Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **3**(1)(2014), 25-37

Numerical solution of nonlinear Hammerstein integral equations through Legendre-Bernstein basis

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Abstract. In this study a numerical method is developed to solve the Hammerstein integral equations. To this end, the kernel has been approximated using the least-squares approximation schemes based on Legendre-Bernstein basis. The Legendre polynomials are orthogonal and this property improves the accuracy of the approximations. Also the nonlinear unknown function has been approximated by using the Bernstein basis. The useful properties of Bernstein polynomials help us to transform Hammerstein integral equation to solve a system of nonlinear algebraic equations.

Keywords: Nonlinear Hammerstein integral equations, Bernstein basis, Legendre basis, Orthogonal polynomials.

2000 Mathematics subject classification: 45B05, 45D05, 45G10, 65D30.

1. Introduction

The Bernstein form of a polynomial offers valuable insight into its geometrical behavior, and has thus won widespread acceptance as the basis for Bézier curves and surfaces. For least-squares approximation problems, on the other hand, the use of orthogonal bases, such as the Legendre polynomials [3,5,7], permits simple and efficient constructions for convergent sequences of approximates.

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Received: 25 Feb 2012
Revised: 25 May 2013
Accepted: 29 May 2013

In this paper, we consider the nonlinear Fredholm-Hammerstein integral equations and nonlinear Volterra-Hammerstein integral equations respectively by the general forms [1,8],

$$g(s) = f(s) + \lambda \int_0^1 k(s,t)F(t,g(t))dt, \quad (1.1)$$

$$g(s) = f(s) + \lambda \int_0^s k(s,t)F(t,g(t))dt \quad ; \quad 0 \leq s \leq 1, \quad (1.2)$$

where the parameter λ and functions $f(s), k(s,t)$ and $F(s,t)$ are known and $g(s)$ is unknown function. It is also assumed that all of these functions are L_2-Functions on $[0,1]$, and $g(s) \in C[0,1]$.

In the following we will introduce the Legendre and Bernstein polynomials and some properties of them used in this article.

1.1. **Legendre polynomials.** To emphasize symmetry properties of Legendre polynomials, they are traditionally defined on the interval $[-1,+1]$, but for our purposes it is preferable to map this to $[0,1]$. The Legendre polynomials $L_k(u)$ on $u \in [0,1]$, can be generated through the recurrence relation [4],

$$(k+1)L_{k+1}(u) = (2k+1)(2u-1)L_k(u)kL_{k1}(u) \quad ; \quad k = 1,2,\cdots, \quad (1.3)$$

commencing with $L_0(u) = 1$ and $L_1(u) = 2u - 1$.
This gives, in the first few instances

$$\begin{aligned} L_0(u) &= 1, \\ L_1(u) &= 2u - 1, \\ L_2(u) &= 6u^2 - 6u + 1, \\ L_3(u) &= 20u^3 - 30u^2 + 12u - 1, \\ &\vdots \end{aligned}$$

The orthogonality of these polynomials is expressed by the relation

$$\int_0^1 L_j(u)L_k(u)du = \begin{cases} \frac{1}{2k+1} & j = k \\ 0 & j \neq k \end{cases} .$$

Now for arbitrary function $f(u)$ on $[0,1]$, we can express it in the Legendre form,

$$f(u) \simeq P_N(u) = \sum_{j=0}^{N} l_j L_j(u), \quad (1.4)$$

where the coefficients l_j, for Legendre polynomials are obtained from following relation

$$l_k = (2k+1)\int_0^1 L_k(u)f(u)du \quad ; \quad k = 0, 1, \cdots, N. \tag{1.5}$$

1.2. **Bernstein polynomials.** $(N+1)$-Bernstein basic function on $[0,1]$, are defined by using the following relation [2],

$$B_{i,N}(u) = \binom{N}{i} u^i (1-u)^{N-i} \quad ; \quad i = 0, 1, \cdots, N. \tag{1.6}$$

In the follow, some properties of Bernstein polynomials have been expressed and used in this article,

- The product of a power basic function and a Bernstein basic function,

$$u^m B_{i,N}(u) = \frac{\binom{N}{i}}{\binom{N+m}{i+m}} B_{i+m,N+m}(u). \tag{1.7}$$

