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A Uniqueness Theorem of the Solution of an Inverse Spectral Problem

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Abstract. This paper is devoted to the proof of the unique solvability of the inverse problems for second-order differential operators with regular singularities. It is shown that the potential function can be determined from spectral data, also we prove a uniqueness theorem in the inverse problem.

Keywords: Inverse spectral problem, Eigenvalues, Uniqueness theorem.

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1. Introduction and Preliminaries

We consider the eigenvalue problems generated by the differential equation

$$\ell_j(w) := -w'' + q_j(x)w = \mu w, \quad x \in [0,T], \quad j = 1,2, \tag{1.1}$$

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and boundary conditions

$$U_{j0}(w) = 0, \quad U_{j1}(w) = 0. \tag{1.2}$$

Here $\mu = \eta^2$ is the eigenvalue parameter, the potential functions $q_j(x)$ are real and have regular singularities.
Inverse spectral problems, in particularly self-adjoint ones, received extensive in last years such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences (see [1],[2],[3]).
In 1952 and 1978, Marchenko and Levitan used the transformation to show that the eigenvalues and norming constants uniquely determine potential function [10],[8]. On the other hand, a finite number of eigenvalues in one spectrum is unknown, $q(x)$ is not uniquely determined by one full spectrum and one partial spectrum. This problem was investigated by Gesztesy and Simon in [6]. In later years, these problems was studied for regular and singular problem by some authors, [7], [9], [12], [13]. Freiling and yurko studied an inverse problem of synthesizing parameters of differential systems having a finite number of arbitrary order singularities and turning points, and established properties of the spectral characteristics, also, by using the weyl-function method, they proved a uniqueness theorem for the solution of the inverse problem (see[4]).
In [5], the authors considered a second-order differential equation having singularities at the end-points of the interval and gave formulations of the inverse problems both for the case of separated and non-separated singular boundary conditions. Also, some second-order differential operators with arbitrary regular nonseparable boundary conditions were studied in [5], and it is shown that the operator can be recovered from three of its spectra. In [11], the authors studied differential systems having a finite number of arbitrary order singularities and turning points, and they fined the asymptotic approximation of the eigenvalues and the infinite product representation of solutions of the Sturm-Liouville problem.
The main purpose of this paper is to study inverse spectral problems for singular Sturm-Liouville expressions in the cases that the potential function has a regular singularity at $x = 0$ or inside the interval $(0, T)$. Sections 2,3 are devoted to inverse problems for the first case, and in section 4, we study the inverse problems for boundary value problems in the second case. For each class of these inverse problems we show that the potential function can be determined from spectral data and we prove uniqueness theorems in the inverse problems.

2. Inverse problems I,II: Boundary value problems with singularity inside the interval

Let us consider the boundary value problem $L_1 = L_1(q_1(x), H)$ defined by equation

$$\ell_1 y \equiv -w'' + q_1(x)w = \mu w, \quad x \in [0,T], \tag{2.1}$$

and the boundary conditions

$$w'(0,\mu) + i\eta w(0,\mu) = 0, \quad w'(T,\mu) + hw(T,\mu) = 0. \tag{2.2}$$

where $\mu = \eta^2$ is the spectral parameter, H is a real number, and the potential function $q_1(x)$ is the form

$$q_1(x) = \frac{a}{(x-x_1)^2} + q_0(x),$$

where a is a real. We assume that $q_0(x) \in L(0,T)$. Denote

$$\Omega := \{(\eta, x) : \ |\eta(x-x_1)| \geq 1\}.$$

Let $F(x,\eta)$ be the solution of (3) under the initial conditions

$$F(0,\eta) = 1, \quad F'(0,\eta) = -i\eta. \tag{2.3}$$

From [4], there are fundamental systems of solutions $\{H_k(x,\mu)\}_{k=1,2}$ of equation (3), where for each $x \in [0,T]\backslash\{x_1\}$, the functions $H^{(\nu)}(x,\mu)$, $\nu = 0,1$, are entire in μ. Also for $x \in [0,T]\backslash\{x_1\}$, $(\eta,x) \in \Omega, k,m = 1,2$, $|\eta| \to \infty$

$$H_k^{(m-1)}(x,\mu) = \frac{1}{2}(i\eta)^{m-k}\{\exp(i\eta x)[[1]] + \exp(-i\eta x)[[1]] \tag{2.4}$$

$$+(-1)^k 2i\cos\pi\nu\exp(i\eta(x-2x_1))[[1]]\},$$

Where $[[1]] = 1 + O((\eta(x-x_1))^{-1})$. Moreover, $H_k^{(m-1)}(0,\mu) = \delta_{k,m}$ ($\delta_{k,m}$ is the Kronecker delta). We will call the functions $H_k(x,\mu)$ the Bessel-type solutions for equation (3). Thus, according to (5) and (6) we have the following theorem.

