Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **1**(2)(2012), 88-93

1-Soliton Solution of the Biswas-Milovic Equation With Log Law Nonlinearity

Fayequa Majid [1]
Department of Physics, Chemistry & Mathematics
Alabama A & M University, Normal, AL-35762,USA.

Abstract. This paper studies the Biswas-Milovic equation with log law nonlinearity. The Gausson solution is obtained by the ansatz method. Subsequently, the conservation laws are derived and the conserved quantities are computed using the Gausson solution.

Keywords: Biswas-Milovic equation, Gausson solution, Ansatz method.

1. Introduction

The study of nonlinear waves is governed by several kinds of nonlinear evolution equations (NLEEs) [1-10]. These are studied in various contexts, namely fluid dynamics, nonlinear optics, plasma physics, nuclear physics, mathematical biosciences and various other areas. There are various types of waves that formulate out of these NLEEs. Some of the commonly studied waves are cnoidal waves, snoidal waves, compactons, solitary waves, shock waves, peakons, cuspons, stumpons and many others. Of all these variety of waves, it is the solitons that are primarily focused in the area of the study of waves. This is because solitons are “visible” essentially in our everyday lives. They appear in shallow water waves, fiber optic communications, Langmuir and Alfven waves in plasmas, quantum Hall effect and many others. Thus, in this paper the focus is going to be on the study of a special kind of solitons, also known as *Gaussons* from the Biswas-Milovic equation (BME).

[1]Received: 19 Feb 2012
Accepted: 10 April 2012

2. Governing Equation

The Biswas-Milovic equation (BME) first appeared in 2010 [2]. Subsequently, BME was studied by several authors over time [9, 10]. This equation is a generalized version of the nonlinear Schrödinger's equation (NLSE) that is primarily studied in the context of solitons in nonlinear fiber optics. The dimensionless form of the BME that was first studied in 2010 is given by [2]

$$i\left(q^{m}\right)_{t}+a\left(q^{m}\right)_{xx}+bF\left(|q|^{2}\right)q^{m}=0 \tag{2.1}$$

Here, the dependent variable is $q(x,t)$ which is a complex valued function, with x and t being the two independent variables. The variable x is the spatial variable while t is the temporal variable. The first term represents the temporal evolution of pulses as they propagate down the optical fiber. The constant parameters a and b respectively represent the coefficients of group velocity dispersion (GVD) and nonlinearity. The exponential parameter m, where $m \geq 1$, transforms the NLSE to BME. When NLSE is studied in the context of nonlinear fiber optics, there are several imperfections that arise. The fiber imperfections and its geometry may lead to a departure from linear evolution of pulses after the laser injection from the initial end of the fiber. This leads to a generalized evolution as dictated by the exponential parameter m. Furthermore, the GVD may also be affected and this in turn might lead to its compromise from linearity. Finally the nonlinear term is also generalized to adjust and match the phase component.

Biswas and Milovic only studied the BME with regards to four types of nonlinearity. They are the Kerr law, power law, parabolic law and the dual power law. In all of these cases, the soliton solution was obtained. It needs to be noted that the case of log law nonlinearity was skipped in that paper in 2010 [2]. This paper therefore addresses the BME with log law nonlinearity and a 1-soliton solution will be obtained by the ansatz method.

It is important and preferable to consider the log law nonlinearity. This type of nonlinearity as will be seen gives a Gausson solution. The advantage of these Gaussons is that there is no soliton radiation. This simply implies that there will be no radiation and thus there will be no shedding of energy during the propagation of pulses through optical fibers for long distances [3-6].

3. Mathematical Analysis

In order to integrate the BME given by (2.1), the starting hypothesis is taken to be [2]

$$q(x,t) = g(x-vt)e^{i\phi(x,t)} \tag{3.1}$$

where $g(x,t)$ is the amplitude component while $\phi(x,t)$ is the phase component of the wave that is given by

$$\phi(x,t) = -\kappa x + \omega t + \theta \tag{3.2}$$

where, κ represents the soliton wave number, ω is the soliton frequency and θ is the phase constant. Thus, from (3.1)

$$(q^m)_t = \left(-mvg^{m-1}g' + im\omega g^m\right) e^{im\phi} \tag{3.3}$$

$$(q^m)_x = \left(mg^{m-1}g' - im\kappa g^m\right) e^{im\phi} \tag{3.4}$$

$$F\left(|q|^2\right) q^m = F\left(g^2\right) g^m e^{im\phi} \tag{3.5}$$

and

$$(q^m)_{xx} = \left\{mg^{m-1}g'' - 2im^2\kappa g^{m-1}g' + m(m-1)g^{m-2}\left(g'\right)^2 - m^2\kappa^2 g^m\right\} e^{im\phi} \tag{3.6}$$