- The product of two Bernstein basic functions,

$$B_{i,j}(u)B_{k,m}(u) = \frac{\binom{j}{i}\binom{m}{k}}{\binom{j+m}{i+k}} B_{i+k,j+m}(u). \tag{1.8}$$

- The expression of power basic functions in the Bernstein form and vice versa,

$$B_{k,N}(u) = \sum_{i=k}^{N} (-1)^{i-k} \binom{N}{i}\binom{i}{k} u^i. \tag{1.9}$$

  Let $B_s^t = [B_{0,N}(s), B_{1,N}(s), \cdots, B_{N,N}(s)]$ and $S^t = [1, s, s^2, \cdots, s^N]$ then

$$B_s = MS \quad \text{and} \quad S = M^{-1}B_s, \tag{1.10}$$

  where

$$M = \begin{bmatrix} (-1)^0\binom{N}{0}\binom{0}{0} & (-1)^1\binom{N}{1}\binom{1}{0} & \cdots & (-1)^N\binom{N}{N}\binom{N}{0} \\ 0 & (-1)^0\binom{N}{1}\binom{1}{1} & \cdots & (-1)^{N-1}\binom{N}{N}\binom{N}{1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (-1)^0\binom{N}{N}\binom{N}{N} \end{bmatrix} \tag{1.11}$$

- All the basic functions have the same definite integral over $[0,1]$, namely

$$\int_0^1 B_{i,N}(u)du = \frac{1}{N+1} \quad ; \quad i = 0, 1, \cdots, N. \tag{1.12}$$

- The produced matrix from the integration over the product of two bases in form $T = \int_0^1 B_s B_s^t ds$,
  where T is a $(N+1)\times(N+1)$ matrix by elements in the following form

$$T_{i+1,j+1} = \frac{\binom{N}{i}\binom{N}{j}}{(2N+1)\binom{2N}{i+j}} \quad ; \quad i,j = 0,1,\cdots,N. \tag{1.13}$$

- Operational matrix of integration
  Let $B_t^t = [B_{0,N}(t), B_{1,N}(t), \cdots, B_{N,N}(t)]$, and $\tau^t = [1, t, t^2, \cdots, t^N]$, then the integration of vector B_t is given by

$$\int_0^s B_t dt \simeq PB_s, \tag{1.14}$$

where P is the $(N+1)\times(N+1)$ operational matrix for integration and is given in [10]. By using of (1.11), we have

$$\int_0^s B_t dt = \int_0^s M\tau dt = M\int_0^s \tau dt = M\begin{bmatrix} s \\ \frac{1}{2}s^2 \\ \vdots \\ \frac{1}{N+1}s^{N+1} \end{bmatrix} = MM_pS_p, \tag{1.15}$$

where $S_p^t = [s, s^2, \cdots, s^{N+1}]$, and M_p is the following matrix

$$M_p = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{N+1} \end{bmatrix}_{(N+1)\times(N+1)} , \tag{1.16}$$

According to (1.11), we had $S = M^{-1}B_s$. Therefore for $k = 0, 1, \cdots, N$, we have

$$s^k = M^{-1}_{[k+1]}B_s, \tag{1.17}$$

where $M^{-1}_{[k+1]}$ is $(k+1)$-th row of M^{-1} for $k = 0, 1, \cdots, N$. We just need to approximate $s^{N+1} \simeq B_s^t C_{N+1}$. Having the product both sides of it at B_s and integration on $[0,1]$, we have

$$C_{N+1} = T^{-1}\int_0^1 s^{N+1}B_s ds = T^{-1}\begin{bmatrix} \int_0^1 s^{N+1}B_{0,N}(s)ds \\ \int_0^1 s^{N+1}B_{1,N}(s)ds \\ \vdots \\ \int_0^1 s^{N+1}B_{N,N}(s)ds \end{bmatrix} = \frac{T^{-1}}{2N+2}\begin{bmatrix} \frac{\binom{N}{0}}{\binom{2N+1}{N+1}} \\ \frac{\binom{N}{1}}{\binom{2N+1}{N+2}} \\ \vdots \\ \frac{\binom{N}{N}}{\binom{2N+1}{2N+1}} \end{bmatrix}. \tag{1.18}$$

now assume

$$B = \begin{bmatrix} M_{[2]}^{-1} \\ M_{[3]}^{-1} \\ \vdots \\ M_{[N+1]}^{-1} \\ C_{N+1}^t \end{bmatrix}, \tag{1.19}$$

then $S_p \simeq BB_s$. Therefore we have the operational matrix of integration $P = MM_pB$.

