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Existence of a positive solution for a p-Laplacian equation with singular nonlinearities

S. Khademloo [1] and F. Yosefzade [2]
[1] Faculty of Basic Sciences, Babol University of Technology, Babol, Iran
[2] Faculty of Basic Sciences, Babol University of Technology, Babol, Iran

Abstract. In this paper, we study a class of boundary value problem involving the p-Laplacian oprator and singular nonlinearities. We analyze the existence of a critical parameter λ^* such that the problem has at least one solution for $\lambda \in (0, \lambda^*)$ but no solution for $\lambda > \lambda^*$. We find lower bounds of a critical parameter λ^*. We use the method of sub-supersolution to establish our results.
Keywords: Singular nonlinearities , Positive solution, Sub-supersolution

1. INTRODUCTION

We consider the following boundary value problem

$$\begin{cases} -\triangle_p u = \frac{\lambda f(x)}{(M-u)^\beta}, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where λ and β are positive parameters, Ω is a bounded domain in R^N, $N > 1$ with $0 < u < M$, $M > 1$, $f \in C(\bar{\Omega})$ is a nonnegative function and $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian oprator. We are interested in the study of positive solution to (1.1) in the space $W_0^{1,p}(\Omega)$. We

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will use the method of sub-supersolution to establish our results.
We define $\psi \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ to be a subsolution of (1.1) if:

$$\int_\Omega |\nabla \psi|^{p-2} \nabla \psi \nabla h \leq \lambda \int_\Omega \frac{\lambda f(x)}{(M-u)^\beta} h, \forall h \in C_0^\infty(\bar{\Omega}), h \geq 0,$$

and

$$\psi \leq 0 \ on \ \partial\Omega,$$

and $z \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ to be a supersolution of (1.1) if:

$$\int_\Omega |\nabla z|^{p-2} \nabla z \nabla h \geq \lambda \int_\Omega \frac{\lambda f(x)}{(M-u)^\beta} h, \forall h \in C_0^\infty(\bar{\Omega}), h \geq 0,$$

and

$$z \geq 0 \ on \ \partial\Omega.$$

Then the following result holds:

Lemma 1.1. *([4]). Suppose there exists a subsolution ψ and supersolution z of (1.1) such that $\psi \leq z$; then there exists a solution of (1.1) such that $\psi \leq u \leq z$.*

In this work, we first show that there exists a critical parameter $\lambda^* = \lambda^*(\Omega, \beta, f)$ such that (1.1) has at least one solution for $\lambda \in (0, \lambda^*)$ but no solution for $\lambda > \lambda^*$. Secondly, we establish lower bounds of critical parameter λ^*.
Also we extend some results of [9]. In the case $p = 2$, $\beta = 2$, $M = 1$ such results were established by Ghoussoub and Guo in [3,4,5]. This equation models a simple electrostatic Micro Electro Mechanical Systems (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at a below a rigid plate located at $+1$. When a voltage (represented by λ) is applied, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value λ^*(pull in voltage).This creates a so- called pull- in instability, which greatly affects the design of many devices (see[8]). MEMS devices have therefore become key components of many commercial systems, including accelerometers for airbag deployment in automobiles ink jet printer heads, optical switches, chemical sensors, and so on. Relative research on this model are also discussed in [2,6,7,9].
It is often convenient to take change of variable $\nu = M - u$ in considering problem(1.1). Then we have to arrive the following problem

$$\begin{cases} \triangle_p \nu = \frac{\lambda f(x)}{\nu^\beta}, & in\ \Omega, \\ 0 < \nu < M, & in\ \Omega, \\ \nu = M, & on\ \partial\Omega. \end{cases} \tag{1.2}$$

Definition 1.2. A weak solution of problem (1.1) is a function $u \in W_0^{1,p}(\Omega)$ satisfying $0 < u < M(a.e)$ in Ω and,
$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi dx = \int_\Omega \frac{\lambda f(x)}{(M-u)^\beta}\phi dx, \forall \phi \in W^{1,p}(\Omega).$ (1.3)

It is easy to see that if (1.1) has a weak solution, then taking $\phi = u$ in (1.3), we get
$+\infty > \int_\Omega |\nabla u|^p = \int_\Omega \frac{\lambda f(x)}{(M-u)^\beta}u \geq \lambda \int_\Omega \frac{f(x)}{M^\beta}u.$
Therefore we see that the problem (1.1) has no solution for the large λ.

