + \psi'\sin\phi\sin\theta\cos\psi + \tau(-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi)\\
&\quad -\sigma\sin\phi\cos\theta)B_1(s)\\
&\quad +(\phi'\cos\phi\cos\theta - \theta'\sin\phi\sin\theta\\
&\quad +\sigma(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi))B_2(s)
\end{aligned}$$

$$
\begin{aligned}
M_3'(s) = & -\varkappa(\sin\phi\sin\psi+\cos\phi\sin\theta\cos\psi)T(s) \\
& +(\phi'\cos\phi\sin\psi+\psi'\sin\phi\cos\psi-\phi'\sin\phi\sin\theta\cos\psi \\
& +\theta'\cos\phi\cos\theta\cos\psi-\psi'\cos\phi\sin\theta\sin\psi-\tau(-\sin\phi\cos\psi+\cos\phi\sin\theta\sin\psi))N(s) \\
& +(-\phi'\cos\phi\cos\psi+\psi'\sin\phi\sin\psi-\phi'\sin\phi\sin\theta\sin\psi+\theta'\cos\phi\cos\theta\sin\psi \\
& +\psi'\cos\phi\sin\theta\cos\psi+\tau(\sin\phi\sin\psi+\cos\phi\sin\theta\cos\psi) \\
& +\sigma(\cos\phi\cos\theta))B_1(s) \\
& +(-\phi'\sin\phi\cos\theta-\theta'\cos\phi\sin\theta \\
& +\sigma(-\sin\phi\cos\psi+\cos\phi\sin\theta\sin\psi))B_2(s).
\end{aligned}
$$

Since the M_1, M_2 and M_3 are relatively parallel vector fields then normal components of the M_1', M_2' and M_3' must be zero. Also, we consider that $k_1=\langle T',M_1\rangle$, $k_2=\langle T',M_2\rangle$, $k_3=\langle T',M_3\rangle$, we can easily see that

$$
\begin{aligned}
\varkappa(s) &= \sqrt{k_1^2+k_2^2+k_3^2}, \\
\tau(s) &= -\psi'+\phi'\sin\theta, \\
\sigma(s) &= \frac{\theta'}{\sin\psi},
\end{aligned}
$$

$$\phi'\cos\theta+\theta'\cot\psi=0,$$

and

$$
\begin{aligned}
k_1 &= \varkappa\cos\theta\cos\psi, \\
k_2 &= \varkappa(-\cos\phi\sin\psi+\sin\phi\sin\theta\cos\psi), \\
k_3 &= \varkappa(\sin\phi\sin\psi+\cos\phi\sin\theta\cos\psi).
\end{aligned}
$$

If we choose $\theta'=\dfrac{\sigma}{\sqrt{\varkappa^2+\tau^2}}$ then we obtain $\psi'=-\tau-\sigma\dfrac{\sqrt{\sigma^2-\theta'^2}}{\sqrt{\varkappa^2+\tau^2}}$, $\phi'=-\dfrac{\sqrt{\sigma^2-\theta'^2}}{\cos\theta}$. So, we can write

$$
\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix}
=
\begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix},
$$

which complete the proof. □

Corollary 3.2. *If we choose that $\theta=\phi=0$ in the above equations then we get the Bishop frame in $\mathbb{E}^3$.*

Now, we give an example for the curve which has not a Frenet frame at some points but it has parallel transport frame at these points.

Example 3.3. Let $\alpha(s) = (\sin s, 2s+1, 2s-1, s)$ be a curve in Euclidean 4−space. It is provided that $\alpha''(0) = (0,0,0,0)$ at the point $s = 0$, then we can not calculate the Frenet vector fields at the point. However, we can calculate the parallel transport vector fields as follows

$$\begin{aligned} M_1(s) &= \sin\psi(s)B_1(s), \\ M_2(s) &= \cos\phi(s)\cos\psi(s)B_1(s), \\ M_3(s) &= -\sin\phi(s)\cos\psi(s)B_1(s), \end{aligned}$$

where ψ and ϕ constant angles.

