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Improving the convergence order of Steffensen's method from two to four and its dynamic

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Abstract. In this paper, the degree of convergence of Newton's method has been increased from two to four using two function evaluations. For this purpose, the weakness of Newton's method, derivative calculation has been eliminated with a proper approximation of the previous data. Then, by entering two self-accelerating parameters, the family new with-memory methods with Steffensen-Like memory with convergence orders of 2.41, 2.61, 2.73, 3.56, 3.90, 3.97, and 4 are made. This goal has been achieved by approximating the self-accelerator parameters by using the secant method and Newton interpolation polynomials. Finally, we have examined the dynamic behavior of the proposed methods for solving polynomial equations.

Keywords: With-memory method, Accelerator parameter, Basin of attraction, Efficiency index, Newton's interpolatory polynomial.

2000 Mathematics subject classification: 65H99, 65H05.

1. Introduction

1.1. **definition.** Suppose f be a real one-valued function of a real variable. If $f(\xi) = 0$ then ξ is told to be a zero of f

$$f(x) = 0. \tag{1.1}$$

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This equation does not have an exact root. Therefore, we obtain an approximation of it by using iterative methods. Newton's method is one of the oldest methods for finding the roots of nonlinear equations.

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)},\ m = 0, 1, 2, \cdots . \tag{1.2}$$

This method finds the root approximation of the equations in which the exact roots are unclear.The order of convergence and the efficiency index is equal to 2 and 1.41,respectively. The weak point of this method is the derivative function for its root approximation.

1.2. **Existing iterative methods.** Argyros and Hilout made algorithms to find the roots of the equations using a Newton-like method [2]. Yu et al. obtained Newton's method and approximated the nonlinear root of the equation. They used the suitable Lipschitz condition approximating the second derivative [35]. Also, Wu and Wu proposed a class of quadratic convergence iteration formulae derivative-free [34]. Cordero and Torregrosa [6], Džunić and Petković [8], and Soleymani and Karimi solved nonlinear equations utilizing Steffensen-like methods [24]. In addition, Kung and Traub used Newton-Steffensen-like methods in their work [13]. Amat et al. made the Chebyshev-type iterative method [1]. Kou and Li improved Chebyshev-Halley-like methods until fifth-order convergence [12]. Petković et al. generalized Halley-like methods for solving nonlinear equations [17]. Cordero et al. proposed two weighted-order classes of iterative root-finding methods [5, 7]. Sharma et al. solved nonlinear equations using a Newton-Steffensen-like method [20, 21]. In addition, Soleymani made optimal fourth-order iterative methods free from derivatives [25]. Steffensen [26] presented optimal secend-order iterative methods.
Fariborzi Araghi et al. [10] and Torkashvand et al. [29] have solved nonlinear equations using adaptive methods. The authors in references [22] and [23] used repeat techniques for the first time to solve differential equations. Biazar et al. developed a system of ordinary differential equations and used the Adomian decomposition method [3].

1.3. **Motivation and organization.** Our model transforms Newton-Raphson's method into Steffensen's type with two function evaluations and fourth-order convergence. The adaptive method proposed has the highest efficiency index. Self-accelerator parameters have an essential role in the absorption area and increase the convergence order. In addition, the calculation error is less than in other methods.
Different sections of this work are as follows. Section 2 self-accelerator parameters are added to Newton to create a with-memory. Section 3 has been divided into three sub-sections.In the first part,single-parametric

methods have developed with $20.5\%, 30.5\%,$ and 50% convergence order improvement. The second part presents two-parameter methods with memory with 36.5% and 78% convergence order improvement. In the following, adaptive methods with 100% convergence order improvement have been presented. Section 4 confirms the accuracy of the remark in Sections 2 and 3 with numerical examples. In Section 5, we describe the importance of the self-accelerator parameters in the absorption and stability of the proposed method. Finally, the conclusion of the paper is given in Section 6.