2. CRITICAL PARAMETER

In this section, we establish the existence for critical parameter λ^*, which is defined as
$\lambda^*(\Omega, \beta, f) =$ $sup\{\lambda >$
$0; the\ problem\ (1.1)\ possesses\ atleast\ on\ solution\}.$ (2.1)

For any bounded domain Γ in R^N, we denote by μ_Γ the first eigenvalue of $-\triangle_p$ on $W_0^{1,p}(\Omega)$ and by ψ_Γ the corresponding positive eigenfunction normalized with $sup_{x\in\Omega}\psi_\Gamma = M$. We also associate with any domain Ω in R^N the following parameter
$\upsilon_\Omega = sup\{\mu_\Gamma H(inf \psi_\Gamma); \Gamma\ is\ bounded\ domain\ of\ R^N, \Gamma \supset \bar{\Omega}\},$ (2.2)
where H is the function

$$H(t) = \frac{M^{\beta+p-1}(M^{\frac{p-1}{\beta}} - t^{\frac{p-1}{\beta}})^{\ p-1}((M-1)M^{\frac{p-1}{\beta}} + (1-t)t^{\frac{p-1}{\beta}})^\beta}{(M^{\frac{p-1+\beta}{\beta}} - t^{\frac{p-1+\beta}{\beta}})^{\beta+p-1}}.$$

Theorem 2.1. *Suppose that λ^* is defined in (2.1). Then there exists a finite critical parameter $\lambda^* = \lambda^*(\Omega, \beta, f) > 0$ such that*
1. If $\lambda < \lambda^*$, there exists at least one solution for (1.1);
2. If $\lambda > \lambda^*$, there is no solution for (1.1) .
Moreover, with υ_Ω defined by (2.2), we can estimate λ^* by the lower bound

$$\frac{\upsilon_\Omega}{sup_{x\in\Omega}\ f(x)} \leq \lambda^*.$$

Proof. We need to show that (1.1) has at least one solution when $\lambda < \frac{\upsilon_\Omega}{sup_{x\in\Omega}\ f(x)}$. It is clear $u = 0$ is a subsolution of (1.1) for all $\lambda > 0$.
To constract a supersolution, to consider a bouned domain $\Gamma \supset \bar{\Omega}$ with smooth boundary, and to let $(\mu_\Gamma, \psi_\Gamma)$ be its first the eigenpair normalized in such way that
$sup_{x\in\Gamma}\psi_\Gamma(x) = M$ and $inf_{x\in\Omega}\psi_\Gamma(x) = s_1 > 0$.
Now we have :
$0 < s_1 = inf_{x\in\Omega}\psi_\Gamma(x) < sup_{x\in\Omega}\psi_\Gamma(x) < sup_{x\in\Gamma}\psi_\Gamma(x) = M$.
We construct a supersolution in from $\psi = A\psi_\Gamma$ where A is a scalar to be chosen later. First, we must have $A\psi_\Gamma \geq 0$ on $\partial\Omega$ and $0 < M - A\psi_\Gamma < M$ in Ω, which requires that $0 < A < 1$.
We also require

$$-\triangle_p\psi - \frac{\lambda f(x)}{(M-\psi)^\beta} \geq 0\ in\ \Omega.$$

In this case we have

$$\mu_\Gamma A\psi_\Gamma|A\psi_\Gamma|^{p-2} \geq \frac{\lambda f(x)}{(M-A\psi_\Gamma)^\beta}\ in\ \Omega,$$