Theorem 3.4. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a curve with nonzero curvatures k_i $(i = 1,2,3)$ according to parallel transport frame in Euclidean 4−space $\mathbb{E}^4$. Then the curve α lies on a sphere if and only if $ak_1+bk_2+ck_3+1 = 0$ where a, b and c are non-zero constants.*

Proof. Let α be a curve on a sphere with center P and radius r,

then $\langle \alpha - P, \alpha - P\rangle = r^2$. Differentiating this equation, we obtain that $\langle T, \alpha - P\rangle = 0$. We can write $\alpha - P = aM_1 + bM_2 + cM_3$ for some function a, b, c and where $a' = \langle \alpha - P, M_1\rangle' = \langle T, M_1\rangle + \langle -k_1T, \alpha - P\rangle = 0$. So, the function a is a constant. Similarly, we can easily say that the functions b and c are constants. Then differentiating the equation $\langle T, \alpha - P\rangle$, we get

$$\langle k_1M_1 + k_2M_2 + k_3M_3, \alpha - P\rangle + \langle T, T\rangle = 0.$$

Consequently, between the curvature functions k_1, k_2 and k_3 of the curve α have the linear relation such as

$$ak_1 + bk_2 + ck_3 + 1 = 0.$$

Moreover, $r^2 = \langle \alpha - P, \alpha - P\rangle = a^2 + b^2 + c^2 = \dfrac{1}{d^2}$ where d is the distance of the plane $ax + by + cz + 1 = 0$ from the origin.

Conversely, we suppose that the equation

$$ak_1 + bk_2 + ck_3 + 1 = 0.$$

If the center P denoted by $P = \alpha - aM_1 - bM_2 - cM_3$ then differentiating the last equation we have $P' = T + (ak_1 + bk_2 + ck_3)T = 0$. So, the center P of the sphere is a constant. Similarly, we show that $r^2 = \langle \alpha - P, \alpha - P\rangle$ is a constant. So, the curve α lies on a sphere with center P and radius r. □

Example 3.5. Let $\alpha(s) = (\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s)$ be a curve in Euclidean 4−space. According the Frenet frame there are lots of formulas for showing that this curve is a spherical curve. But the formulas have some disadvantages which were explained the above chapters. Then we calculate curvature functions of the curve α according to parallel transport frame

$$k_1 = 0,\ k_2 = -\cos\phi,\ k_3 = \sin\phi$$

where ϕ is constant. The curve α satisfy the fallowing equation

$$ak_1 + bk_2 + ck_3 + 1 = 0.$$

Consequently, the curve α is a spherical curve. Because of the curve α has a zero torsion, we can not show that α is a spherical curve by using the Frenet curvatures.

4. Normal, rectifying and osculating curves according to parallel transport frame

In this section, we define normal, rectifying and osculating curves according to parallel transport frame and obtain some characterizations for such curves.

4.1. **Normal curves according to parallel transport frame.** The normal space according to parallel transport frame of the curve α as the orthogonal complement $T^{\perp}(s)$ of its tangent vector field $T(s)$. Hence, the normal space is given by $T^{\perp}(s) = \{X \in \mathbb{E}^4 \mid \langle X, T(s)\rangle = 0\}$. Normal curves are defined in [8] as a curve whose position vector always lies in its normal space. Consequently, the position vector of the normal curve α with parallel transport vector fields M_1, M_2 and M_3 satisfies the equation

$$\alpha(s) = \lambda(s)M_1(s) + \mu(s)M_2(s) + \upsilon(s)M_3(s), \tag{4.1}$$

where $\lambda(s), \mu(s)$ and $\upsilon(s)$ differentiable functions.

Then, we have the following theorem which characterize normal curves according to parallel transport frame.

Theorem 4.1. *Let α be a unit speed curve in Euclidean 4−space with parallel transport vector fields T, M_1, M_2, M_3 and its curvature functions k_1, k_2, k_3 of the curve α. Then the curve α is a normal curve if and only if α is a spherical curve.*

Proof. Let us first assume that α is a normal curve. Then its position vector satisfies the equation (4.1). By taking the derivative of equation (4.1) and using the parallel transport frame in equation (3.1), we obtain

$$\begin{aligned} T(s) &= -(\lambda(s)k_1(s) + \mu(s)k_2(s) + \upsilon(s)k_3(s))T(s) \\ &\quad +\lambda'(s)M_1(s) + \mu'(s)M_2(s) + \upsilon'(s)M_3(s) \end{aligned}$$

and so

$$\begin{aligned} &-(\lambda(s)k_1(s) + \mu(s)k_2(s) + \upsilon(s)k_3(s)) = 1 \\ &\lambda'(s) = 0,\ \mu'(s) = 0,\ \upsilon'(s) = 0. \end{aligned}$$

From the last equations we find

$$\lambda(s)k_1(s) + \mu(s)k_2(s) + \upsilon(s)k_3(s) + 1 = 0,$$

where $\lambda(s), \mu(s)$ and $\upsilon(s)$ are non zero constant functions. From Theorem (2), the curve α is a spherical curve.