1.3. **The expression of the Legendre polynomials in the Bernstein form.** In this scale, we expand a favorite polynomial such as $P_N(s)$ in terms of Legendre-Bernstein basis. That is, we combine two bases Legendre and Bernstein, and then calculate expansion coefficients. The Legendre polynomials $L_k(s)$ can be expressed in the Bernstein basis B_s of degree N as

$$L_k(s) = \sum_{j=0}^{N} \Lambda_{k,j} B_{j,N}(s) \quad ; \quad k = 0, 1, \cdots, N, \tag{1.20}$$

where [4],

$$\Lambda_{k,j} = \frac{1}{\binom{N}{j}} \sum_{i=max(0,j+k-N)}^{min(j,k)} (-1)^{k+i}\binom{k}{i}\binom{k}{i}\binom{N-k}{j-i} \quad ; \quad j, k = 0, 1, \cdots, N. \tag{1.21}$$

Now considering the polynomial $P_N(s)$ of degree N, as expressed in (1.4), we can transform it in the Bernstein form as

$$P_N(s) = \sum_{k=0}^{N} l_k L_k(s) = \sum_{k=0}^{N} l_k \left(\sum_{j=0}^{N} \Lambda_{k,j} B_{j,N}(s)\right) = \sum_{j=0}^{N} b_j B_{j,N}(s),$$

where

$$b_j = \sum_{k=0}^{N} l_k \Lambda_{k,j} \quad ; \quad j, k = 0, 1, \cdots, N \qquad or \qquad b = l^t \Lambda.$$

That b_j are expansion coefficients of $P_N(s)$, in terms of Legendre-Bernstein basis. Similarly, we can calculate expansion coefficients of least squares approximation of kernel $k(s,t)$, based on Legendre-Bernstein basis. Let $L_s^t = [L_0(s), L_1(s), \cdots, L_N(s)]$, then for $k(s,t)$ we have

$$\begin{aligned} k(s,t) &= L_s^t K L_t \\ &= \sum_{m=0}^{N}\sum_{n=0}^{N} L_m(s) k_{m,n} L_n(t) \\ &= \sum_{m=0}^{N}\sum_{n=0}^{N} \left(\sum_{i=0}^{N} \Lambda_{m,i} B_{i,N}(s)\right) k_{m,n} \left(\sum_{j=0}^{N} \Lambda_{n,j} B_{j,N}(t)\right) \\ &= \sum_{i=0}^{N}\sum_{j=0}^{N} B_{i,N}(s) \left(\sum_{m=0}^{N}\sum_{n=0}^{N} \Lambda_{m,i} k_{m,n} \Lambda_{n,j}\right) B_{j,N}(t), \end{aligned}$$

where

$$\begin{aligned} k_{m,n} &= \frac{\langle \langle k(s,t), L_n(t)\rangle, L_m(s)\rangle}{\langle L_n(t), L_n(t)\rangle \langle L_m(s), L_m(s)\rangle} \\ &= (2n+1)(2m+1) \int_0^1 \int_0^1 L_m(s) L_n(t) k(s,t) dt ds \\ &= (2n+1)(2m+1) \sum_{i=0}^{N}\sum_{j=0}^{N} \Lambda_{m,i}\Lambda_{n,j} \int_0^1 \int_0^1 B_{i,N}(s) B_{j,N}(t) k(s,t) dt ds \\ &\qquad ; \quad i,j = 0,1,\cdots,N. \end{aligned}$$

Let

$$C_{i,j} = \sum_{m=0}^{N}\sum_{n=0}^{N} \Lambda_{m,i} k_{m,n} \Lambda_{n,j} \quad ; \quad i,j = 0,1,\cdots,N, \tag{1.22}$$

or

$$C = \Lambda^t K \Lambda. \tag{1.23}$$

Then

$$k(s,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} B_{i,N}(s) C_{i,j} B_{j,N}(t) = B_s^t C B_t. \tag{1.24}$$

