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A result on fixed points for weakly quasi-contraction maps in metric spaces

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Abstract. In this paper, I provide a new fixed point theorem for Weakly quasi-contraction maps in metric spaces. Our results extend and improve some fixed point and theorems in literature.

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1. INTRODUCTION

The common Banach's fixed point theorem asserts that if (X, d) is a complete metric space and $f : X \to X$is a map such that

$$d(Tx, Ty) \leq cd(x, y), \text{ for each } x, y \in X,$$

where $0 \leq c < 1$. Then f has a unique fixed point $\overline{x} \in X$ and for any $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to $\overline{x}$.
In recent years, a number of generalizations of the above Banach's contraction principle have appeared. Of all these, the following generalization of Ćirić [1] stands on the top.

Let (X, d) be a complete metric space. Let $T : X \to X$ be a quasi-contraction map there exists $c < 1$ such that

$$d(Tx, Ty) \leq c \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \qquad (qc),$$

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for any $x, y \in X$. Then T has a unique fixed point $\overline{x} \in X$ and for any $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to $\overline{x}$.

2. MAIN RESULTS

Now,we introduce the concept of a *weakly quasi-contraction* map in metric spaces. Let (X, d) be a metric space. The self-map $T : X \to X$ is said to be a *weakly quasi-contraction* if there exists $\alpha : [0, \infty) \to [0, 1]$, with $\theta(a, b) = sup\{\alpha(d(x, y)) : a \leq d(x, y) \leq b\} < 1$ for every $0 < a \leq b$ such that, for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha(d(x, y)) \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

(wc).

As the following simple example ,due to Sastry and Naidu [2], shows that Theorem 1.1 is not true for weakly quasi-contraction maps even we suppose α is continuous and increasing. Let $X = [1, \infty)$ with the usual metric, $T : X \to X$ be given by $Tx = 2x$. Define $\alpha : [0, \infty) \to [0, 1)$ by $\alpha(t) = \frac{2t}{1+2t}$. Then, clearly, α is continuous and increasing, and

$$|Tx - Ty| \leq \alpha(|x - y|) \max\{|x - y|, |x - Tx|, |y - Ty|, |x - Ty|, |y - Tx|\},$$

for each $x, y \in X$, but T has no fixed point. Now, a natural question iswhat further conditions are to be imposed on T or α to guarantee the existence of a fixed point for T? For some partial answers to this question and application of quasi-contraction maps to variational inequalities see [3] and references there in.
Now, we are ready to state our main result.

Let (X, d) be a complete metric space. Let $T : X \to X$ be a weakly quasi-contraction map such that α satisfying

$$\limsup_{t \to 0^+} \alpha(t) < 1.$$

Assume there is an $x_0 \in X$ such that , $\lim d(T^n x_0, T^{n+1} x_0) = 0$. Then,T has a unique fixed point .

Proof. Let $x_1 = Tx_0$,and $x_n = T(x_{n-1}) = T^n x_0$ for $n_0 \in \mathbb{N}$, we shall prove $\{x_n\}$ is a Cauchy sequence ,and its limit is a fixed point for T. To do it let us prove the that for each $k, n \in \mathbb{N}$

$$\begin{aligned} d(x_{n+k+1}, x_{n+1}) \leq \alpha(d(x_{n+k}, x_n))(d(x_{n+k}, x_n) + d(x_{n+k+1}, x_{n+k}) \\ + d(x_{n+k+1}, x_{n+k})) \end{aligned} \tag{2.1}$$

observe that, for all $n > 0$ we have

$$d(x_{n+k+1}, x_{n+1}) \leq \alpha(d(x_{n+k}, x_n))u \tag{2.2}$$

where

$u \in \{d(x_n, x_{n+k}), d(x_n, x_{n+1}), d(x_{n+k+1}, x_n), d(x_{n+k+1}, x_{n+k}), d(x_{n+k}, x_{n+1})\}$.

If $u = d(x_n, x_{n+1})$or$u = d(x_{n+k+1}, x_{n+k})$ or $u = d(x_n, x_{n+k})$, it is trivial that (2.1) holds.

If $u = d(x_{n+k}, x_{n+1})$, then we have

$$d(x_{n+k}, x_{n+1}) \leq d(x_{n+k}, x_n) + d(x_n, x_{n+1})$$
$$\leq d(x_{n+k}, x_n) + d(x_n, x_{n+1}) + d(x_{n+k+1}, x_{n+k}). \qquad (2.3)$$

By (2.3) and (2.2),

we see that (2.1) holds for this case .