or

$$\lambda sup_{x\in\Omega}f(x) \leq |A\psi_\Gamma|^{p-2}\mu_\Gamma A\psi_\Gamma(\mu - A\psi_\Gamma)^\beta,$$

or

$$\lambda sup_{x\in\Gamma}f(x) \leq A^{P-1}\psi_\Gamma^{p-1}(M - A\psi_\Gamma)^\beta,$$

or

$$\lambda sup_{x\in\Omega}f(x) \leq \mu_\Gamma(A\psi_\Gamma)^{p-1}(M - A\psi_\Gamma)^\beta,$$

or

$$\lambda sup_{x\in\Omega}f(x) \leq \beta(A,\Gamma) := \mu_\Gamma inf\{g(As); s\ \in\ [s_1, 1]\},$$

where $g(s) = s^{p-1}(M-1)^\beta$.
By Theorem 2.1 we know that problem (1.1) has a weak solution and λ^* exists. For λ^* by 2.1 there holds $\lambda^* \geq sup\{\beta(A,\Gamma); 0 < A < M, \Gamma \supset \bar{\Omega}\}$. Hence in oder to prove 2.1 , it shows $\upsilon_\Omega = sup\{\beta(A,\Gamma); 0 < A < M, \Gamma \supset \bar{\Omega}\}$.

For that, we first note $inf g(As) = min\{g(As_1), g(AM)\}$.
We also have that $g(As_1) \leq g(AM)$ if and only if $0 < A < A_1$ with A_1 satisfying $\frac{(p-1)M}{p-1+\beta} \leq A_1 \leq \frac{(p-1)M}{p-1+\beta} s_1$, $A_1 := \frac{M^{\frac{p-1+\beta}{\beta}} - M s_1^{\frac{p-1}{\beta}}}{M^{\frac{p-1+\beta}{\beta}} - s_1^{\frac{p-1+\beta}{\beta}}}$.
Therefore, we get that

$$G(A) = inf_{s\in[s_1,1]} g(As) = \begin{cases} g(As_1) & g(As_1) \leq g(AM) \quad 0 \leq A \leq A_1, \\ g(AM) & g(AM) \leq g(As_1) \quad A_1 \leq A \leq 1. \end{cases}$$

We now have that $\frac{dG}{dA} = g'(As_1)s_1 \geq 0$ for all $0 \leq A \leq A_1$ and since $A_1 \leq A \leq M$, $\frac{(p-1)M}{(p-1+\beta)} \leq A_1$, we have $\frac{dG}{dA} = Mg'(AM) \leq 0$, for all $A_1 \leq A \leq 1$. It follows that
$sup_{0<A<M} inf_{s\in[s_1,1]} g(As) = sup_{0<A<M} G(A) = G(A_1) = g(A_1)$
$= (\frac{M^{\frac{p-1+\beta}{\beta}} - M s_1^{\frac{p-1}{\beta}}}{M^{\frac{p-1+\beta}{\beta}} - s_1^{\frac{p-1+\beta}{\beta}}})^{p-1}$
$(M - \frac{M^{\frac{p-1+\beta}{\beta}} - M s_1^{\frac{p-1}{\beta}}}{M^{\frac{p-1+\beta}{\beta}} - s_1^{\frac{p-1+\beta}{\beta}}})^{\beta} = \frac{M^{\beta+P-1}(M^{\frac{p-1}{\beta}} - s_1^{\frac{p-1}{\beta}})^{p-1}((M-1)M^{\frac{p-1}{\beta}} + (1-s_1)s_1^{\frac{p-1}{\beta}})^{\beta}}{(M^{\frac{p-1+\beta}{\beta}} - s_1^{\frac{p-1+\beta}{\beta}})^{\beta+p-1}}$.
Therefore