2. Approximation of integral equations

Consider the equation (1.1), as follows

$$g(s) = f(s) + \lambda \int_0^1 k(s,t) F(t, g(t)) dt, \tag{2.1}$$

set $w(s) = F(s, g(s))$, then we have

$$w(s) = F\left(s, f(s) + \lambda \int_0^1 k(s,t)w(t)dt\right),$$

now if we approximate $w(s)$, by Bernstein basis as $w(s) = B_s^t A$, where $A^t = [a_0, a_1, \cdots, a_N]$, and by using of relations (1.13),(1.24), we can write

$$\begin{aligned} B_s^t A &= F\left(s, f(s) + \lambda \int_0^1 B_s^t C B_t B_t^t A dt\right) \\ &= F\left(s, f(s) + \lambda B_s^t C \left(\int_0^1 B_t B_t^t dt\right) A\right) \\ &= F\left(s, f(s) + \lambda B_s^t C T A\right). \end{aligned} \tag{2.2}$$

So by putting the nodes $\{s_i = \frac{i}{N} \mid i = 0, 1, \cdots, N\}$ in (2.2), we get a system of nonlinear algebraic equations of $(N+1) \times (N+1)$ degree, with unknown coefficients $\{a_i \mid i = 0, 1, \cdots N\}$.
After solving this nonlinear system by using of Newton method and by software Matlab, we can approximate the solution of equation (2.1), as follows

$$g(s) = f(s) + \lambda B_s^t C T A. \tag{2.3}$$

3. Apprximation of Volterra integral equations

Consider the equation (1.2), as follows

$$g(s) = f(s) + \lambda \int_0^s k(s,t)F(t, g(t))dt \quad ; \quad 0 \le s \le 1, \tag{3.1}$$

such as Fredholm kind that we set $w(s) = F(s, g(s))$ then

$$w(s) = F\left(s, f(s) + \lambda \int_0^s k(s,t)w(t)dt\right).$$

Now if we approximate $w(s)$, by Bernstein basis as $w(s) = B_s^t A$, that $A^t = [a_0, a_1, \cdots, a_N]$, and by using of (1.24), we can write

$$\begin{aligned} B_s^t A &= F\left(s, f(s) + \lambda \int_0^s B_s^t C B_t B_t^t A dt\right) \\ &= F\left(s, f(s) + \lambda B_s^t C \left(\int_0^s B_t B_t^t A dt\right)\right). \end{aligned} \tag{3.2}$$

Now it's only necessary to express the integration $\int_0^s B_t B_t^t A dt$ in Bernstein basis form. By using of equation (1.11), we have

$$\begin{aligned}\int_0^s B_t B_t^t A dt &= \int_0^s M\tau\left(\sum_{k=0}^{N} a_k B_{k,N}(t)\right)dt \\ &= M\int_0^s \begin{bmatrix} \sum_{k=0}^N a_k B_{k,N}(t) \\ \sum_{k=0}^N a_k t B_{k,N}(t) \\ \vdots \\ \sum_{k=0}^N a_k t^N B_{k,N}(t)\end{bmatrix} dt. \qquad (3.3)\end{aligned}$$

Now, we approximate all functions $t^j B_{k,N}(t)$ in terms of B_t. Namely

$$t^j B_{k,N}(t) \simeq B_t^t e_{j,k} \quad ; \quad j,k = 0,1,\cdots,N, \qquad (3.4)$$

where $e_{j,k}$, is a approximation coefficients vector as follows

$$e_{j,k} = \begin{bmatrix} e_0^{j,k} \\ e_1^{j,k} \\ \vdots \\ e_N^{j,k}\end{bmatrix}. \qquad (3.5)$$

