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Linear fractional programming problem in hesitant fuzzy environment

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Abstract. In recent years, the hesitant fuzzy set as a proper generalization of the fuzzy set theory, which can imply in situations that the decision maker hesitates in determining the membership of parameters, has been introduced. Several applications of such sets have been revealed in the multi-criteria decision-making, graph theory and clustering methods, but there is little research on hesitant fuzzy programming problems and solving process for them in the literature. However, recently some research has been carried out in the field of linear programming under hesitant fuzzy information. However, less research can be found that has developed this perspective in nonlinear mode and especially for linear fractional programming under hesitant fuzzy data. Hence, our main focus is to propose the modeling of the linear fractional programming problem with hesitant fuzzy parameters along with the introduction of a method to solve this type of structure. For this aim, two kinds of linear fractional programming with hesitancy in different values are introduced. Then, a novel method is suggested to determine the optimal solutions for them. Some numerical examples show the reliability and validity of the models.

Keywords: Hesitant fuzzy sets, linear fractional programming problem, fuzzy linear fractional programming problem.

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1. Introduction

Here we focus on the fractional linear programming problem as a type of fractional programming problem in which the objective function is considered as a ratio of two linear functions (for example, we may want the ratio of the cost function to the time function which is expressed as two linear functions and minimize them). In addition, in this case, it is assumed that the constraints of the problem have a linear structure. Issues such as financial and corporate planning (debt-to-equity ratio), production planning (inventory/sales, output/employee), and health care and hospital planning (cost/patient, nurse-patient ratio) among other issues, applications are based on relative proportions of physical or economic values that can be modeled as linear fractional programming (LFP) problems. In 1962, Charnes and Cooper [9] showed that an LFP problem with a finite set of feasible solutions can be transformed into a linear programming problem by appropriate variable transformations. Various methods have been introduced to solve the LFP problem over the years. For example, the LFP problem was solved by several researchers [15,17,24] using different types of solution processes based on the simplex method developed [11]. Two different methods based on the feasible direction method and the duality method for solving the LFP problem have been presented by Tantawy [26,27].

The complexities governing most of the economic, engineering and dynamic systems problems are one of the main reasons for the expansion of nonlinear models and the importance of examining their corresponding solution approaches from different perspectives. To ensure solvability, the extension of methods based on linear approximations as well as metaheuristic algorithms have attracted the attention of many researchers. For example, Ahmad et al. [1] proposed a modified method to the variational iteration algorithm-II (MVIA-II) for the numerical treatment of diffusion as well as convection-diffusion equations. For this, they defined an auxiliary parameter which makes sure of a fast convergence of the standard VIA-II iteration algorithm. In another paper, Ahmad et al. [2] used a developed computational scheme for the nonlinear predator-prey model for forming new computational findings that introduced a prototype of an excitable system. More efforts have been made to develop mathematical modeling based on complex physical phenomena, along with presenting and analyzing the solution processes and checking the convergence of methods by some other researchers [3-6].

In a special type of nonlinear programming problem, the goal is maximization or minimization of functions that are written as a ratio of two functions that are not necessarily linear. Specifically, in economic optimization, problems of ratios such as profitability for investment,

profitability to cost and income to risk under existing organizational conditions can play a key role in determining constructive decisions to increase system efficiency. Various models of data envelopment analysis can be considered as a complete example of fractional structures that provide a suitable analytical perspective for the managers of organizations to improve and promote productivity regarding the ratio of data output(s) to input(s). However, deficit planning is not only limited to economic issues and increasing productivity in organizations. For instance, Ahmad et al. [7] introduced a novel method called the fractional iteration algorithm-I without using transformation, small perturbation, Adomian polynomials, and linearization for finding the conventional solution of nonlinear non-integer order partial differential equations.
With the emergence of fuzzy theory, the concept of fuzzy sets was used for greater flexibility and reliability in modeling LFP problems to real-world problems [8]. To solve the FLFp problem when all parameters and variables are defined by triangular fuzzy numbers. Pop and Stancu-Minasian [18] designed a method in which they transformed the original fuzzy problem into a crisp multi-objective LFP problem with quadratic constraints. The generalization method based on Bellman and Zadeh's principle has been used to define this transformation to evaluate the fuzzy constraints. Das et al. [12] formulated an equivalent three-objective LFP problem for solving the triangular fuzzy linear fractional programming problem. In 2019, Sirinivasan [23] presented a ranking method based on triangular fuzzy numbers to solve FLFP in wood company. Recently, Loganathan and Ganesan [16] introduced a method to solve the fully FLFP problem without converting to its equivalent deterministic format. Mehlawat and Kumar [18] proposed a method to calculate an acceptable optimal value (,), where [0,1] and [0,1] are the satisfaction scores associated with the fuzzy objective function and they show fuzzy limits respectively. Also, the authors in [10] used the acceptable optimal value (,r) for a linear fractional programming problem with fuzzy coefficients and fuzzy decision variables and presented a method for their calculation. Ebrahimnejad et al. [13]. proposed a new method to solve FLFP with all parameters as well as decision variables like triangular fuzzy numbers by transforming it into a bi-objective linear programming problem.

Over time, new extensions of fuzzy numbers with different structural features of membership function and non-membership function were invented to better match the modeling of problems arising from the real world with human understanding and knowledge. However, the problem arises from the lack of an answer to the question that sometimes a decision maker (due to his relative knowledge) hesitates in assigning a value as a membership degree to an element or that different experts are required to assign a membership degree to an element. In the sense that they do not have the same opinion, what should be done? It has led to the introduction of hesitant fuzzy sets by Torra and Narukawa, [28,29]. In fact, hesitant fuzzy sets are stated to explain those situations where decision makers hesitate about some values to evaluate different parameters.

In hesitant fuzzy sets (HFSs), one can consider multiple membership values for an element of the set, while other fuzzy number extensions lack this feature. Hence, HFSs can provide decision makers with a suitable and complete structure for modeling problems. Since its introduction, HFSs have found many extensions in theoretical and practical fields. However, very little research has been done in the field of linear programming, nonlinear programming, and fractional linear programming. For linear programming under hesitant fuzzy data, Renjber and Effati [20] defined hesitant fuzzy decision environment as a generalization of the fuzzy decision space and presented methods for solving symmetric and right-sided fuzzy linear programming problems.

Farnam and Darehmiraki [14] formulated and solved the linear programming structure for a three-level multi-objective supply chain problem under hesitant fuzzy numbers. In the following, Ranjbar et al. [21] presented an approach to solve a fully hesitant fuzzy linear programming (FHFLP) problem by using an (,)-cut for the hesitant fuzzy numbers. According to this attitude, the main problem is converted into some interval linear programming (ILP) problems and hence the final approximate solutions are computed through one of the available algorithms to solve the ILP problems.