2. Constructing a family of schemes

2.1. **Derivation.** Fundamental problems of the numerical analysis are solving the nonlinear equation $f(x) = 0$. Iterative methods are one of the powerful tools to solve such equations. The equation root is found by using the fixed point method.

$$x_{m+1} = g(x_m),\ m = 0, 1, 2, \cdots . \tag{2.1}$$

Newton's method considered a fixed point method with a suitable initial approximation. But the weakness is the function derivative's estimation. Hence, the function's first derivative can approximate using the Lagrange interpolator polynomial. Using the points x_m and $w_m = x_m + f(x_m)$ approximating $f'(x)$, we have

$$p(s) = \frac{(s-w)f(x)}{x-w} + \frac{(s-x)f(w)}{w-x}. \tag{2.2}$$

By putting $f'(x) \approx p'(x)$ and $s = x$ we have

$$p'(s) = \frac{f(x)}{x-w} - \frac{f(w)}{x-w} = f[x, w]. \tag{2.3}$$

Therefore, Newton's method can be rewritten as follows

$$w_m = x_m + f(x_m),\ x_{m+1} = x_m - \frac{f(x_m)}{f[x_m, w_m]},\ m = 0, 1, 2, \cdots . \tag{2.4}$$

The error equation of this method without memory is as follows

$$e_{m+1} = (1 + f'(\xi))c_2 e_m^2 + O(e_m^3). \tag{2.5}$$

To create a with-memory method, you must enter the parameter or parameters of the self-accelerator in the without memory method. Entering a self-accelerator parameter into Steffensen's method, we can rewrite it as follows:

$$w_m = x_m + \gamma f(x_m),\ x_{m+1} = x_m - \frac{f(x_m)}{f[x_m, w_m]}. \tag{2.6}$$

Enter another parameter, a without-memory method with two self-accelerator parameters is obtained.

$$w_m = x_m + \gamma f(x_m),\ x_{m+1} = x_m - \frac{f(x_m)}{f[x_m, w_m] + \beta f(w_m)}. \tag{2.7}$$

For convergence study of the proposed methods (2.6) and (2.7), we state the following Theorem.

Theorem 2.1. *(Theorem 1 [9, 37]) Suppose that $f(x)$ is a sufficiently differentiable function in an open interval D and $\xi \in D$ is a simple zero of $f(x)$. If the initial estimation x_0 is close enough to ξ, then iterative schemes (2.6) and (2.7) yield a family of second-order methods satisfying the error equation below, respectively.*

$$e_{m+1} = (1 + \gamma f'(\xi))c_2 e_m^2 + O(e_m^3), \tag{2.8}$$

and

$$e_{m+1} = (1 + \gamma f'(\xi))(\beta + c_2)e_m^2 + O(e_m^3). \tag{2.9}$$

3. Convergence analysis

3.1. **A self-accelerator parameter.** It is clear from (2.8) that the convergence order of the family (2.6) is two when $\gamma \neq \frac{-1}{f'(\xi)}$. Thus, it is possible to improve the convergence speed of the suggested class (2.6) if $\gamma = \frac{-1}{f'(\xi)}$. This order improve from 2 to $1+\sqrt{2}$, $\frac{1}{2}(3+\sqrt{5})$, and 3, by taking $\gamma = \frac{-1}{f'(\xi)}$, but root ξ is not known. To improve the order of convergence of (2.6), we re-calculate the value of parameter γ in each iterate by taking $\gamma \approx \frac{-1}{f'(\xi)}$, while $f'(\xi)$ is not provided. We represent this estimation through γ_m using current and previous iteration satisfying.

$$\lim_{m\to\infty} \gamma_m = \frac{-1}{f'(\xi)}. \tag{3.1}$$

So, it can be approximated as follows

$$f'(\xi) \approx \bar{f}'(\xi),\ \gamma_m = \frac{-1}{\bar{f}'(\xi)}. \tag{3.2}$$

Now, we propose the following iterative method with memory based on the method (2.6)

$$\begin{cases} \gamma_m = \frac{-1}{\bar{f}'(\xi)},\ m = 1, 2, 3, \cdots, \\ w_m = x_m + \gamma_m f(x_m),\ x_{m+1} = x_m - \frac{f(x_m)}{f[x_m, w_m]},\ m = 0, 1, 2, \cdots. \end{cases} \tag{3.3}$$

Therefore, one of the following methods can be used to approximate the self-accelerator parameter γ_m

(1) Secant approach

$$\bar{f}'(\xi) = \frac{f(x_m) - f(x_{m-1})}{x_m - x_{m-1}}. \tag{3.4}$$

(2) Best secant approach

$$\bar{f}'(\xi) = \frac{f(x_m) - f(w_{m-1})}{x_m - w_{m-1}}. \tag{3.5}$$

(3) Newton's interpolatory approach with second-degree polynomial

$$N_2(t) = N_2(t; x_m, x_{m-1}, w_{m-1}),\ m \geqslant 1,$$
$$\bar{f}'(\xi) = N_2'(x_m). \tag{3.6}$$