Theorem 2.1. *Let $F(x,\eta)$ be the solution of (3) under the initial conditions (5). Then, for $x \in [0,T]\backslash\{x_1\}$, $(x,\eta) \in \Omega$, $|\eta| \to \infty$, $Im\eta \geq 0$ and $m = 0,1$,*

$$F^{(m)}(x,\eta) = (-i\eta)^m \exp(-i\eta x)[[1]] - 2i(i\eta)^m \cos\pi\nu\exp(i\eta(x-2x_1))[[1]],$$

The functions $x \longmapsto F(x,\mu)$ are eigenfunctions of L_1. The boundary value problem L_1 has a countable set of eigenvalues $\{\eta_n(q,h)\}_{n\geq 1}$. It follows from (2) and theorem 1 that of the problem L_1 in the from

$$\eta_n = \sqrt{\eta_n(q,h)} = \frac{n\pi + \frac{\pi}{4}}{T - x_1} + O(\frac{1}{n}). \tag{2.5}$$

Now, we consider the boundary value problem $L_2 = L_2(q_1(x))$ for equation (3) with the boundary conditions

$$w'(0,\mu) + i\eta w(0,\mu) = 0, \quad w'(T,\mu) = 0. \tag{2.6}$$

Using theorem 1 and (8) we can calculate the spectrum $\{\widetilde{\mu}_n(q_1)\}_{n\geq 1}$, corresponding set of eigenvalues of the problem L_2 . The spectrum $\{\widetilde{\mu}_n(q_1)\}_{n\geq 1}$ has the asymptotic

$$\widetilde{\eta}_n = \sqrt{\widetilde{\mu}_n(q_1)} = \frac{n\pi - \frac{3\pi}{4}}{T - x_1} + O(\frac{1}{n}). \tag{2.7}$$

3. Inverse problems and Uniqueness Theorem

In this section, it will be given that one potential $q_1(x)$ can be determined from $\mu_n(q_1, h_k)$, where n is fixed and $h_k, k \geq 1$, are distinct.

Firstly, according to (7) and (9) we have the following lemma.

Lemma 3.1. *Let h is a real number, then*

$$\widetilde{\mu}_n(q_1) < \mu_n(q,h) \leq \widetilde{\mu}_{n+1}(q_1).$$

Now, we give the following lemma that required for proof of the uniqueness theorem. This lemma is a statement of the inverse problem of the singular Sturm-Liouville operator.

Lemma 3.2. *Let $\mu_n(q_i^0, h_i)$, $i = 1, 2$, are eigenvalues of the problems*

$$-w'' + (q_i^0(x) + \frac{a}{(x - x_1)^2})w = \mu w,$$

$$w'(0,\mu) + i\eta w(0,\mu) = 0, \quad w'(T,\mu) + h_i w(T,\mu) = 0.$$

If for $n = 1, 2, 3, ...$ these eigenvalues satisfy

$$\mu_n(q_1^0, h_1) = \mu_n(q_2^0, h_1)$$
$$\mu_n(q_1^0, h_2) = \mu_n(q_2^0, h_2),$$

then $q_1^0 = q_2^0$.

The following uniqueness theorem is the main result of this section.

Theorem 3.1. *Let $q_1^0(x), q_2^0(x) \in L^2(0,T)$. Assume that h_k for $k = 1, 2, 3, ...$ are real distinct numbers and*

$$\mu_n(q_1^0, h_k) = \mu_n(q_2^0, h_k),$$

then $q_1^0 = q_2^0$.