$$sup_{0<A<M} inf_{s\in[s_1,1]} g(As) = H_{\Omega}(inf \psi_{\Gamma}),$$

and therefore

$$sup\ \beta(A, \Gamma) = \mu_{\Gamma} H(s_1),$$

which proves our lower estimate.
We know that λ^* is positive, pick $\lambda \in (0, \lambda^*)$ and use the difinition of λ^* to find $\hat{\lambda} \in (\lambda, \lambda^*)$ such that (1.1) has a solution $u_{\hat{\lambda}}$, i.e,
$-\triangle u_{\hat{\lambda}} = \frac{\hat{\lambda} f(x)}{(M-u_{\hat{\lambda}})^{\beta}},\ x \in\ \Omega,\ u_{\hat{\lambda}} = 0,\ x \in\ \partial\Omega.$
And in particular $-\triangle u_{\hat{\lambda}} \geq \frac{\hat{\lambda} f(x)}{(M-u_{\hat{\lambda}})^{\beta}},\ x \in \Omega$ witch then implies that $u_{\hat{\lambda}}$ is a supersolution of (1.1) for every $\lambda \in (0, \lambda^*)$.
The difinition of λ^* in (2.1) implies that there is no solution of (1.1) for every $\lambda > \lambda^*$. □

3. LOWER BOUNDS OF CRITICAL PARAMETER λ^*

Generally it is difficult to find out a proper Γ for detrmining the value of v_{Ω} and hence to therefore it is desirable to seek for better lower bound on λ^*. We proceed this work as follows.
Consider the problem

$-\triangle_p u = \frac{\lambda f(x)}{(M-u^{\beta})},\ x \in\ \Omega,$
$u = 0,\ x \in\ \partial\Omega,$

with $0 < u < M$ on $\Gamma \subset R^N$, and denote $\lambda^*(\Gamma)$ as its critical parameter. The following result is similar to Theorem 4.10 of [1].

Proposition 3.1. *For any bounded domain Γ in R^N and any function f on Γ such that $0 \leq f \leq 1$, we have $\lambda^*(\Gamma, f) \geq \lambda^*(B_R, f^*)$, where $(B_R = B_R(0))$ is the Euclidean ball in R^N with radius $R > 0$ and with volume $|B_R| = |\Gamma|$, and where f^* is the Schwaz symmetrization of f.*

Proposition 3.2. *Assume that f satisfies $0 \leq f \leq 1$ on Ω, then critical parameter λ^* satisfies*

$$\lambda^* \geq \frac{2(p-2+N)\beta^{\beta}M^{\beta+1}}{(\beta+1)^{\beta+1}sup_{x\in\Omega}f}(\frac{\omega_N}{|\Omega|})^{\frac{2}{N}}. \tag{3.1}$$

In patticular, if $f(x) = |x|^{\alpha}$ on Ω with $\alpha \geq 0$, then we have the lower bound

$$\lambda^* \geq \frac{(2+\alpha)M^{\beta+1}\beta^{\beta}((\alpha+1)(p-2)+(N+\alpha))}{(\beta+1)^{\beta+1}}(\frac{\omega_N}{|\Omega|})^{\frac{2+\alpha}{N}} \tag{3.2}$$

Proof. setting $R = (\frac{|\Omega|}{\omega_N})^{\frac{2}{N}}$, it suffies-inview of proposition (3.1) and since $sup_{B_R}f^* = sup_{\Omega}f$ to show that

$$\lambda^* \geq \frac{(p-2+N)\beta^{\beta}M^{\beta+1}}{(\beta+1)^{\beta+1}R^2 sup_{x\in\Omega}f}. \tag{3.3}$$

For the case $\Omega = B_R$. In fact, the function $\omega(x) = \frac{M}{\beta+1}(1-\frac{|x|^2}{R^2})$ satisfies on B_R. It is easy to see $\nabla\omega = -\frac{2M}{(\beta+1)R^2}x, |\nabla\omega| = \frac{2M}{(\beta+1)R^2}|x|$.
Therefor

$$-\triangle_p\omega = (p+N-2)|x|^{p-2}(\tfrac{2M}{(\beta+1)R^2})^{p-1}$$
$$= (p+N-2)|x|^{p-2}(\tfrac{2M}{(\beta+1)R^2})^{p-1}\frac{(M-\frac{M}{\beta+1})^{\beta}}{(M-\frac{M}{\beta+1})^{\beta}}. \tag{3.4}$$