The self-accelerator parameter γ_m can be designated recursively as follows:

(1) Secant approach

$$\gamma_m = \frac{-1}{\bar{f}'(\xi)} = -\frac{x_m - x_{m-1}}{f(x_m) - f(x_{m-1})}. \tag{3.7}$$

(2) Best secant approach

$$\gamma_m = \frac{-1}{\bar{f}'(\xi)} = -\frac{x_m - w_{m-1}}{f(x_m) - f(w_{m-1})}. \tag{3.8}$$

(3) Newton's interpolatory approach with second-degree polynomial

$$\gamma_m = \frac{-1}{\bar{f}'(\xi)} = \frac{-1}{N_2'(x_m)}. \tag{3.9}$$

If we determine the parameter γ_m using one of the above methods, we have the following with-memory methods.

Theorem 3.1. *If an initial approximation x_0 is sufficiently close to the zero ξ of $f(x)$ and the parameter γ_m in the iterative method* (3.3) *is recursively calculated by the forms given in* (3.7)- (3.9)*). Then, the R-order of convergence of the one-point with-memory methods* (3.3) *with the corresponding expressions* (3.7)-(3.9) *of is at least* $1+\sqrt{2}$, $\frac{1}{2}(3+\sqrt{5})$, *and 3.*

Proof. We only obtained the convergence order of the method (3.3) that self-accelerator parameter is earned by using the relationship (3.8).
First, we assume that the $R-$order of convergence in sequence x_m and w_m is at least ρ and ρ_1, respectively. Hence

$$e_{m+1} \sim e_m^{\rho} \sim e_{m-1}^{\rho^2}, \tag{3.10}$$

also

$$e_{m,w} \sim e_m^{\rho_1} \sim e_{m-1}^{\rho\rho_1}. \quad (3.11)$$

The Taylor's series expansion of $f(x)$ about $x = \xi$ is given as

$$f(x) = f(\xi) + (x-\xi)f'(\xi) + \frac{(x-\xi)^2 f'(\xi)}{2!} + \frac{(x-\xi)^3 f'(\xi)}{3!} + \cdots . \quad (3.12)$$

Now, using the relations $x_m - \xi = e_m, x_m - w_{m-1} = e_m - e_{m-1,w}$ and also the relation (3.12), we obtain

$$f(x_m) = f(\xi) + (x_m - \xi)f'(\xi) + \frac{(x_m-\xi)^2 f'(\xi)}{2!} + \frac{(x_m-\xi)^3 f'(\xi)}{3!} + \cdots , \quad (3.13)$$

and

$$f(w_{m-1}) = f(\xi) + (w_{m-1} - \xi)f'(\xi) + \frac{(w_{m-1}-\xi)^2 f'(\xi)}{2!} + \frac{(w_{m-1}-\xi)^3 f'(\xi)}{3!} + \cdots . \quad (3.14)$$

Also, we have

$$\begin{aligned}\frac{f(x_m) - f(w_{m-1})}{x_m - w_{m-1}} &= ((e_m - e_{m-1,w})f'(\xi) + (e_m^2 - e_{m-1,w}^2)\frac{f''(\xi)}{2!} \quad (3.15)\\ &+ (e_m^3 - e_{n-1,w}^3)\frac{f''(\xi)}{3!} + ...)(e_m - e_{m-1,w})^{-1}\\ &= f'(\xi) + (e_m + e_{m-1,w})\frac{f''(\xi)}{2!}\\ &+ (e_m^2 + e_m e_{m-1,w} + e_{m-1,w}^2)\frac{f'''(\xi)}{3!} + \cdots .\end{aligned}$$

Using the relation (3.15), we have

$$\begin{aligned}\gamma_m &= -\frac{x_m - w_{m-1}}{f(x_m) - f(w_{m-1})} \quad (3.16)\\ &= \frac{-1}{f'(\xi) + (e_m + e_{m-1,w})\frac{f''(\xi)}{2!} + (e_m^2 + e_m e_{m-1,w} + e_{m-1,w}^2)\frac{f'''(\xi)}{3!} + ...}.\end{aligned}$$