The lack of research related to hesitant fuzzy linear fractional problem (HFLFP) and the wide application of fractional problems in physical and economic models is the primary motivation of the authors for the current research. Converting the main problem (HFLFP) into several sub-problems in the form of a fuzzy linear fractional problem (FLFP) and using Bellman and Zadeh's principle in solution approach guaranteed the robustness structure of the proposed method. In section 2: we review some fundamental notations and terminologies related to hesitant fuzzy sets. In section 3: firstly, the problem of LFP and the expressions of some required contents are considered. secondly, the formulation of the LFP problem and the solution process define under the hesitant fuzzy environment for two types of such problem. In section 4: three numerical examples illustrate the capability of proposed models and solution procedures. In section 5: the conclusions are provided in this part.

2. Preliminaries and notations

This section involves some basic notions of HFS and its operations.

Definition 2.1. Xia and Xu in [25] defined the HFS as follows:

$$H = \{(x, \tilde{h}_H(x)) | x \in X\}$$

where $h_H(x)$ is a set of values in [0,1] indicating the possible membership degrees of element like x belong to X for the set H. Most of the time, $\tilde{h}_H(x)$ is defined an hesitant fuzzy element (HFE). They defined some of operations on HFE as follows:

a) $\tilde{h}_1(x) \cup \tilde{h}_2(x) = \bigcup_{\gamma_1 \in \tilde{h}_1(x), \gamma_2 \in \tilde{h}_2(x)} max\{\gamma_1, \gamma_2\}$,
b) $\tilde{h}_1(x) \cap \tilde{h}_2(x) = \bigcap_{\gamma_1 \in \tilde{h}_1(x), \gamma_2 \in \tilde{h}_2(x)} min\{\gamma_1, \gamma_2\}$,
c) $(\tilde{h}_1(x))^\lambda = \bigcup_{\gamma_1 \in \tilde{h}_1(x)} \{\gamma_1^\lambda\}$,
d) $\lambda(\tilde{h}_1(x)) = \bigcup_{\gamma_1 \in \tilde{h}_1(x)} \{1 - (1 - \gamma_1)^\lambda\}$.

In next definition we generalize Bellman and Zadeh theory which need to state a decision in hesitant fuzzy spaces [8].

Definition 2.2. [20] Let we have a hesitant fuzzy objective function $\tilde{\tilde{G}}$ and a hesitant constraint $\tilde{\tilde{C}}$ in X, which means the combintion of $\tilde{\tilde{G}}$ and $\tilde{\tilde{C}}$ make a decition like $\tilde{\tilde{D}}$, as a hesitant fuzzy decision caused by the intersection of $\tilde{\tilde{G}}$ and $\tilde{\tilde{C}}$. In this case, $\tilde{\tilde{D}} = \tilde{\tilde{G}} \cap \tilde{\tilde{C}}$ is corresponded to $\tilde{h}_{\tilde{\tilde{D}}} = \tau(\tilde{h}_{\tilde{\tilde{G}}}, \tilde{h}_{\tilde{\tilde{C}}})$, such that:

$$\tilde{h}_{\tilde{\tilde{C}}} = \{\tilde{h}^1_{\tilde{\tilde{C}}}, \tilde{h}^2_{\tilde{\tilde{C}}}, \cdots, \tilde{h}^{qc}_{\tilde{\tilde{C}}}\} \qquad , \qquad \tilde{h}_{\tilde{\tilde{G}}} = \{\tilde{h}^1_{\tilde{\tilde{G}}}, \tilde{h}^2_{\tilde{\tilde{G}}}, \cdots, \tilde{h}^{qc}_{\tilde{\tilde{G}}}\},$$

where the numbers of the decision makers that establish different aspiration level for the goal and constraints are shown by q_G and q_C, respectively. Furthermore τ as a T-norm is applied to detemine the membership degree values of intersectin the hesitant fuzzy elements. This idea can extend for multi objective programming.

Definition 2.3. [22] Let $\tau : H^{(m)} \times H^{(m)} \to H^{(m)}$, that $H^{(m)}$ is a HFS with m elements and τ is a hesitant triangular norm such that, if $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3 \in H^{(m)}$ then the following axioms hold:

a) Commutative property: $\tau(\tilde{h}_1, \tilde{h}_2) = \tau(\tilde{h}_2, \tilde{h}_1)$,
b) Associative property: $\tau(\tilde{h}_1, \tau(\tilde{h}_2, \tilde{h}_3)) = \tau(\tau(\tilde{h}_1, \tilde{h}_2), \tilde{h}_3)$,
c) Monotone property: if $\tilde{h}_2 \leq_{H^{(m)}} h_3$ *then* $\tau(\tilde{h}_1, \tilde{h}_2) \leq_{H^{(m)}} \tau(\tilde{h}_1, \tilde{h}_3)$,
d Neutral element: $\tau(\tilde{h}_1, 1_{H^{(m)}}) = \tilde{h}_1$.

where $1_{H(m)} = \{1, 1, \cdots, 1\}$ is a full HFE with m elements.
In this study, for the hesitant triangular norm on HFE we use the minimum operator as follows:

$$\tau(\tilde{h}_1, \tilde{h}_2) = \bigcup_{\gamma_1 \in \tilde{h}_1(x), \gamma_2 \in \tilde{h}_2(x)} \min\{\gamma_1, \gamma_2\}$$

3. Developing LFP problem under hesitant fuzzy decision making environment

In this section, at first the problem of LFP and the expression of some required properties and theorems are reviewed. Then, we investigate the formulation of the LFP problem under the hesitant fuzzy environment for two types of uncertainties in the hesitant fuzzy environment. Also the solution processes are developed efficiently.

3.1. **LFP problem.** The LFP problem, which objective function is expressed as a ratio of two linear functions, is introduced generally as follows:

$$LFP : \max \frac{f(x)}{g(x)} = \frac{c^T x + \alpha}{d^T x + \beta}, \tag{3.1}$$
$$s.t \quad x \in X(A, b) = \{x \in R^n : Ax(\leq = \geq)b, x \geq 0\}$$

where $A \in R^{mn}, \alpha, \beta \in R, c, d \in R^n$, and $b \in R^m$. Due to the convex space of the feasible solution space, $X(A, b)$, if the denominator is non-zero in the objective function, the sign of the denominator will not change. (i.e. for all $x \in X(A, b)$, we have: $d^T x + \beta > 0$ or $d^T x + \beta < 0$. Therefore, the sign of the denominator will always be positive or negative. In this study, without loss of generality, we always consider the sign of the dominator is positive. (If the denominator is negative, we multiply the numerator and the denominator by a negative, so for all $x \in X(A, b)$, we have $d^T x + \beta > 0$).
The solution process of LFP:
According to the prior assumptions, for solving the classic form of LFP problem that is defined in (1), Charnes and Cooper [9] used the change of variable technique. They defined $t \in R$, as follows:

$$t = \frac{1}{d^T x + \beta}. \tag{3.2}$$

Then by applying this transformation to the objective function and defining $y = tx$ where $y \in R^n$ and $\gamma \in R$ is nonzero (as a scaler number), Problem (1), for the case that the constraints are as equality form can be converted to the following equivalent form:

$$\begin{aligned} &\max c^T y + \alpha t \\ &s.t \quad Ay - bt = 0, \\ &d^T y + \beta t = \gamma, \\ &y, t \leq 0. \end{aligned} \tag{3.3}$$

It is clear that using this method, the number of variables and the number of constraints increase from n to $n+1$ and from m to $m+1$, respectively.