If $u = d(x_{n+k+1}, x_n)$, then we have

$$d(x_{n+k+1}, x_n) \leq d(x_{n+k+1}, x_{n+k}) + d(x_{n+k}, x_n)$$
$$\leq d(x_{n+k+1}, x_{n+k}) + d(x_n, x_{n+1}) + d(x_{n+k}, x_n). \qquad (2.4)$$

By (2.2) and (2.4) we see that (2.1) holds for this case. Thus, (2.1) is proved. To prove$\{x_n\}$ is a Cauchy sequnce,suppose that $\epsilon > 0$ is given. Since $\lim d(T^n x_0, T^{n+1} x_0) = 0$, we can obtain $N \in \mathbb{N}$ such that $\forall n \geq N$

$$d(x_n, x_{n+1}) \leq \frac{1}{6}[1 - \theta(\frac{\epsilon}{2}, \epsilon)]\epsilon. \qquad (2.5)$$

We will prove inductively that$d(x_N, x_{N+k}) \leq \epsilon$.It is obvious for $k = 1$, and assuming that $d(x_{N+k}, x_N) < \epsilon \; \forall k \in \mathbb{N}$, let us show $d(x_{N+k+1}, x_N) < \epsilon$. Note using (2.1) we get

$$d(x_{N+k+1}, x_N) \leq d(x_{N+k+1}, x_{N+1}) + d(x_{N+1}, x_N)$$
$$\leq \alpha(d(x_{N+k}, x_N))[(d(x_{N+k}, x_N) + (d(x_{N+1}, x_N) + (d(x_{N+k+1}, x_{N+k})] + d(x_{N+1}, x_N)$$
$$\leq \alpha(d(x_{N+k}, x_N)[d(x_{N+k}, x_N) + d(x_{N+k+1}, x_{N+k})] + 2d(x_{N+1}, x_N) \qquad (2.6)$$

Thus, if $d(x_{N+k}, x_N) \leq \frac{\epsilon}{2}$ it follows from (2.5) and (2.6),

$$d(x_{N+k+1}, x_N) \leq \frac{\epsilon}{2} + 3\frac{1}{6}[1 - \theta(\frac{\epsilon}{2}, \epsilon)]\epsilon.$$

Now if $d(x_{N+k}, x_N) \geq \frac{\epsilon}{2}$, since T is a weakly quasicontraction,applying the induction hypothesis,

$$\frac{\epsilon}{2} \leq d(x_N, x_{N+k}) \leq \epsilon$$

so

$$\alpha(d(x_N, x_{N+k})) \leq \theta(\frac{\epsilon}{2}, \epsilon) < 1. \qquad (2.7)$$

Then from (2.5) and (2.7), we conclude that

$$d(x_{N+k+1}, x_N) \leq \alpha(d(x_{N+k}, x_N))[d(x_{N+k}, x_N) + d(x_{N+k+1}, x_{N+k})] + 2d(x_{N+1}, x_N)$$

$$\leq \theta(\frac{\epsilon}{2}, \epsilon).\epsilon + 3\frac{1}{6}[1 - \theta(\frac{\epsilon}{2}, \epsilon)]\epsilon.$$

Since (X,d) ia complete, then $\{x_n\}$ is a convergent, say, to $y \in X$. We also know

$$\lim d(Ty, x_n) = d(Ty, y)$$

.Then we have

$$d(Ty, x_{n+1}) \leq \alpha(d(y, x_n))max\{d(y, x_n), d(Ty, y), d(x_{n+1}, x_n), d(y, x_{n+1}), d(Ty, x_n)\}$$

so

$$d(Ty, y) \leq \limsup \alpha(d(x_n, y))d(Ty, y).$$

Since $\limsup_{t\to 0^+} \alpha(t) < 1$ we get $Ty = y$.

In order to see y is the only fixed point of T, suppose $Tz = z$ then

$$d(y, z) = d(Ty, Tz) \leq \alpha(d(y, z))max\{d(y, z), d(Ty, y), d(Tz, z), d(Ty, z), d(y, Tz)\}$$
$$= \alpha(d(y, z))d(y, z)$$

so we have $d(y, z) = 0$ then $y = z$

□

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