Substituting (3.16) in $(1 + f'(\xi)\gamma_m)$ yields

$$\begin{aligned}&1 + f'(\xi)\gamma_m \quad (3.17)\\ &= 1 + \frac{-f'(\xi)}{f'(\xi) + (e_m + e_{m-1,w})\frac{f''(\xi)}{2!} + (e_m^2 + e_m e_{m-1,w} + e_{m-1,w}^2)\frac{f'''(\xi)}{3!} + ...}\\ &= \frac{(e_m + e_{m-1,w})c_2 + (e_m^2 + e_m e_{m-1,w} + e_{m-1,w}^2)c_3 + ...}{1 + (e_m + e_{m-1,w})c_2 + (e_m^2 + e_m e_{m-1,w} + e_{m-1,w}^2)c_3 + ...}\\ &\sim c_2 e_{m-1,w}.\end{aligned}$$

Consequently, by (3.3), we find the following error relations

$$e_{m,w} \sim (1+\gamma_m f'(\xi))e_m \sim e_{m-1,w}e_m \sim e_{m-1}^{\rho_1+\rho}, \tag{3.18}$$

$$e_{m+1} \sim (1+\gamma_m f'(\xi))e_m^2 \sim e_{m-1,w}e_m^2 \sim e_{m-1}^{\rho_1+2\rho}. \tag{3.19}$$

Comparing the right and left sides of error equations (3.10), (3.19) and (3.11), (3.18), we obtain

$$\begin{cases} \rho_1+\rho-\rho\rho_1=0, \\ \rho_1+2\rho-\rho^2=0. \end{cases} \tag{3.20}$$

Therefore, the non-trivial solution of this system of equations is given by $\rho_1=\frac{1}{2}(1+\sqrt{5})\approx 1.62$ and $\rho=\frac{1}{2}(3+\sqrt{5})\approx 2.62$. Thus, we can conclude that the lower bound of the R-order of the with-memory methods (3.3) and (3.8) is $\rho=\frac{1}{2}(3+\sqrt{5})\approx 2.62$. We show this method with TM2.6. We use the secant of approach in (3.7) and obtain Traub's method. The proof of the convergence order of 2 to 2.41 is given in [27]. Furthermore, Džunić and Petković showed the convergence of this method is 3. Hence, it is omitted. (by replacing $\gamma_m=\frac{-1}{N_2'(x_m)}$) [8]. □

3.2. **Two self-accelerator parameters.** In the following, we propose single-step methods based on (2.7) that have two parameters accelerator

$$\begin{cases} \gamma_m=\frac{-1}{\bar{f}'(\xi)},\ \beta_m=\frac{-\bar{f}''(\xi)}{2\bar{f}'(\xi)},\ m=1,2,3,\cdots, \\ w_m=x_m+\gamma_m f(x_m),\ x_{m+1}=x_m-\frac{f(x_m)}{f[x_m,w_m]+\beta_m f(w_m)},\ m=0,1,2,\cdots. \end{cases} \tag{3.21}$$

We construct two methods through the following forms of γ_m and β_m

(*i*)

$$\begin{cases} \gamma_m \approx \frac{-1}{\bar{f}'(\xi)}=\frac{-1}{N_1'(x_m)}, \\ \beta_m \approx \frac{-\bar{f}''(\xi)}{2\bar{f}'(\xi)}=\frac{-N_2''(w_m)}{2N_2'(w_m)}. \end{cases} \tag{3.22}$$

where $N_1(t)=N_1(t;w_{m-1},x_{m-1}), N_2(t)=N_2(t;x_m,w_{m-1},x_{m-1})$,

(*ii*)

$$\begin{cases} \gamma_m \approx \frac{-1}{\bar{f}'(\xi)}=\frac{-1}{N_2'(x_m)}, \\ \beta_m \approx \frac{-\bar{f}''(\xi)}{2\xi\bar{f}'(\xi)}=\frac{-N_3''(w_m)}{2N_3'(w_m)}. \end{cases} \tag{3.23}$$

where $N_2(t)=N_2(t;x_m,w_{m-1},x_{m-1}), N_3(t)=N_3(t;w_m,x_m,w_{m-1},x_{m-1})$.

Theorem 3.2. *If an initial approximation x_0 is adequately close to the zero ξ of $f(x)$ and the parameters γ_m and β_n in the iterative method* (3.21) *has recursively calculated by the forms furnished in* (3.22) *and* (3.23). *Then, the R-order of convergence of the one-point with-memory* *methods* (3.21) *with the analogous expressions* (3.22) *and* (3.23) *of is at least* $1+\sqrt{3}$, *and* $\frac{1}{2}(3+\sqrt{17})$.