Also we have that

$$(M-\tfrac{M}{\beta+1}(1-\tfrac{|x|^2}{R^2}))^{\beta} \geq (M-\tfrac{M}{\beta+1})^{\beta}, \tag{3.5}$$

therefore

$\frac{1}{(M-\frac{M}{\beta+1})^\beta} \geq \frac{1}{(M-\frac{M}{\beta+1}(1-\frac{|x|^2}{R^2}))^\beta}$. (3.6)

Also we have

$(M - \frac{M}{\beta+1})^\beta = \frac{M^\beta \beta^\beta}{(\beta+1)^\beta}$. (3.7)

From (3.5), (3.6) and (3.7)we deduce that

$-\triangle_p \omega \geq (p+N-2)|x|^{p-2}(\frac{2M}{(\beta+1)R^2})^{p-1} \frac{M^\beta \beta^\beta}{(\beta+1)^\beta sup_{x\in\Omega} f} \frac{f}{(M-\frac{M}{\beta+1}(1-\frac{|x|^2}{R^2}))^\beta}$

$= (p+N-2)|x|^{p-2}(\frac{2M}{(\beta+1)R^2})^{p-2} \frac{2M}{(\beta+1)R^2} \frac{M^\beta \beta^\beta}{(\beta+1)^\beta sup_{x\in\Omega} f} \frac{f}{(M-\omega)}$

$= (p+N-2)|\nabla\omega|^{p-2} \frac{2M^{\beta+1}}{(\beta+1)^{\beta+1} R^2 sup_{x\in\Omega} f} \frac{f}{(M-\omega)^\beta}$. (3.8)

Also note that if $x \in B_R(0)$ then, $|x| < R$ and therefore $\frac{|x|}{R} < 1$.

For $M > \frac{(\beta+1)R}{2}$,we have

$-\triangle_p \omega \geq \frac{2(p+N-2)M^{\beta+1}\beta^\beta}{(\beta+1)^{\beta+1} R^2 sup_{x\in\Omega} f} \frac{f}{(M-\omega)^\beta}$. (3.9)

Thus, if

$$\lambda \leq \frac{2(p+N-2)M^{\beta+1}\beta^\beta}{(\beta+1)^{\beta+1} R^2 sup_{x\in\Omega} f}, \qquad (3.10)$$

then

$\frac{\lambda f(x)}{(M-\omega)^\beta} \leq \frac{f(x)}{(M-\omega)^\beta} \frac{2(p+N-2)M^{\beta+1}\beta^\beta}{(\beta+1)^{\beta+1} R^2 sup_{x\in\Omega} f} \leq -\triangle_p \omega$.

Therfore ω is a supersolution of (1.1) in B_R. Since $\omega_0 \equiv 0$ is a subsolution of (1.1) and $\omega_0 \leq \omega$ in B_R, there exists a solution of (1.1) in B_R. This proves (3.3) and hence (3.1) follows.

In order of prove (3.2), as above it suffices to note that $\frac{M}{\beta+1}(1 - \frac{|x|^{2+\alpha}}{R^{2+\alpha}})$ in B_R is a supersolution of (1.1) if

$$\lambda \leq \frac{M^{\beta+1}\beta^\beta(2+\alpha)((\alpha+1)(p-2)+(N+\alpha))}{(\beta+1)^{\beta+1} R^{2+\alpha}}.$$

This completes the proof of proposition (3.2). □

Remark 3.3. In proposition (3.2) if $M \to \infty$ then $\lambda^* \to \infty$, therefore (1.1) has a solution for $\lambda > \lambda^*$.

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