Lemma 3.1. *For every (y,t) where satisfied in the constraints of model (3), we have, $t > 0$.*

Theorem 3.2. *If x^* is an optimum solution of the problem (1), such that $sgn(\gamma) = sgn(d^T x^* + \beta t) > 0$ and (y^*, t^*) is an optimum solution of the problem (4), then $\frac{y^*}{t^*}$ is an optimum solution of the problem (1).*

Theorem 3.3. *Assume that there is not any solution such $(y, 0)$ with $y \geq 0$ for model (3). Also, let denominator is positive in x, then problems (1) and (3) are equivalent.*

Remark: If the sign of denominator in optimal solution for model (1) is negative, then we have similar problem such as model (3) with some partial variations [9].

Notice: In this study we assume that the sign of denominator in optimal solution is positive.

Corollary [9]**:**For every nonempty and bounded set of feasible solutions of the problem (1), we can solve the following equivalent problem:

$$\begin{aligned} &\max c^T y + \alpha t \\ s.t \quad &Ay - bt = 0, \\ &d^T y + \beta t = 1, \\ &y, t \geq 0. \end{aligned} \tag{3.4}$$

3.2 Hesitant linear fractional programming (HLFP) problem

Due to the uncertainty in the opinions of decision makers and their hesitation to determine the value of parameters, uncertainty can be occurring in different part of modeling. In this study we consider specially two cases of these uncertainties and state solving procedures as follows.

Model 1 of HFLFP problem:

$$max \frac{f(x)}{g(x)} = \frac{c^T x + \alpha}{d^T x + \beta} \tag{3.5}$$
$$s.t \quad x \in X_{\leq}(A, \tilde{\tilde{b}}) = \{x \in R^n : A_i x \leq \tilde{\tilde{b}}_i, for\ i = 1, 2, \cdots, m, x \geq 0\}$$

where, A_i indicates the i^{th} row $(i = 1, 2, , m)$ of the A. Assuming that $d^{k_i}_{\tilde{\tilde{b}}_i}, k_i = 1, 2, \cdots, q_i$, is the maximum permissible deviation from the value of $b^{k_i}_i$ (right hand side value of the i^{th} constraint) which is determined by the k^{th}_i decision maker. Also consider q_i as the number of decision makers who have established their evaluations for i^{th} constraint.

The solution process for model 1 of HFLFP problem:

According to definition 2.2. we should determine the hesitant fuzzy decision space of presented model. The corresponding membership function of each constraint for $k_i = 1, 2, \cdots, q_i$ can be introduced by the following linear formulation:

$$\mu^{k_i}_{\tilde{\tilde{b}}_i}(A_i x) = \begin{cases} 1 & A_i x \leq b^{k_i}_i,\ for\ i = 1, 2, \cdots, m \\ 1 - \frac{A_i x - b^{k_i}_i}{d^{k_i}_{\tilde{\tilde{b}}_i}} & b^{k_i}_i \leq A_i x \leq b^{k_i}_i + d^{k_i}_{\tilde{\tilde{b}}_i} \\ 0 & A_i x \geq b^{k_i}_i + d^{k_i}_{\tilde{\tilde{b}}_i}. \end{cases} \tag{3.6}$$

As a result, for the hesitant decision space caused by each constraint, we have:

$$\tilde{h}_{\tilde{\tilde{b}}_i}(x) = \{\mu^1_{\tilde{\tilde{b}}_i}(x), \mu^2_{\tilde{\tilde{b}}_i}(x), \cdots, \mu^{q_i}_{\tilde{\tilde{b}}_i}(x)\}.$$

And hence the hesitant fuzzy decision space of all model constraints is as follows:

$$\tilde{h}_{\tilde{\tilde{b}}}(x) = \{\tilde{h}_{\tilde{\tilde{b}}_1}(x), \tilde{h}_{\tilde{\tilde{b}}_2}(x), \cdots, \tilde{h}_{\tilde{\tilde{b}}_m}(x)\} = \{\mu^{k_i}_{\tilde{\tilde{b}}_i}(x) : i = 1, 2, \cdots, m\ ,\ k_i = 1, 2, \cdots, q_i\}.$$

Since the constraints of model (6) are HFSs, the decision space resulting from the objective function is also preferably HFS. Hence the HFEs of the hesitant fuzzy objective should be determined. To form a suitable membership function for the objective function, we first solve the following two fractional programming problem for $s = 1, 2, \cdots, (q_1 q_2 q_m)$:

$$p^{z_l^s}: \quad z_l^s = max \frac{c^T x + \alpha}{d^T x + \beta} \tag{3.7}$$
$$s.t \quad A_i x \leq b_i^{k_i} \; for \; i = 1, 2, \cdots, m,$$
$$x \geq 0,$$

and

$$p^{z_u^s}: \quad z_u^s = max \frac{c^T x + \alpha}{d^T x + \beta} \tag{3.8}$$
$$s.t \quad A_i x \leq b_i^{k_i} + d_{\tilde{\tilde{b}}_i}^{k_i} \; for \; i = 1, 2, \cdots, m,$$
$$x \geq 0.$$

If z_u^s and z_l^s are non-negative and $z_l^s \leq z_u^s$, the optimal values obtained from (8) and (9) using the transformation variable of the Charnes and Cooper transformation, then the $s^t h$ membership function for the objective function is defined as follows:

$$\mu_{\tilde{\tilde{z}}_0}^s(y,t) = \mu_{\tilde{\tilde{z}}_0}^s(tf(\frac{y}{t})) = \begin{cases} 1 & tf(\frac{y}{t}) > z_u^s, \\ 1 - \frac{z_u^s - tf(\frac{y}{t})}{z_u^s - z_l^s} & z_l^s \leq tf(\frac{y}{t}) \leq z_u^s, \\ 0 & tf(\frac{y}{t}) < z_l^s. \end{cases} \tag{3.9}$$

Therefore, the hesitant fuzzy decision space related to the objective function can propose as follows:

$$\tilde{h}_{\tilde{\tilde{z}}_o}(y,t) = \{\mu_{\tilde{\tilde{z}}_0}^1(y,t), \mu_{\tilde{\tilde{z}}_0}^2(y,t), \cdots, \mu_{\tilde{\tilde{z}}_0}^{(q_1,q_2,\cdots,q_m)}(y,t)\}.$$

Additionally, by applying Charnes and Cooper's transformation the constraint $A_i x \leq \tilde{\tilde{b}}_i$ is converted to $A_i y - \tilde{\tilde{b}}_i t \leq 0$, and hence the hesitant fuzzy membership function correspond to the i^{th} constraint for $k_i = 1, 2, \cdots, q_i$ obtains as follows:

$$for \; i = 1, 2, \cdots, m.$$

$$\mu_{\tilde{\tilde{b}}_i}^{k_i}(y,t) = \mu_{\tilde{\tilde{b}}_i}^{k_i}(A_i y - \tilde{\tilde{b}}_i^{k_i} t) = \begin{cases} 1 & A_i y - b_i^{k_i} t < 0 \\ 1 - \frac{A_i y - b_i^{k_i} t}{d_{\tilde{\tilde{b}}_i}^{k_i}} & 0 \leq A_i y - b_i^{k_i} t \leq d_{\tilde{\tilde{b}}_i}^{k_i} \\ 0 & A_i y - b_i^{k_i} t > d_{\tilde{\tilde{b}}_i}^{k_i} \end{cases} \tag{3.10}$$