Proof. For each λ, let $\psi_i(x) = \psi(x, q_i^0, \mu)$, $i = 1, 2$, be the solution of problem

$$-w'' + (q_i^0(x) + \frac{a}{(x-x_1)^2})w = \mu w, , \tag{3.1}$$

with the conditions

$$w'(0,\mu) + i\eta w(0,\mu) = 0, \quad w'(T,\mu) = 0, \tag{3.2}$$

Then, by using (11) it has been the Sturm identity for Sturm-Liouville problem

$$\psi(x, q_1^0, \mu)\{\psi''(x, q_2^0, \mu) - [q_2^0(x) + \frac{a}{(x-x_1)^2}]\psi(x, q_2^0, \mu)\} \tag{3.3}$$

$$-\psi(x, q_2^0, \mu)\{\psi''(x, q_1^0, \mu) - [q_1^0(x) + \frac{a}{(x-x_1)^2}]\psi(x, q_1^0, \mu)\}$$

$$= \quad \{q_1^0(x) - q_2^0(x)\}\psi(x, q_1^0, \mu)\psi(x, q_2^0, \mu) + [\psi(x, q_1^0, \mu)\psi'(x, q_2^0, \mu)$$

$$- \quad \psi(x, q_2^0, \mu)\psi'(x, q_1^0, \mu)]' = 0.$$

Now, to facilitate some softwares , we use the simplified notation

$$\lambda_k = \mu_n(q_1^0, H_k) = \mu_n(q_2^0, H_k), \ k = 1, 2, ...$$

Inserting $\mu = \lambda_k$ in (11) and integrating from 0 to T, we get for $k = 1, 2, ...$,

$$\int_0^T (q_1^0 - q_2^0)\psi(x, q_1^0, \lambda_k)\psi(x, q_2^0, \lambda_k)dx = 0.$$

For fixed x, it can be shown that

$$S(\mu) := \int_0^T (q_1^0 - q_2^0)\psi(x, q_1^0, \mu)\psi(x, q_2^0, \mu)dx$$

Is analytical function of μ then $S(\mu) \equiv 0$. Now, we will show that all of the eigenvalues of problem L_1 and all of the eigenvalues of problem L_2 are the same for the $q^0(x) = q_i^0(x)$, i.e., we will show that for $n = 1, 2, ...$,

$$\widetilde{\eta}_n(q_1^0) = \widetilde{\eta}_n(q_2^0), \tag{3.4}$$

$$\eta_n(q_1^0, 0) = \eta_n(q_2^0, 0). \tag{3.5}$$

then from lemma 2 we would be able to conclude that $q_1^0 = q_2^0$. For proving (14), (15),we return to the identity (11) and get that when $\eta = \eta_n(q_1^0, 0)$, then $\psi(1, q_1^0, \eta_n(q_1^0, 0)) \neq 0$ while $\psi'(1, q_1^0, \eta_n(q_1^0, 0)) = 0$. Integrating (11) from 0 to T and using $S(\mu) \equiv 0$ we must have
$\psi'(1, q_2^0, \eta_n(q_1^0, 0)) = 0, n = 1, 2, ...$, this implies that each $\eta_n(q_1^0, 0)$ is an eigenvalue for L. Similarly set $\eta = \widetilde{\eta}_n(q_1^0)$ in the identity (11) and doing the above process we conclude that (11) holds. Then, the proof is complete.

4. Inverse problems III,IV: The inverse problem for the singularity type $\frac{A}{x^2}$

In this section we consider the eq.(3) in the case the potential functions $\frac{A}{x^2} + p_1(x)$ where A is a real number and $p_1(x) \in L^2(0, T)$. In this case from [?], for the interval $(0, T]$ there exist two linearly independent solutions $z_1(x, \eta), z_2(x, \eta)$ of the equation (3) satisfying

$$z_k(x, \eta) = d_k \eta^{-\mu_k} (e^{-i\pi\mu_k + i\eta x}[1]_0 + e^{-i\eta x}[1]_0), \tag{4.1}$$

where $\mu_k := (-1)^k \nu + \frac{1}{2},\ k = 1, 2, \quad d_1^0 d_2^0 = -\frac{1}{4i \sin \pi\nu}$.