Conversely, suppose that the curve α is a spherical curve. Let us consider the vector $m \in \mathbb{E}^4$ given by

$$m(s) = \alpha(s) - (\lambda(s)M_1(s) + \mu(s)M_2(s) + \upsilon(s)M_3(s)). \tag{4.2}$$

Differentiating equation (4.2) and applying equation (3.1), we get

$$\begin{aligned} m'(s) &= (1 + \lambda(s)k_1(s) + \mu(s)k_2(s) + \upsilon(s)k_3(s))T(s) \\ &\quad + \lambda'(s)M_1(s) + \mu'(s)M_2(s) + \upsilon'(s)M_3(s). \end{aligned}$$

Since the curve α is a spherical curve and $\lambda(s)$, $\mu(s)$ and $\upsilon(s)$ are non zero constants then we have $m'(s) = 0$. Consequently, the curve α is a normal curve. □

4.2. **Rectifying curves according to parallel transport frame.** The rectifying space according to parallel transport frame of the curve α as the orthogonal complement $M_1^{\perp}(s)$ of its first normal vector field $M_1(s)$. Hence, the normal space is given by

$$M_1^{\perp}(s) = \{X \in \mathbb{E}^4 \mid \langle X, M_1(s)\rangle = 0\}.$$

Rectifying curves are defined in [6] as a curve whose position vector always lies in its rectifying space. Consequently, the position vector of the rectifying curve α with respect to parallel transport frame satisfies the equation

$$\alpha(s) = c_1(s)T(s) + c_2(s)M_2(s) + c_3(s)M_3(s), \tag{4.3}$$

where $c_1(s)$, $c_2(s)$ and $c_3(s)$ are differentiable functions.

Then, we have the following theorem which characterize rectifying curves according to parallel transport frame.

Theorem 4.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a curve with non zero curvatures k_i $(i = 1, 2, 3)$ according to parallel transport frame in $\mathbb{E}^4$. Then α is a rectifying curve if and only if $c_2k_2 + c_3k_3 + 1 = 0$, $c_2, c_3 \in \mathbb{R}$.*

Proof. Let us first assume that α be a rectifying curve. With the help of the definition of rectifying curve, the position vector of the curve α satisfies

$$\alpha(s) = c_1T(s) + c_2M_2(s) + c_3M_3(s). \tag{4.4}$$

Differentiating the last equation and using the parallel transport frame formulas we get

$$T = (c_1' - c_2k_2 - c_3k_3)T + c_1k_1M_1 + (c_2' + c_1k_2)M_2 + (c_3' + c_1k_3)M_3 \tag{4.5}$$

and therefore

$$\begin{aligned} c_1 &= 0, \\ c_2 &= \text{constant}, \\ c_3 &= \text{constant}. \end{aligned} \tag{4.6}$$

Then using the last equation, we can easily find that the curvatures k_1, k_2 and k_3 satisfies the equation

$$c_2 k_2 + c_3 k_3 + 1 = 0 \tag{4.7}$$

where c_2 and c_3 are constants.

Conversely, we suppose that the curvatures k_1, k_2 and k_3 satisfy the equation $c_2 k_2 + c_3 k_3 + 1 = 0$, $c_2, c_3 \in \mathbb{R}$. Let us consider the vector $X \in \mathbb{E}^4$ given by

$$X(s) = \alpha(s) - c_2 M_2(s) + c_3 M_3(s) \tag{4.8}$$

then using the equations (4.6) and (4.7) we can easily see that $X'(s) = 0$, that is, X is constant vector field. So, the curve α is a rectifying curve. □

Corollary 4.3. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a curve with non zero curvatures k_i $(i = 1, 2, 3)$ according to parallel transport frame in $\mathbb{E}^4$. Then α is a rectifying curve if and only if*

$$(\frac{k_3}{k_2})' \frac{k_2^2}{k_3'} = constant \text{ or } (\frac{k_3}{k_1})' \frac{k_1^2}{k_3'} = constant.$$

Proof. The proof is clear from equation (4.7). □

4.3. **Osculating curves according to parallel transport frame.** The osculating space according to parallel transport frame of the curve α as the orthogonal complement $M_2^{\perp}(s)$ of its second binormal vector $M_2(s)$ of the parallel transport frame. Hence, the osculating space is given by $M_2^{\perp}(s) = \{X \in \mathbb{E}^4 \mid \langle X, M_2(s) \rangle = 0\}$. Osculating curves are defined in [7] as a curve whose position vector always lies in its osculating space. Thus, the position vector of the osculating curve α with respect to parallel transport frame satisfies the equation

$$\alpha(s) = \lambda_1(s) T(s) + \lambda_2(s) M_1(s) + \lambda_3(s) M_3(s), \tag{4.9}$$

where $\lambda_1(s)$, $\lambda c_2(s)$ and $\lambda_3(s)$ differentiable functions.