Therefore, the hesitant fuzzy decision space of the constraints is represented as follows:

$$\tilde{h}_{\tilde{\tilde{b}}}(y,t) = \{\tilde{h}_{\tilde{\tilde{b}}_1}(y,t), \tilde{h}_{\tilde{\tilde{b}}_2}(y,t), \cdots, \tilde{h}_{\tilde{\tilde{b}}_m}(y,t)\} = \{\mu_{\tilde{\tilde{b}}_i}^{k_i}(y,t) : i = 1, 2, \cdots, m,$$

$$k_i = 1, 2, \cdots, q_i\}$$

In other words, for each constraint $(i = 1, 2, , m)$ we have:

$$\tilde{h}_{\tilde{\tilde{b}}_i}(y,t) = \{\mu^1_{\tilde{\tilde{b}}_i}(y,t), \mu^2_{\tilde{\tilde{b}}_i}(y,t), \cdots, \mu^{q_i}_{\tilde{\tilde{b}}_i}(y,t)\}.$$

Now, using the membership functions introduced in (10) and (11) and based on definition 2.2, the hesitant fuzzy decision space resulting from the model (1), can be defined by:

$$\tilde{h}_{\tilde{\tilde{D}}} = \tau(\tilde{h}_{\tilde{\tilde{z}}_0}, \tilde{h}_{\tilde{\tilde{b}}}).$$

Now apply the Bellman and Zadeh principle to solve $(q_1, q_2, \cdots, q_m)^2$ number of programming problems by introducing variable $\lambda^r, (r = 1, 2, \cdots, (q_1, q_2, \cdots, q_m)^2)$ as follows:

$$\begin{aligned}
&p^r : max\ \lambda^r \\
&1 - \frac{z_u^r - t^r f(\frac{y^r}{t^r})}{z_u^r - z_l^r} \geq \lambda^r, \\
&1 - \frac{A_i y^r - b_i^{k_i} t^r}{d^{k_i}_{\tilde{\tilde{b}}_i}} \geq \lambda^r, \qquad for\ i = 1, 2, \cdots, m \\
&d^T y^r + \beta t^r = 1, \\
&\lambda^r \in [0,1], \quad y^r \geq 0, t^r \geq 0.
\end{aligned} \tag{3.11}$$

If $(y^{r*}, t^{r*}, \lambda^{r*})$ denotes the hesitant fuzzy optimal solution of (12), where λ^{r*} indicates the highest degree of aspiration level that objective function and constraints together can be achieved in the r^{th}problem with the optimal solution (y^{r*}, t^{r*}), then $x^{r*} = \frac{y^{r*}}{t^{r*}}$ shows the hesitant fuzzy optimal solution of (6). Actually by assuming that the number of variables is n in the main problem, hesitant fuzzy decision space can be written as:

$$\tilde{h}_{\tilde{\tilde{D}}}(y^*, t^*) = \{\lambda^{1*}, \lambda^{2*}, \cdots, \lambda^{(q_1 q_2 \cdots q_m)^2 *}\},$$

where

$$(y^*, t^*) = (y_1^*, y_2^*, \cdots, y_n^*, t^*)$$

and

$$\begin{aligned}
y_1^* &= \{y_1^{1*}, y_1^{2*}, \cdots, y_1^{(q_1 q_2 \cdots q_m)^2 *}\}, \\
&\vdots \\
y_n^* &= \{y_n^{1*}, y_n^{2*}, \cdots, y_n^{(q_1 q_2 \cdots q_m)^2 *}\}, \\
t^* &= \{t^{1*}, t^{2*}, \cdots, t^{(q_1 q_2 \cdots q_m)^2 *}\}.
\end{aligned}$$

Therefore, the hesitant fuzzy optimal solution of the main problem is $x^* = \{x_1^*, x_2^*, \cdots, x_n^*\}$ where

$$x_1^* = \frac{y_1^*}{t^*} = \{\frac{y_1^{1*}}{t^{1*}}, \frac{y_1^{2*}}{t^{2*}}, \cdots, \frac{y_1^{(q_1q_2\cdots q_m)^2*}}{t^{(q_1q_2\cdots q_m)^2*}}\},$$

$$\vdots$$

$$x_n^* = \frac{y_n^*}{t^*} = \{\frac{y_n^{1*}}{t^{1*}}, \frac{y_n^{2*}}{t^{2*}}, \cdots, \frac{y_n^{(q_1q_2\cdots q_m)^2*}}{t^{(q_1q_2\cdots q_m)^2*}}\}.$$

If the decision maker is not satisfied with the hesitant fuzzy solution, the solutions obtained from solving the hesitant fuzzy problem based on the desired achievement degree or achieve a moderate estimate using the score operators can be used.

Model 2 of HFLFP problem:

$$m\tilde{\tilde{a}}x \ \frac{f(x)}{g(x)} = \frac{c^Tx + \alpha}{d^Tx + \beta} \tag{3.12}$$

$$s.t \quad x \in X_{\leq}(A, \tilde{\tilde{b}}) = \{x \in R^n : A_ix \leq \tilde{\tilde{b}}_i, for\ i = 1, 2, \cdots, m, x \geq 0\}.$$

In this model, the right hand side values are similar to the previous hesitant fuzzy model, so the membership function is obtained after applying the Charnes and Cooper's transformation. In addition, assume that the number of decision makers can record the different aspiration level for the objective function which they want to achieve. The symbol $m\tilde{\tilde{a}}x$ means that q_0 number of decision makers have been recorded their opinion to achieve the desired level of goal. For example, the decision maker $k_0^{th}, (k_0 = 1, 2, \cdots, q_0)$ sets a certain lower level such as $b_0^{k_0}$ to achieve the acceptable deviation $d_{\tilde{\tilde{b}}_0}^{k_0}$ for the objective function.