Let $L_3 = L_3(p_1(x), r_1, \zeta_1)$ be the inverse problem of the from

$$\ell_2 y \equiv -y'' + (p_1(x) + \frac{A}{x^2})y = \beta y, \tag{4.2}$$

with the conditions

$$y'(0, \beta) - r_1 y(0, \beta) = 0, \quad y'(T, \beta) + \zeta_1 y(T, \beta) = 0, \tag{4.3}$$

where $r_1, \zeta_1 \in R$. Now, we consider $u(x, \beta)$ be the solution of (17) that

$$u(T, \beta) = 1, \quad u'(T, \beta) = -\zeta_1. \tag{4.4}$$

Theorem 4.1. *Let $u(x, \tau)$ be the solution of (17) now, with the boundary conditions $u'(0) - r_1 u(0) = 0$,*

$$\begin{aligned} u(x, \tau) = \frac{d_{10}\tau^{-\mu_{10}}}{2\tau^2(\exp(-i\pi\mu_{10}) - \exp(-i\pi\mu_{20}))} &\{(-\exp(-i\tau x) - \exp(i\tau x - i\pi\mu_{10}))(r_1 + i\tau) \\ &+(\exp(i\tau x) - \exp(-i\tau x - i\pi\mu_{20}))(r_1 - i\tau) \\ &+(\exp(-i\tau x) + \exp(i\tau x - i\pi\mu_{20}))(r_1 + i\tau) \\ &+(\exp(-i\tau x - i\pi\mu_{10}) - \exp(i\tau x))(r_1 - i\tau)\}. \end{aligned}$$

Proof. For fixed $x \in (0, T]$, use (15) and (18) we determine the connection coefficient a_1, a_2 with

$$u(x, \tau) = a_1 z_1(x, \tau) + a_2 z_2(x, \tau), \tag{4.5}$$

where $\beta = \tau^2$. Substituting the estimates of z_1 and z_2 from (15) we get the solution $u(x, \tau)$ for $x \in (0, T]$.

Now, let us consider $L_3 := L_3(p_1(x), r_1, \zeta_1)$ with the boundary condition

$$u'(0,\tau) - r_1 u'(0,\tau) = 0. \tag{4.6}$$

Applying theorem 3 and (21), we obtain the following estimates for the eigenvalues of inverse problem L_3

$$\tau_n = \sqrt{\beta_n(p_1, r_1, \zeta_1)} = \frac{n\pi + \frac{\pi}{4} + \frac{\pi}{2}\nu}{T} + O(\frac{1}{n}).$$

Similarly, for the problem $L_4 := L_4(p_1, r_2, \zeta_2)$, $r_2, \zeta_2 \in R$, for the differential equation

$$-y'' + (\frac{A}{x^2} + p_1(x))y = \widetilde{\beta} y, \tag{4.7}$$

with the conditions

$$y'(0,\widetilde{\beta}) - r_2 y(0,\widetilde{\beta}) = 0, \quad y'(T,\widetilde{\beta}) + \zeta_2 y(T,\widetilde{\beta}) = 0, \tag{4.8}$$

there exists a countable set of eigenvalues $\{\widetilde{\beta}_n\}_{n\geq 1}$ of the from

$$\widetilde{\tau}_n = \sqrt{\widetilde{\beta}_n(p_1, r_2, \zeta_2)} = \frac{n\pi + \frac{\pi}{4} - \frac{\pi}{2}\nu}{T} + O(\frac{1}{n}).$$

Corollary 4.2. *Let $p_1(x) \in L^2(0,T)$, then for all $\zeta_2 \in R$,*

$$\widetilde{\tau}_n(p_1, r_1, \zeta_1) < \tau_n(p_1, r_2, \zeta_2) \leq \widetilde{\tau}_{n+1}(p_1, r_1, \zeta_1). \tag{4.9}$$

Thus, we have the following theorem.

Theorem 4.3. *Let $\zeta^k, k \in N$, are real numbers and for $p_1^1(x), p_2^1(x) \in L^2(0,T)$,*

$$\beta_n(p_1^1, r_1, \zeta_2^k) = \beta_n(p_2^1, r_2, \zeta_2^k), k \in N$$

then $p_1^1 = p_2^1$.