By multiplying B_t, in both sides of (3.4), and integration of them, and by using of (1.13), we have

$$\begin{aligned} e_{j,k} &= T^{-1}\int_0^1 t^j B_{k,N}(t) B_t dt \\ &= T^{-1}\begin{bmatrix} \int_0^1 t^j B_{k,N}(t) B_{0,N}(t)dt \\ \int_0^1 t^j B_{k,N}(t) B_{1,N}(t)dt \\ \vdots \\ \int_0^1 t^j B_{k,N}(t) B_{N,N}(t)dt \end{bmatrix} = \frac{T^{-1}\binom{N}{k}}{2N+j+1}\begin{bmatrix} \frac{\binom{N}{0}}{\binom{2N+j}{k+j}} \\ \frac{\binom{N}{1}}{\binom{2N+j}{k+j+1}} \\ \vdots \\ \frac{\binom{N}{N}}{\binom{2N+j}{k+j+N}} \end{bmatrix}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=0}^N a_k t^j B_{k,N}(t) &\simeq \sum_{k=0}^N a_k B_t^t e_{j,k} = \sum_{k=0}^N a_k\left(\sum_{i=0}^N e_i^{j,k} B_{i,N}(t)\right) = \sum_{i=0}^N B_{i,N}(t)\left(\sum_{k=0}^N a_k e_i^{j,k}\right) \\ &= \begin{bmatrix} \sum_{k=0}^N a_k e_0^{j,k} \\ \sum_{k=0}^N a_k e_1^{j,k} \\ \vdots \\ \sum_{k=0}^N a_k e_N^{j,k}\end{bmatrix} B_t = A^t \begin{bmatrix} e_{j,0}^t \\ e_{j,1}^t \\ \vdots \\ e_{j,N}^t \end{bmatrix} B_t = A^t E_{j+1} B_t,\end{aligned}$$

that E_{j+1} is a $(N+1)\times(N+1)$ matrix that, it has vectors $e_{j,k}^t$, $j = 0,1,\cdots,N$, for each row. Therefore we define $\widehat{E_{j+1}} = A^t E_{j+1}$ for $j =$

$0, 1, \cdots, N$. So

$$\sum_{k=0}^{N} a_k t^j B_{k,N}(t) \simeq \widehat{E_{j+1}} B_t \quad ; \quad j = 0, 1, \cdots, N. \tag{3.6}$$

Now by substituting (3.6), into (3.3), we have

$$B_s^t A = F\left(s, f(s) + \lambda B_s^t CM \int_0^s \begin{bmatrix} \widehat{E_1} B_t \\ \widehat{E_2} B_t \\ \vdots \\ \widehat{E_{N+1}} B_t \end{bmatrix} dt \right). \tag{3.7}$$

If we define matrix G as follows

$$G = \begin{bmatrix} \widehat{E_1} \\ \widehat{E_2} \\ \vdots \\ \widehat{E_{N+1}} \end{bmatrix}, \tag{3.8}$$

that G is a $(N+1) \times (N+1)$ matrix that, it has vectors $\widehat{E_{j+1}}, \quad j = 0, 1, \cdots, N$, for each row. Therefore we can write the equation (3.1), as

$$\begin{aligned} B_s^t A &= F\left(s, f(s) + \lambda B_s^t CM \int_0^s G B_t dt \right) \\ &= F\left(s, f(s) + \lambda B_s^t CMG \int_0^s B_t dt \right), \end{aligned}$$

by using of (1.13), we have

$$B_s^t A = F\left(s, f(s) + \lambda B_s^t CMGPB_s \right). \tag{3.9}$$

So by putting nodes $\{s_i = \frac{i}{N} \mid i = 0, 1, \cdots, N\}$ in (3.9), we get a system of nonlinear algebraic equations of $(N+1) \times (N+1)$ degree, with unknown coefficients $\{a_i \mid i = 0, 1, \cdots N\}$.
After solving this nonlinear system by using of Newton method and by software Matlab, we can approximate the solution of equation (3.1), as follows

$$g(s) = f(s) + \lambda B_s^t CMGPB_s. \tag{3.10}$$

3.1. **Error bound for approximation.** Degree of approximation of a function $f(x), a \leq x \leq b$ by polynomials may be simply described in terms of its modulus of continuity $\omega(\delta) = \omega^f(\delta)$. For each $\delta > 0, \omega(\delta)$ is the maximum of $|f(x) - f(y)|$ for all $a \leq x \leq b,\ a \leq y \leq b,\ |x - y| < \delta$; $\omega(\delta)$ and clearly decreases to 0 with δ if $f(x)$ is continuous.