The solution process for model 2 of HFLFP problem:

First, we should determine the hesitant fuzzy decision space of presented model. The corresponding membership function of the objective function for $k_0 = 1, 2, \cdots, q_0$ can be presented by the following construction:

$$\mu_{\tilde{\tilde{b}}_0}^{k_0}(y, t) = \mu_{\tilde{\tilde{b}}_0}^{k_0}(tf(\frac{y}{t}))$$

$$= \begin{cases} 1 & tf(\frac{y}{t}) > b_0^{k_0}, \\ 1 - \frac{b_0^{k_0} - tf(\frac{y}{t})}{d_{\tilde{\tilde{b}}_0}^{k_0}} & b_0^{k_0} - d_{\tilde{\tilde{b}}_0}^{k_0} \leq tf(\frac{y}{t}) \leq b_0^{k_0}, \\ 0 & tf(\frac{y}{t}) < b_0^{k_0} - d_{\tilde{\tilde{b}}_0}^{k_0}. \end{cases} \tag{3.13}$$

Therefore, the hesitant fuzzy decision space related to the objective function can be shown by

$$\tilde{h}_{\tilde{\tilde{b}}_0}(y,t) = \{\mu^1_{\tilde{\tilde{b}}_0}(y,t), \mu^2_{\tilde{\tilde{b}}_0}(y,t), \cdots, \mu^{q_0}_{\tilde{\tilde{b}}_0}(y,t)\}.$$

So according to the membership functions (11), (14) and Definition 2.2, we have:

$$\tilde{h}_{\tilde{\tilde{D}}} = \tau(\tilde{h}_{\tilde{\tilde{b}}_0}, \tilde{h}_{\tilde{\tilde{b}}}).$$

Now we solve $(q_0 q_1 q_2 \cdots q_m)$ number of programming problems by applying the Bellman and Zadeh principle and introducing variable λ^r, $(r = 1, 2, \cdots, (q_0 q_1 q_2 \cdots q_m))$ as follows:

$$\begin{aligned}
&\max \lambda^r \\
&1 - \frac{b_0^{k_0} - t^r f(\frac{y^r}{t^r})}{d^{k_0}_{\tilde{\tilde{b}}_0}} \geq \lambda^r \\
&1 - \frac{A_i y^r - b_i^{k_i} t^r}{d^{k_i}_{\tilde{\tilde{b}}_i}} \geq \lambda^r, \quad for\ i = 1, 2, \cdots, m \\
&d^T y^r + \beta t^r \leq 1 \\
&\lambda^r \in [0,1], y^r \geq 0, t^r \geq 0
\end{aligned} \tag{3.14}$$

The solution of problem (15) leads us to the solution of problem (13). When the main problem have n variables and m constraints, Table 1 Shows a comparison among LFP and FLFP problems related to models (1) and (2) and HFLFP models (1) and (2).

Table 1. Comparison among LFP problem, FLFP problems and HFLFP problems

Type problem	of Number variables	of Number constraints	of Number of LPs
LFP	$n+1$	$m+1$	1
FLFP model (1)	$n+1$ $n+2$	$m+1$ $m+2$	2 1
HFLFP model (1)	$n+1$ $n+2$	$m+1$ $m+2$	$2(q_1 q_2 \cdots q_m)$ $(q_1 q_2 \cdots q_m)^2$
FLFP model (2)	$n+1$	$m+1$	1
HFLFP model (2)	$n+2$	$m+2$	$(q_0 q_1 q_2 \cdots q_m)$

4. Numerical examples

Now, three examples are presented to show the performance of the presented method.

Example 4.1. Consider the following HFLP model

$$\begin{aligned} & max\ \frac{x_1+x_2}{3x_1-x_2+4} \\ s.t \quad & -x_1+2x_2 \leq \tilde{\tilde{b}}_1, \\ & 3x_1+x_2 \leq \tilde{\tilde{b}}_2, \\ & x_1, x_2 \geq 0, \end{aligned} \tag{4.1}$$

where $\tilde{\tilde{b}}_1$ and $\tilde{\tilde{b}}_2$ are hesitant sets with HFEs. Suppose that two decision makers give aspiration levels of the constraints where provided in Table 2. It should be noted that the $b_i^{k_i}$ and $d_{\tilde{\tilde{b}}_i}^{k_i} (i = 1, 2, k_i = 1, 2)$ comprehended as the acceptable value and admissible violation for the constraints, respectively.

Table 2. HFEs of the constraints by two DMs for Example 1

DM	DM_1	DM_2
$(b_1^{k_1}, d_{\tilde{\tilde{b}}_1}^{k_1})$	$(4, 1)$	$(1.1, 3)$
$(b_2^{k_2}, d_{\tilde{\tilde{b}}_2}^{k_2})$	$(2, 4)$	$(1, 2)$

Firstly, by using Charnes and Cooper's transformation we have:

$$\begin{aligned} & \max y_1+y_2 \\ s.t \quad & -y_1+2y_2-\tilde{\tilde{b}}_1 t \leq 0, \\ & 3y_1+y_2-\tilde{\tilde{b}}_2 t \leq 0, \\ & 3y_1-y_2+4t = 1, \\ & y, t \geq 0. \end{aligned}$$

Then we should determine the hesitant fuzzy objective function by solving problems of (8) and (9). By solving these problems, we find lower and upper bounds of the hesitant fuzzy objective function. The results of the optimal values of these lower and upper bounds summarized in Table 3.

Table 3. Bounds of the optimal values based on the values of the HFEs of constraints (four problems) for Example 1.

Bounds of the objective function → Space of the constraints ↓	Lower bound	Upper bound
$(b_1^1, d^1_{\tilde{\tilde{b}}_1})$ *and* $(b_2^1, d^1_{\tilde{\tilde{b}}_2})$	$z_l^1 = 1$	$z_u^1 = 1.6667$
$(b_1^1, d^1_{\tilde{\tilde{b}}_1})$ *and* $(b_2^2, d^2_{\tilde{\tilde{b}}_2})$	$z_l^2 = 0.1970$	$z_u^2 = 1.0513$
$(b_1^2, d^2_{\tilde{\tilde{b}}_1})$ *and* $(b_2^1, d^1_{\tilde{\tilde{b}}_2})$	$z_l^3 = 0.3333$	$z_u^3 = 1.6667$
$(b_1^2, d^2_{\tilde{\tilde{b}}_1})$ *and* $(b_2^2, d^2_{\tilde{\tilde{b}}_2})$	$z_l^4 = 0.2611$	$z_u^4 = 1.0513$

According to the values of Table 3, the membership functions of objective function are constructed using (10) as follows:

$$\mu^1_{\tilde{\tilde{z}}_0}(y,t) = \begin{cases} 1 & y_1 + y_2 > 1.6667, \\ 1 - \frac{1.6667-(y_1+y_2)}{1.6667-1} & 1 \le y_1 + y_2 \le 1.6667, \\ 0 & y_1 + y_2 < 1, \end{cases} \tag{4.2}$$

$$\mu^2_{\tilde{\tilde{z}}_0}(y,t) = \begin{cases} 1 & y_1 + y_2 > 1.0513, \\ 1 - \frac{1.0513-(y_1+y_2)}{1.0513-0.197} & 0.197 \le y_1 + y_2 \le 1.0513, \\ 0 & y_1 + y_2 < 0.197, \end{cases} \tag{4.3}$$

$$\mu^3_{\tilde{\tilde{z}}_0}(y,t) = \begin{cases} 1 & y_1 + y_2 > 1.6667, \\ 1 - \frac{1.6667-(y_1+y_2)}{1.6667-0.3333} & 0.3333 \le y_1 + y_2 \le 1.6667, \\ 0 & y_1 + y_2 < 0.3333, \end{cases} \tag{4.4}$$

and,

$$\mu^4_{\tilde{\tilde{z}}_0}(y,t) = \begin{cases} 1 & y_1 + y_2 > 1.0513, \\ 1 - \frac{1.0513-(y_1+y_2)}{1.0513-0.2611} & 0.2611 \le y_1 + y_2 \le 1.0513, \\ 0 & y_1 + y_2 < 0.2611, \end{cases} \tag{4.5}$$