Proof. For each λ, let $\theta_i(x) = \theta(x, p_i^1, \beta)$, $(i = 1,2)$ be the solution of problem

$$-y'' + (p_i^1(x) + \frac{A}{x^2})y = \beta y, \tag{4.10}$$

$$y'(0,\beta)) = 0, \quad y'(T,\beta) + \zeta_1 y(T,\beta) = 0, \tag{4.11}$$

then, we substitute for $\theta_i(x) = \theta(x, p_i^1, \beta)$, in (24) and (25), The calculations have

$$= \{p_1^1(x) - p_2^1(x)\}\theta(x, p_1^1, \beta)\theta(x, p_2^1, \beta) \tag{4.12}$$

$$+[\theta(x, p_1^1, \beta)\theta'(x, p_2^1, \beta) - \theta(x, p_2^1, \beta)\theta'(x, p_1^1, \beta)]' = 0.$$

Now, to facilitate some softwares , we use the simplified notation

$$\iota_k = \beta_m(p_1^1, S_k) = \beta_m(p_2^1, S_k), \; k = 1, 2, ... \quad .$$

Inserting $\beta = \iota_k$ in (26) and integrating from 0 to T, we get for $k = 1, 2, ...$,

$$\int_0^T (p_1^1 - p_2^1)\theta(x, p_1^1, \iota_k)\theta(x, p_2^1, \iota_k)dx = 0.$$

Now, we will show that all of the eigenvalues of problem L_3 with $r_1 = 0$ and all of the eigenvalues of problem L_4 are the same for the $p^1 = p_i^1$, i.e., we will show that for $m = 1, 2, ...$,

$$\widetilde{\tau}_m(p_1^1) = \widetilde{\tau}_m(p_2^1), \tag{4.13}$$

$$\tau_m(p_1^1, 0) = \ \tau_m(p_2^1, 0). \tag{4.14}$$

we would be able to conclude that $p_1^1 = p_2^1$.

References

[1] K. Daho and H. Langer, *Sturm-Liouville operators with an indefinite weight function,* Proc. Roy. Soc. Edinburgh, **78A** (1977), 161-191.

[2] A. A. Dorodnicyn, *Asymptotic laws of distribution of the characteristic values for certain special forms of differential equations of the second order,* Tamsui Oxford Journal of Mathematical Sciences., **25** (2009), 277-283.

[3] M. V. Fedoryuk, *Asymptotic Analysis*, Springer-Verlag, Berlin,1993.

[4] G. Freiling and V. Yurko, *On the determination of differential equations with singularityes and turning points*, Results in Mathematics, **41** (2002) 275-290.

[5] G. Freiling and V. Yurko, *Boundary value problems with regular singularities and singular boundary conditions.,* Int. Journal. Math. Math. Sci, (2005), 1481-1495.

[6] F. Gesztesy and B. Simon, *Uniqueness theorems in inverse spectral throry for one dimensional Schrodinger operators*, Trans. Amer. Math. Soc., 348(1996), 349-373.

[7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. 204, Elsevier (North-Holland) Science Publishers, Ameterdam, London and New York, 2006.

[8] B. M. Levitan, *On the determination of the Sturm-Liouville operator from one and two spectra*, Math. USSR Izvestija, **12** (1978), 179-193.

[9] S-D. Lin, W.-C. Ling, K. Nishimoto, and H. M. Srivastava, *A simple fractional-calculus approach to the solutions of the Bessel differential equation of general order and some of its applications*, Comput. Math. Appl., **49** (2005), 1487-1498.

[10] V. A. Marchenko, *Certain problems of the theory of one dimensional linear differential operators of the second order I*, Trudy Mosk. Math Obshch., **1** (1952), 327-340.

[11] A. Neamaty and S. Mosazadeh, *On the canonical solution of Sturm-Liouville problem with singularity and turning point of even order*,Canad. Math. Bull, **54** (2011), no. 3, 506-518.

[12] E. S. Panakhov and H. Koyunbakan, *Inverse problem for singular Sturm-Liouville operator*, Proceeding of IMM of NAS of Azerbaijan, (2003), 113-126.

[13] P.-Y. Wang, S.-D. Lin, and H. M. Srivastava, *Remarks on a simple fractional-calculus approach to the solutions of the Bessel Dierential equation of general order and some of its applications*, Comput. Math. Appl., **51** (2006), 105-114.