Substituting (3.1)-(3.3) and (3.5), (3.6) into (2.1), then decomposing into real and imaginary parts respectively yields

$$m\omega g^m + bF\left(g^2\right) g^m - a\left\{mg^{m-1}g'' + m(m-1)g^{m-2}\left(g'\right)^2 - m^2\kappa^2 g^m\right\} = 0 \tag{3.7}$$

and

$$v = -2am\kappa \tag{3.8}$$

Thus the velocity of the soliton is independent of the type of nonlinearity being considered. Now, equation (3.7) will be integrated for the nonlinearity F that is given by the log law nonlinearity, in the following section.

4. Ansatz Method

The BME with log law nonlinearity is given by

$$i\left(q^m\right)_t + a\left(q^m\right)_{xx} + bq^m \ln|q|^2 = 0 \tag{4.1}$$

Therefore, (8) reduces to

$$m\omega g^m - 2bg^m \ln g + a\left\{mg^{m-1}g'' + m(m-1)g^{m-2}\left(g'\right)^2 - m^2\kappa^2 g^m\right\} = 0 \tag{4.2}$$

The starting hypothesis for log law nonlinearity is given by the Gausson [3, 4]

$$g(x,t) = Ae^{-\tau^2} \tag{4.3}$$

where

$$\tau = B(x - vt) \tag{4.4}$$

and A is the amplitude of the Gausson while B is its inverse width with v being the velocity of the Gausson. Substituting (4.3) into (4.2) yields

$$\omega m - 2b\left(\ln A - \tau^2\right) - 2amB^2 + 4amB^2\tau^2 + 4am(m-1)B^2\tau^2 - am^2\kappa^2 = 0 \tag{4.5}$$

From (4.5) the linearly independent functions are τ^j for $j = 0, 2$. Therefore setting their respective coefficients to zero yields the width of Gausson to be

$$B = \frac{1}{m}\sqrt{\frac{b}{2a}} \tag{4.6}$$

and the frequency ω being

$$\omega = \frac{1}{m}\left(2b\ln A - am^2\kappa^2 - 2amB^2\right) \tag{4.7}$$

The width of the Gausson poses a constraint, namely

$$ab > 0 \tag{4.8}$$

Hence, finally the Gausson solution to the BME is given by

$$q(x,t) = Ae^{-B^2(x-vt)^2}e^{i(-\kappa x + \omega t + \theta)} \tag{4.9}$$

where the amplitude A is arbitrary, while the width B is given by (4.6) and the frequency ω is seen in (4.7). The velocity of the Gausson is given by (3.8). These introduces the constraint relation (4.8) for the Gaussons to exist.

4.1. **Conserved Quantities.** The BME introduces a few conserved quantities. They are the energy (E), linear momentum (M) and the Hamiltonian (H) that are respectively given by

$$E = \int_{-\infty}^{\infty} |q|^{2m}\,dx = \frac{A^{2m}}{B}\sqrt{\frac{\pi}{2m}} \tag{4.10}$$

$$M = i\int_{-\infty}^{\infty} \left\{(q^m)^*(q^m)_x - (q^m)(q^m)^*_x\right\}dx = -\frac{2\kappa A^{2m}}{B}\sqrt{\frac{\pi}{2m}} \tag{4.11}$$

and

$$\begin{aligned} H &= \int_{-\infty}^{\infty}\left\{a\left|(q^m)_x\right|^2 - 2b|q|^2\ln|q| + b|q|^2\right\}dx \\ &= \frac{amA^{2m}}{B}\left(B^2 + m\kappa^2\right)\sqrt{\frac{\pi}{2m}} + \frac{bA^2}{2B}\left\{2(1-\ln A)+1\right\}\sqrt{\frac{\pi}{2}} \end{aligned} \tag{4.12}$$

These conserved quantities are all computed by the Gausson solution that was obtained in the previous part of this section.