Thus, the hesitant fuzzy decision space of the objective function can be written as:

$$\tilde{h}_{\tilde{\tilde{z}}_0}(y,t) = \{\mu^1_{\tilde{\tilde{z}}_0}(y,t), \mu^2_{\tilde{\tilde{z}}_0}(y,t), \mu^3_{\tilde{\tilde{z}}_0}(y,t), \mu^4_{\tilde{\tilde{z}}_0}(y,t)\}$$

furthermore, the fuzzy membership functions corresponding to the constraints for $i = 1, 2,\ k_i = 1, 2$ using (11) are represented as follows:

$$\mu^1_{\tilde{\tilde{b}}_1}(y,t) = \begin{cases} 1 & -y_1 + 2y_2 - 4t < 0, \\ 1 - \frac{-y_1+2y_2-4t}{1} & 0 \le -y_1 + 2y_2 - 4t \le 1, \\ 0 & -y_1 + 2y_2 - 4t > 1, \end{cases} \tag{4.6}$$

$$\mu^2_{\tilde{\tilde{b}}_1}(y,t) = \begin{cases} 1 & -y_1 + 2y_2 - 1.1t < 0, \\ 1 - \frac{-y_1+2y_2-1.1t}{3} & 0 \le -y_1 + 2y_2 - 1.1t \le 3, \\ 0 & -y_1 + 2y_2 - 4t > 3, \end{cases} \tag{4.7}$$

$$\mu^1_{\tilde{\tilde{b}}_2}(y,t) = \begin{cases} 1 & 3y_1 + 2y_2 - 2t < 0, \\ 1 - \frac{3y_1+y_2-2t}{4} & 0 \le 3y_1 + y_2 - 2t \le 4, \\ 0 & 3y_1 + y_2 - 2t > 4, \end{cases} \tag{4.8}$$

and,

$$\mu^2_{\tilde{\tilde{b}}_2}(y,t) = \begin{cases} 1 & 3y_1 + 2y_2 - t < 0, \\ 1 - \frac{3y_1+y_2-t}{2} & 0 \le 3y_1 + y_2 - 2t \le 2, \\ 0 & 3y_1 + y_2 - t > 2. \end{cases} \tag{4.9}$$

So, the hesitant fuzzy decision space of the constraints is defined as follows:

$$\tilde{h}_{\tilde{\tilde{b}}}(y,t) = \{\mu^1_{\tilde{\tilde{b}}_1}(y,t), \mu^2_{\tilde{\tilde{b}}_1}(y,t), \mu^1_{\tilde{\tilde{b}}_2}(y,t), \mu^2_{\tilde{\tilde{b}}_2}(y,t), \}.$$

Now, by the definition 2.2, the hesitant fuzzy decision space, can be stated by:

$$\begin{aligned} &\tilde{h}_{\tilde{\tilde{D}}} = \tau(\tilde{h}_{\tilde{\tilde{z}}_0}, \tilde{h}_{\tilde{\tilde{b}}}) \\ &= \tau(\{\mu^1_{\tilde{\tilde{z}}_0}(y,t), \mu^2_{\tilde{\tilde{z}}_0}(y,t), \mu^3_{\tilde{\tilde{z}}_0}(y,t), \mu^4_{\tilde{\tilde{z}}_0}(y,t)\}, \\ &\{\mu^1_{\tilde{\tilde{b}}_1}(y,t), \mu^2_{\tilde{\tilde{b}}_1}(y,t), \mu^1_{\tilde{\tilde{b}}_2}(y,t), \mu^2_{\tilde{\tilde{b}}_2}(y,t)\}) \end{aligned}$$

Finally, the Bellman and Zadeh principle is applied to solve 16 number of problems due to the (12) by introducing variable λ^r for $(r = 1, 2, \cdots, 16)$. To construct the optimal solution of the HFLFP, we should find the optimal solutions of 16 LP problems. Table 4. Shows the optimal solutions of these problems.

Table 4. Optimal solutions of 16 LP problems for Example 1

LP^r	λ^{r*}	(y^{r*}, t^{r*})	$x^{r*} = \frac{y^{r*}}{t^{r*}}$
LP^1	0.6000	$(0, 1.4000, 0.6000)$	$(0, 2.3333)$
LP^2	0.6000	$(0, 1.4000, 0.6000)$	$(0, 2.3333)$
LP^3	0.5557	$(0.3980, 0.9725, 0.1946)$	$(2.0452, 4.9974)$
LP^4	0.4360	$(0.1366, 1.1541, 0.4360)$	$(0.3133, 2.6470)$
LP^5	0.9723	$(0, 1.0277, 0.5069)$	$(0, 2.0274)$
LP^6	0.7961	$(0, 0.8771, 0.4693)$	$(0, 1.8689)$
LP^7	0.7642	$(0.2546, 0.5953, 0.2079)$	$(1.2246, 2.8634)$
LP^8	0.6942	$(0.0897, 0.7004, 0.3578)$	$(0.2507, 1.9575)$
LP^9	0.7143	$(0, 1.2857, 0.5714)$	$(0, 2.2501)$
LP^{10}	0.6667	$(0, 1.2222, 0.5556)$	$(0, 2.1998)$
LP^{11}	0.6330	$(0.3448, 0.8326, 0.1996)$	$(1.7275, 4.1713)$
LP^{12}	0.5509	$(0.1157, 0.9522, 0.4012)$	$(0.2884, 2.3734)$
LP^{13}	0.9713	$(0, 1.0287, 0.5072)$	$(0, 2.0282)$
LP^{14}	0.7923	$(0, 0.8872, 0.4718)$	$(0, 1.8805)$
LP^{15}	0.7596	$(0.2577, 0.6036, 0.2076)$	$(1.2413, 2.9075)$
LP^{16}	0.6870	$(0.0910, 0.7130, 0.3600)$	$(0.2528, 1.9806)$

According to above table, we have:

$$\tilde{h}_{\tilde{\tilde{D}}} = \{0.6000, 0.6000, 0.5557, 0.4360, 0.9723, 0.7961, 0.76420.6942, 0.7143, 0.6667, 0.6330, 0.5509, 0.9713, 0.7923, 0.7596, 0.6870\}$$

And, hesitant fuzzy optimal solution of the main problem is $x^* = \{x_1^*, x_2^*\}$, where

$$x_1^* = \{0, 0, 2.0452, 0.3133, 0, 0, 0, 1.2246, 0.2507, 0, 0, 1.7275, 0.2884, 0, 0, 1.2413, 0.2528\}$$

$$x_2^* = \{2.3333, 2.3333, 4.9974, 2.6470, 2.0274, 1.8689, 2.8634, 1.9575, 2.2501, 2.1998, 4.1713, 2.3734, 2.0282, 1.8805, 2.9075, 1.9806\}$$