Assume $P_N(x)$ be an approximation polynomial of function $f(x)$ then we have the following theorem.

Theorem 3.1. *(T. Popoviciu [11]). If $f(x)$ is continuous and $\omega(\delta)$ the modulus of continuity of $f(x)$, then*

$$|f(x) - P_N(x)| \leq \frac{5}{4}\omega(N^{-\frac{1}{2}}). \tag{3.11}$$

Now we find error bound for nonlinear Volterra-Hammerstein integral equations and so, for Fredholm kind is as the same. Assume $P_N(s)$ and $g(s)$ be approximate and exact solutions of the integral equation (3.1), respectively, so

$$P_N(s) - \lambda \int_0^s k(s,t)F(t, P_N(t))dt = f(s) + R_N(s), \tag{3.12}$$

where $R_N(s)$ is the perturbation function that depends only on $P_N(s)$. Let $M \equiv \sup_{0 \leq s,t \leq 1} |k(s,t)| < \infty$, and suppose $F(t,s)$ satisfied in Lipschitz condition such that

$$|F(t, s_1) - F(t, s_2)| \leq L|s_1 - s_2|. \tag{3.13}$$

Let $E(s) = |g(s) - P_N(s)|$ be the error function of this method. By subtracting equation (3.12), from equation (3.1), we have

$$|R_N(s)| \leq E(s) + |\lambda|MLE(s) = (1 + |\lambda|ML)E(s), \tag{3.14}$$

where by substituting (3.11), into (3.14), we obtain an error bound for the perturbation function $R_N(s)$ such as

$$|R_N(s)| \leq (1 + |\lambda|ML)\frac{5}{4}\omega(N^{-\frac{1}{2}}). \tag{3.15}$$

4. Illustrations

Example 4.1. Consider nonlinear Fredholm integral equation [11,12]:

$$g(s) = 1 + s + \left(1 - \frac{3}{2}\ln(3) + \frac{\sqrt{3}}{6}\pi\right)s^2 + \int_0^1 2s^2 t \ln(g(t))dt, \tag{4.1}$$

where the exact solution is $g(s) = 1 + s + s^2$. Table 1 and Figure 1 show the numerical results for Example 4.1 in comparison with methods of [11,12].

Table 1: Numerical results for Example 4.1.

Nodes $s_i = \frac{i}{10}$	Method of [6] $N = 6$	Method of [7] $N = 6$	Present method $N = 6$	Exact solution
0.0	1.000000	1.0000000000000000	1.000000000000000	1.00
0.1		1.1099999949939017	1.109999999879360	1.11
0.2	1.238432	1.2399999799756067	1.239999999517438	1.24
0.3		1.3899999549451152	1.389999998914235	1.39
0.4	1.553726	1.5599999199024268	1.559999998069751	1.56
0.5		1.7499998748475423	1.749999996983986	1.75
0.6	1.945884	1.9599998197804611	1.959999995656940	1.96
0.7		2.1899997547011830	2.189999994088613	2.19
0.8	2.414905	2.4399996796097088	2.439999992279004	2.44
0.9		2.7099995945060370	2.709999990228114	2.71
1.0	2.960788	2.9999994993901690	2.999999987935944	3.00