5. Perturbation Term

In this section, the perturbed BME will be considered where the perturbation term will be due to the inter-modal dispersion (IMD) in its generalized format. Thus, the perturbed BME that will be studied in this section will be

$$i\left(q^{m}\right)_{t}+a\left(q^{m}\right)_{xx}+bq^{m}\ln|q|^{2}=i\alpha\left(q^{m}\right)_{x} \tag{5.1}$$

where in (5.1) α represents the coefficient of IMD with full nonlinearity [4]. It needs to be noted that it is only this Hamiltonian perturbation term that makes the BME integrable with log law nonlinearity. However, with other Hamiltonian perturbation terms such as third order dispersion, self-steepening, nonlinear dispersion, BME with log law nonlinearity is not rendered integrable at least by this ansatz method [4].

Thus, to integrate (5.1), the same starting hypothesis as in (3.1) is used. In this case substituting (3.1) into (5.1) and then decomposing into real and imaginary parts respectively yield

$$m\omega g^{m}-2bg^{m}\ln g+a\left\{mg^{m-1}g''+m(m-1)g^{m-2}\left(g'\right)^{2}-m^{2}\kappa^{2}g^{m}\right\}=m\alpha\kappa g^{m} \tag{5.2}$$

and

$$v=-2am\kappa-\alpha \tag{5.3}$$

This shows that the IMD will produce a change in the soliton velocity. For (5.2), proceeding in the same way as in the previous section leads to the same Gausson as in (4.9), where the width is again given by (4.6) while the frequency is now given by

$$\omega=\frac{1}{m}\left(2b\ln A-am^{2}\kappa^{2}-2amB^{2}-m\alpha\kappa\right) \tag{5.4}$$

and consequently the same constraint condition as in (4.8) holds for the existence of Gaussons.

6. Conclusions

This paper studied the BME with log law nonlinearity. The 1-soliton solution was obtained by the ansatz method. This solution is also known as *Gausson* in the context of nonlinear optics. A couple of conserved quantities are also obtained using the Gausson solution. The constraint relation also fell out naturally from the solution. An exact solution with the IMD perturbation term taken into account is also derived.

These solutions are going to be very helpful in future in order to compute the adiabatic parameter dynamics of the Gausson parameters when perturbation terms are considered. Later on, the multiple-scale perturbation analysis will be applied to study the perturbed BME. These results and other new results will be published later.

References

[1] I. Bialynicki-Birula. "Gaussons: Solitons of the logarithmic Schrödinger equation". *Physica Scripta*, 20(3-4) (1979) 539-544.

[2] A. Biswas & D. Milovic. "Bright and dark solitons of the generalized nonlinear Schrödinger's equation". *Communications in Nonlinear Science and Numerical Simulation*, 15(6) (2010) 1473-1484.

[3] A. Biswas & D. Milovic. "Optical solitons with log law nonlinearity". *Communications in Nonlinear Science and Numerical Simulation*, 15(12) (2010) 3763-3767.

[4] A. Biswas, C. Cleary, J. E. Watson Jr. & D. Milovic. "Optical soliton perturbation with time-dependent coefficients in a log law media". *Applied Mathematics and Computation*, 217(6) (2010) 2891-2894.

[5] A. Biswas & C. M. Khalique. "Stationary solutions of the nonlinear Schródinger's equation with log law nonlinearity by Lie symmetry analysis". *Waves in Random and Complex Media*, 21(4) (2010) 554-558.

[6] A. Biswas, D. Milovic & R. Kohl. "Optical soliton perturbation in a log law medium with full nonlinearity by He's semi-inverse variational principle". To appear in *Inverse Problems in Science and Engineering*.

[7] P. Guerrero, J. L. Lopez & J. Nieto. "Global H^1 solvability of the 3D logarithmic Schrödinger equation". *Nonlinear Analysis: Real World Applications*, 11(1) (2010) 79-87.

[8] C. M. Khalique & A. Biswas. "Gaussian soliton solution to nonlinear Schrödinger's equation with log law nonlinearity". *International Journal of Physical Sciences*, 5(3) (2010) 280-282.

[9] C. M. Khalique. "Stationary solutions for the Biswas-Milovic equation". *Applied Mathematics and Computation*, 217(18) (2011) 7400-7404.

[10] B. Sturdevant. "Topological 1-soliton solution of the Biswas-Milovic equation with power law nonlinearity". *Nonlinear Analysis; Real World Applications*, 11(4) (2010) 2871-2874.