Example 4.2. Consider the following HFLP model

$$\begin{aligned} & m\tilde{\tilde{a}}x \ \frac{x_1 + x_2}{3x_1 - x_2 + 4} \\ s.t \quad & -x_1 + 2x_2 \leq \tilde{\tilde{b}}_1, \\ & 3x_1 + x_2 \leq \tilde{\tilde{b}}_2, \\ & x_1, x_2 \geq 0, \end{aligned} \tag{4.10}$$

where $\tilde{\tilde{b}}_1$ and $\tilde{\tilde{b}}_2$ are similar to Example 1. Furthermore, suppose that three decision makers give desired levels of the objective function where provided in Table 5. It should be noted that the $b_0^{k_0}$ and $d_{\tilde{\tilde{b}}_0}^{k_0} (k_0 = 1, 2, 3)$ comprehended as the desired value and admissible violation for the objective function, respectively. This is clear that the hesitant fuzzy constraints space is similar to Example 1. Hence we determine the hesitant fuzzy objective function. So, using (14) we can state:

Table 5. HFEs of the constraint by two DMs for Example 2

DM	DM_1	DM_2	DM_3
$(b_0^{k_0}, d_{\tilde{\tilde{b}}_0}^{k_0})$	$(1.2, 0.4)$	$(1.05, 0.3)$	$(1.35, 0.45)$
$(b_1^{k_1}, d_{\tilde{\tilde{b}}_1}^{k_1})$	$(4, 1)$	$(1.1, 3)$	-
$(b_2^{k_2}, d_{\tilde{\tilde{b}}_2}^{k_2})$	$(2, 4)$	$(1, 2)$	-

$$\mu^1_{\tilde{\tilde{b}}_0}(y,t) = \begin{cases} 1 & y_1 + y_2 > 1.2, \\ 1 - \frac{1.2-(y_1+y_2)}{0.4} & 0.8 \le y_1 + y_2 \le 1.2, \\ 0 & y_1 + y_2 < 0.8, \end{cases} \tag{4.11}$$

$$\mu^2_{\tilde{\tilde{b}}_0}(y,t) = \begin{cases} 1 & y_1 + y_2 > 1.05, \\ 1 - \frac{1.05-(y_1+y_2)}{0.3} & 0.75 \le y_1 + y_2 \le 1.05, \\ 0 & y_1 + y_2 < 0.75, \end{cases} \tag{4.12}$$

And,

$$\mu^3_{\tilde{\tilde{b}}_0}(y,t) = \begin{cases} 1 & y_1 + y_2 > 1.35, \\ 1 - \frac{1.35-(y_1+y_2)}{0.45} & 0.9 \le y_1 + y_2 \le 1.35, \\ 0 & y_1 + y_2 < 0.9. \end{cases} \tag{4.13}$$

Thus, the hesitant fuzzy decision space of the objective function can represent by

$$\tilde{h}_{\tilde{\tilde{b}}_0}(y,t) = \{\mu^1_{\tilde{\tilde{b}}_0}(y,t), \mu^2_{\tilde{\tilde{b}}_0}(y,t), \mu^3_{\tilde{\tilde{b}}_0}(y,t)\}$$

So according to Definition 2.2, for decision space we have

$$\tilde{h}_{\tilde{\tilde{D}}} = \tau(\tilde{h}_{\tilde{\tilde{b}}_0}, \tilde{h}_{\tilde{\tilde{b}}})$$

Finally, we solve 12 number of problems by applying the Bellman and Zadeh principle and defining variable $\lambda^r, (r = 1, 2, \cdots, 12)$ according to (15). To find the optimal solution of the HFLFP, we solve the optimal solutions of 12 LP problems. the optimal solutions of these problems are shown in Table 6.

Table 6. Optimal solutions of 12 LP problems for Example 2.

LP^r	λ^{r*}	(y^{r*}, t^{r*})	$x^{r*} = \frac{y^{r*}}{t^{r*}}$
LP^1	0.8571	$(0, 1.1429, 0.5357)$	$(0, 2.1335)$
LP^2	0.7174	$(0, 1.087, 0.5217)$	$(0, 2.0291)$
LP^3	0.6759	$(0.3153, 0.7551, 0.2023)$	$(1.5586, 3.7326)$
LP^4	0.5712	$(0.1120, 0.9165, 0.3951)$	$(0.2835, 2.3197)$
LP^5	0.9615	$(0, 1.0385, 0.5096)$	$(0, 2.0379)$
LP^6	0.7584	$(0, 0.9775, 0.4944)$	$(0, 1.9771)$
LP^7	0.7179	$(0.2864, 0.6790, 0.2050)$	$(1.3971, 3.3122)$
LP^8	0.6191	$(0.1033, 0.8324, 0.3806)$	$(0.2714, 2.1871)$
LP^9	0.7586	$(0, 1.2414, 0.5603)$	$(0, 2.2156)$
LP^{10}	0.6738	$(0, 1.2032, 0.5508)$	$(0, 2.1845)$
LP^{11}	0.6305	$(0.3465, 0.8372, 0.1994)$	$(1.7377, 4.1986)$
LP^{12}	0.5174	$(0.1218, 1.0110, 0.4114)$	$(0.2961, 2.4575)$

According to information of Table 6, we can be written

$$\tilde{h}_{\tilde{\tilde{D}}} = \{0.8571, 0.7174, 0.6759, 0.5712, 0.9615, 0.7584,$$
$$0.7179, 0.6191, 0.7586, 0.6738, 0.6305, 0.5174\}$$

And, the hesitant fuzzy optimal solution of the main problem is $x^* = \{x_1^*, x_2^*\}$, where

$$x_1^* = \{0, 0, 1.5586, 0.2835, 0, 0, 1.3971, 0.2714, 0, 0, 1.7377, 0.2961\},$$
$$x_2^* = \{2.1335, 2.0291, 3.7326, 2.3197, 2.0379, 1.9771, 3.3122, 2.1871,$$
$$2.2156, 2.1845, 4.1986, 2.4575\}.$$

Example 4.3. Suppose that a company makes two kinds of products A and B with profit \$5 and \$3 per unit respectively. Also, the cost of these products is \$3.35 and \$2 respectively. Furthermore, assume that a fixed cost \$1 should be considered in the cost function. Suppose the raw material which is needed for producing A and B is 3 and 5 per pound respectively, The supply of the raw materials recorded by two decision makers according to their experiences. First decision maker interpreted that the supply of the raw material is sufficient for at least 6 units per pound of the products and it can possibly be increased to 9 units per pound. Also second decision maker interpreted that the supply of the raw material is sufficient for at least 5.5 units per pound of the products and it can possibly be increased to 7.5 units per pound. We should add that the labor hours per unit for the product A and B is 5 and 2 hours per unit. First decision maker considered that the useful total labor hours is at least 4 hours in a day for two products and it may possibly be increased to 8,5 hours in a day. Also second decision maker comprehended that the useful total labor hours is at least 5 hours in a day for two products and it may possibly be increased to 6 hours in a day.