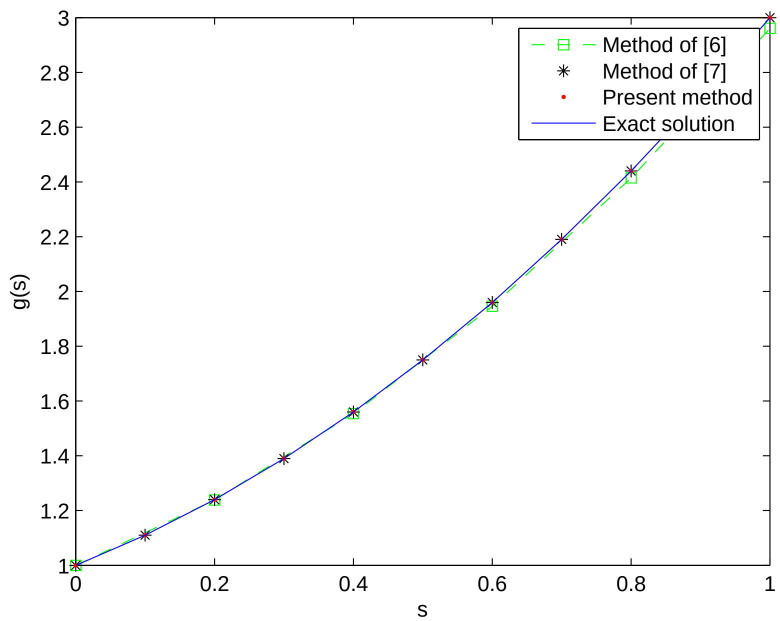

FIGURE 1. Numerical results for Example 4.1.

Example 4.2. Consider nonlinear Volterra integral equation [9]:

$$g(s) = s^2 - 2 - \frac{1}{2}se^{(s^2-2)} + \frac{1}{2}se^{(-2)} + \int_0^s ste^{g(t)}dt, \qquad (4.2)$$

with the exact solution $g(s) = s^2 - 2$. Table 2 and Figure 2 shows the numerical results for Example 4.2 in comparison with method of [9].

Table 2: Numerical results for Example 4.2.

$Nodes\ s_i = \frac{i}{10}$	Method of [8] $(m = 1, n = 5)$	Present method $N = 6$	Exact solution
0.0	−1.999721	−2.000000000000000	−2.00
0.1		−1.990000220506790	−1.99
0.2	−1.959387	−1.959999536138756	−1.96
0.3		−1.909999888083469	−1.91
0.4	−1.839465	−1.840000917327985	−1.84
0.5		−1.750000310331762	−1.75
0.6	−1.639457	−1.639998696711833	−1.64
0.7		−1.509999550677477	−1.51
0.8	−1.359370	−1.360002193879734	−1.36
0.9		−1.189998649656369	−1.19
1.0	−0.999668	−1.000006046653275	−1.00

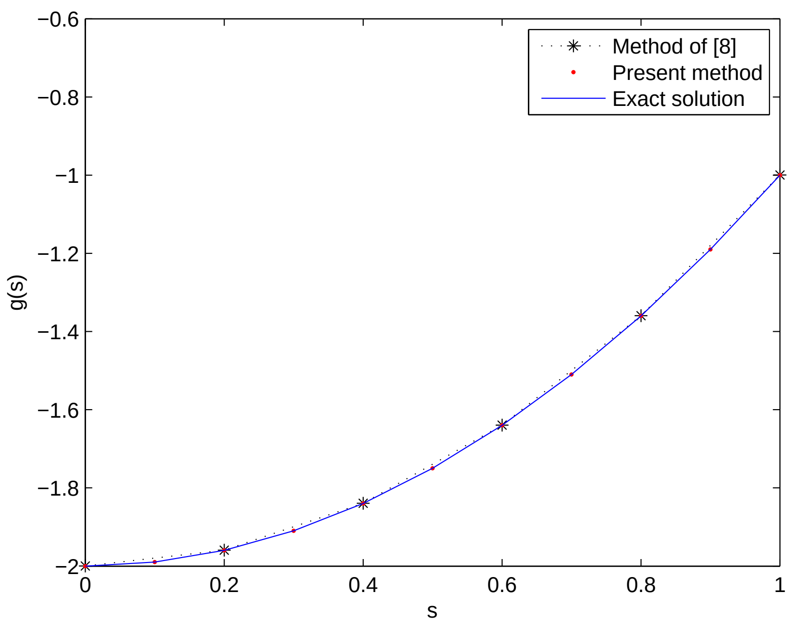

FIGURE 2. Numerical results for Example 4.2.

5. Conclusion

In this paper, the unknown function has been extended in terms of Bernstein basis. And the kernel of nonlinear Hammerstein integral equations has been extended by the least squares approximation of Legendre-Bernstein basis. The advantage of this method is that, both characteristics orthogonality of Legendre polynomials and simplification of Bernstein polynomials are used. Thus, we have accuracy and simplicity together and numerical results obtained from the examples show the accuracy. Therefore, this basis is as a reliable for approximation functions, where coefficients are easily calculated, as it in context.

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