Then we have the following theorem which characterize osculating curves according to parallel transport frame.

Theorem 4.4. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a curve with nonzero curvatures k_i $(i = 1, 2, 3)$ according to parallel transport frame in $\mathbb{E}^4$. Then the curve α is a osculating curve if and only if $\lambda_2 k_1 + \lambda_3 k_3 + 1 = 0$, $\lambda_2, \lambda_3 \in \mathbb{R}$.*

Proof. We assume that α is a osculating curve. With the help of the definition of osculating curve, the position vector of the curve α satisfies

$$\alpha(s) = \lambda_1 T(s) + \lambda_2 M_1(s) + \lambda_3 M_3(s) \tag{4.10}$$

Differentiating the last equation with respect to s and using the parallel transport equations, we get

$$T = (\lambda_1' - \lambda_2 k_1 - \lambda_3 k_3)T + (\lambda_2' + \lambda_1 k_2)M_2 + \lambda_1 k_1 M_1 + (\lambda_3' + \lambda_1 k_3)M_3 \tag{4.11}$$

and therefore

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= \text{constant}, \\ \lambda_3 &= \text{constant}. \end{aligned} \tag{4.12}$$

Then using the last equation , we can easily find that the curvatures k_1, k_2 and k_3 satisfy the equation

$$\lambda_2 k_1 + \lambda_3 k_3 + 1 = 0 \tag{4.13}$$

where λ_2 and λ_3 are constants.

Conversely, we suppose that the curvatures k_1, k_2 and k_3 satisfy the equation $\lambda_2 k_1 + \lambda_3 k_3 + 1 = 0$, $\lambda_2, \lambda_3 \in \mathbb{R}$. Let us consider the vector $X \in \mathbb{E}^4$ given by

$$X(s) = \alpha(s) - \lambda_2 M_1(s) + \lambda_3 M_3(s) \tag{4.14}$$

then using the equations (4.12) and (4.13) we can easily see that $X'(s) = 0$, that is, X is a constant vector field. So, the curve α is a osculating curve. □

Corollary 4.5. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a curve with non zero curvatures k_i $(i = 1, 2, 3)$ according to parallel transport frame in $\mathbb{E}^4$. Then the curve α is a rectifying curve if and only if*

$$\left(\frac{k_3}{k_1}\right)' \frac{k_1^2}{k_3'} = constant \text{ or } \left(\frac{k_3}{k_1}\right)' \frac{k_1^2}{k_2'} = constant.$$

Proof. The proof is clear from equation (4.13). □

References

[1] L. R. Bishop, There is more than one way to frame a curve, *Amer. Math. Mont. 3(1975), 246-251.*

[2] B. Bükcü and M. K. Karacan, The Bishop Darboux rotation axis of the spacelike curve in Minkowski 3-space, *Ege University J. Fac. Sci. 3(2007), 1-5.*

[3] B. Bükcü and M. K. Karacan, Bishop motion and Bishop Darboux rotation axis of the timelike curve in Minkowski 3-space, *Kochi J. Math. 4(2009), 109-117.*

[4] H. Gluk, Higher curvatures of curves in Euclidean space, *Amer. Math. Month.* ***73****(1966), 699-704.*

[5] A. J. Hanson and H. Ma, Parallel Transport Aproach to curve Framing, *Tech. Math. Rep.* ***425****(1995).*

[6] K. İlarslan, E. Nešović and M. Petrovic-Torgasev, Some Characterizations of Rectifying Curves in the Euclidean Space E^4, *Turk J. Math.* ***32****(2008), 21-30.*

[7] K. İlarslan and E. Nešović, Some characterizations of osculating curves in the Euclidean Space, *Demo. Math.* ***4****(2008), 931-939.*

[8] K. İlarslan and E. Nešović, Spacelike and timelike normal curves in Minkowskispace-time, *Publ. Inst. Math. Belgrade* ***85****(2009), 111-118.*

[9] M. K. Karacan, B. Bükcü and N. Yüksel, On the dual Bishop Darboux rotation axis of the dual space curve, *Appl. Sci.* ***10****(2008), 115-120.*

[10] http://en.wikipedia.org/wiki/Rotation_matrix, Euler angles.