The manufacturer wants to determine how many units of products A and B should be produced such that satisfy restrictions and to give the maximum profit gained per unit to the total cost of production.
For solution, let x_1 and x_2 are the number of units of A and B that are produced in one day. Thus the presented Example 3 can be formulated as

$$\begin{aligned} &\max \frac{5x_1 + 3x_2}{3.25x_1 + 2x_2 + 1} \\ s.t \quad &3x_1 + 5x_2 \leq \tilde{\tilde{b}}_1, \\ &5x_1 + 2x_2 \leq \tilde{\tilde{b}}_2, \\ &x_1, x_2 \geq 0, \end{aligned} \tag{4.14}$$

where $\tilde{\tilde{b}}_1$ and $\tilde{\tilde{b}}_2$ are hesitant fuzzy right hand side values which is determined by two decision makers.
For this practical example the results of the objective function bounds in hesitant fuzzy environment summarize in Table 7. According to information of this example we should use model (1) to find the optimal answers of HFLFP problems, Table 8 shows the optimal solutions for corresponded LP problems.

Table 7. Bounds of the optimal values based on the values of the HFEs of constraints (four problems) for Example 3.

Bounds of the objective function → Space of the constraints ↓	Lower bound	Upper bound
$(b_1^1, d^1_{\tilde{\tilde{b}}_1})$ *and* $(b_2^1, d^1_{\tilde{\tilde{b}}_2})$	$z_l^1 = 1.1605$	$z_u^1 = 1.3152$
$(b_1^1, d^1_{\tilde{\tilde{b}}_1})$ *and* $(b_2^2, d^2_{\tilde{\tilde{b}}_2})$	$z_l^2 = 1.0942$	$z_u^2 = 1.2506$
$(b_1^2, d^2_{\tilde{\tilde{b}}_1})$ *and* $(b_2^1, d^1_{\tilde{\tilde{b}}_2})$	$z_l^3 = 1.1025$	$z_u^3 = 1.2589$
$(b_1^2, d^2_{\tilde{\tilde{b}}_1})$ *and* $(b_2^2, d^2_{\tilde{\tilde{b}}_2})$	$z_l^4 = 1.1546$	$z_u^4 = 1.3107$

Table 8. Optimal solutions of 16 LP problems for Example 3.

LP^r	λ^{r*}	(y^{r*}, t^{r*})	$x^{r*} = \frac{y^{r*}}{t^{r*}}$
LP^1	0.8787	$(0.1684, 0.1514, 0.1497)$	$(1.1249, 1.0114)$
LP^2	0.7664	$(0.0938, 0.2700, 0.1551)$	$(0.6048, 1.7408)$
LP^3	0.8707	$(0.1984, 0.1011, 0.1531)$	$(1.2959, 0.6604)$
LP^4	0.7363	$(0.1329, 0.2034, 0.1615)$	$(0.8229, 1.2594)$
LP^5	0.9294	$(0.1393, 0.1811, 0.1852)$	$(0.7522, 0.9779)$
LP^6	0.9043	$(0.1225, 0.2076, 0.1865)$	$(0.6568, 1.1131)$
LP^7	0.9228	$(0.1642, 0.1391, 0.1880)$	$(0.8734, 0.7399)$
LP^8	0.8861	$(0.1462, 0.1672, 0.1903)$	$(0.7683, 0.1852)$
LP^9	0.9229	$(0.1430, 0.1773, 0.1807)$	$(0.7914, 0.9812)$
LP^{10}	0.8867	$(0.1189, 0.2156, 0.1825)$	$(0.6515, 1.1814)$
LP^{11}	0.9161	$(0.1686, 0.1343, 0.1835)$	$(0.9188, 0.7319)$
LP^{12}	0.8670	$(0.1445, 0.1718, 0.1866)$	$(0.7744, 0.9207)$
LP^{13}	0.8823	$(0.1663, 0.1536, 0.1523)$	$(1.0919, 1.0085)$
LP^{14}	0.7767	$(0.0960, 0.2653, 0.1574)$	$(0.6099, 1.6855)$
LP^{15}	0.8745	$(0.1959, 0.1038, 0.1556)$	$(1.2590, 0.6671)$
LP^{16}	0, 7475	$(0.1339, 0.2007, 0.1636)$	$(0.8185, 1.2268)$

According to the results of the optimal values of Table 8 we have

$$\tilde{h}_{\tilde{\tilde{D}}} = \{0.8787, 0.7664, 0.8707, 0.7363, 0.9294, 0.9043, 0.9228, 0.8861, 0.9229, 0.8867, 0.9161, 0.8670, 0.8823, 0.7767, 0.8745, 0.7475\}$$

And, $x^* = \{x_1^*, x_2^*\}$, where

$$x_1^* = \{1.1249, 0.6048, 1.2959, 0.8229, 0.7522, 0.6568, 0.8734, 0.7683, 0.7914, 0.6515, 0.9188, 0.7744, 1.0919, 0.6099, 1.2590, 0.8185\},$$

$$x_2^* = \{1.0114, 1.7408, 0.6604, 1.2594, 0.9779, 1.1131, 0.7399, 0.1852, 0.9812, 1.1814, 0.7319, 0.9207, 1.0085, 1.6855, 0.6671, 1.2268\},$$

Due to the application of proposed method in Example 3, it be concluded that the proposed method can be used as a useful implementation in any real life problems of form LFP.

Conclusion

There are situations where it is not possible to accurately determine the amount of membership due to the uncertainty of the decision maker. In such a situation, the use of hesitant fuzzy sets can lead to more logical and effective results. Hesitant fuzzy sets allow an element to have not only one membership value, but multiple membership values. Ranjbar

and Effati [20] for the first time, formulated and solved linear programming problems under hesitant fuzzy data in different modes. However, most of the mathematical and physical problems are nonlinear and especially fractional. Therefore, in this research, our main focus has been on introducing hesitant fuzzy linear fractional programming along with a hybrid idea to solve it. In the procedure of solving it, we find a set of solutions based on different expert levels for LFP in a hesitant fuzzy decision environment, which is a significant output of the proposed method. As mentioned, in the proposed method for HFLFP, the number of sub-problems according to the comparative analysis presented in Table 1 is different and has more variety than the previous cases. On the other hand, according to the transformation of the main problem into a number of sub-problems and the application of Bellman and Zadeh's principle, it can be confirmed that the proposed method introduces a comprehensive and consistent structure to solve these types of problems. In fact, modeling fractional linear programming problems under hesitant fuzzy set information can make mathematical models more compatible and similar to current real-world systems.
Since the sub-problems include all the states related to the membership values, finally, the decision maker has a suitable range of answers. Hence, the decision maker chooses the answer based on the expected level (type of view from the most optimistic to the most pessimistic perspective) and makes the final decision. The novelty of this work can be a light for the development of the proposed method in order to solve other nonlinear modeling problems under hesitant fuzzy data. In this regard, the following studies are recommended by the authors for future research:
1. Considering the uncertainties arising from hesitant fuzzy numbers over the more parameters of the model.
2. Improving the presented solution method presented in this study to deal with uncertainties caused by hesitant fuzzy data.
3. Considering more objectives to solve the problem.
4. Constructing the problem with higher-dimensional models, and solving them using heuristic and meta-heuristic methods, and comparing responses with deterministic approaches.

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