Proof. By a similar argument to that of Theorem 3.1, we obtain

$$1+f'(\xi)\gamma_m \sim e_{m-1}, \quad \beta_m + c_2 \sim e_{m-1}. \tag{3.24}$$

From (2.7), (3.10), (3.11) and (3.24), it can be obtained easily

$$e_{m,w} \sim (1+\gamma_m f'(\xi))e_m \sim e_{m-1}e_m \sim e_{m-1}^{1+\rho}, \tag{3.25}$$

and

$$e_{m+1} \sim (1+\gamma_m f'(\xi))(\beta_m + c_2)e_m^2 \sim e_{m-1}e_{m-1}e_m^2 \sim e_{m-1}^{2+2\rho}. \tag{3.26}$$

Now, we achieve the following system of equations

$$\begin{cases} 1+\rho-\rho\rho_1 = 0, \\ 2+2\rho-\rho^2 = 0. \end{cases} \tag{3.27}$$

Since that the only positive answer to this system equation is $\rho_1 = \frac{1}{2}(1+\sqrt{3})$ and $\rho = 1+\sqrt{3}$, then the convergence order of this method is $\rho = (1+\sqrt{3}) = 2.73$. We show this method with TM2.7.
In 2013, Džunić [8] showed that the convergence of this single-step two-parametric is $\frac{1}{2}(3+\sqrt{17})$. □

In the third part of Section 3, we design the with-memory methods using two self-accelerator parameters with 100% convergence improvement.

3.3. **Maximum improvement.** Now, using all previous and current information, method (3.21) can be rewritten as follows

$$\begin{cases} \gamma_m = \frac{-1}{N'_{2m}(x_m)}, \ \beta_m = \frac{N''_{2m+1}(w_m)}{-2N'_{2m+1}(w_m)}, \ m = 1,2,\cdots, \\ w_m = x_m + \gamma_m f(x_m), \ x_{m+1} = x_m - \frac{f(x_m)}{f[x_m,w_m]+\beta_m f(w_k)}, \ m = 0,1,2,\cdots, \end{cases} \tag{3.28}$$

where $N'_{2m}(x_m), N'_{2m+1}(w_m)$ and $N''_{2m+1}(w_m)$ are Newton's interpolating polynomials of $2m$ and $2m+1$ degrees, set through $2m+1$ and $2m+2$ best available approximations (nodes) $(x_m, x_{m-1}, w_{m-1}, \ldots, w_1, x_1, w_0, x_0)$ and $(w_m, x_m, x_{m-1}, w_{m-1}, \ldots, w_1,$
$x_1, w_0, x_0)$, respectively. The following result determines the convergence order of the one-point adaptive method (3.28).

Theorem 3.3. *If an initial estimation* x_0 *is close adequate to a simple root* ξ *of* $f(x) = 0$, *and* γ_0 *and* β_0 *have uniformly bounded above, being* $f(x)$ *a real sufficiently differentiable function, then, the R-order of convergence of the one-point method adaptive with memory* (3.28) *for* $m \geq 4$ *is 3.994*

Proof. Let ρ and ρ_1 are the convergence order of the sequences $\{x_m\}$ and $\{w_m\}$, respectively. Therefore

$$e_{m+1} \sim e_m^{\rho}, \tag{3.29}$$

and

$$e_{m,w} \sim e_m^{\rho_1}. \tag{3.30}$$

Consequently

$$e_{m+1} \sim e_m^{\rho} \sim e_{m-1}^{\rho^2} \sim e_{m-2}^{\rho^3} \sim e_{m-3}^{\rho^4} \sim e_{m-4}^{\rho^5} \sim \cdots , \tag{3.31}$$

and

$$e_{m,w} \sim e_m^{\rho_1} \sim e_{m-1}^{\rho_1\rho} \sim e_{m-2}^{\rho_1\rho^2} \sim e_{m-3}^{\rho_1\rho^3} \sim e_{m-4}^{\rho_1\rho^4} \sim \cdots . \tag{3.32}$$

Using (2.8), (2.9) and self-accelerating parameters γ_m and β_m, we get the corresponding error relations for the with-memory methods (3.28)

$$e_{m,w} \sim (1 + \gamma_m f'(\xi))e_m, \tag{3.33}$$

and

$$e_{m+1} \sim (1 + \gamma_m f'(\xi))(\beta_m + c_2)e_m^2. \tag{3.34}$$

Also, we have

$$(1 + \gamma_m f'(\xi)) \sim c_9 e_{m-4}e_{w,m-4}e_{m-3}e_{w,m-3}e_{m-2}e_{w,m-2}e_{m-1}e_{w,m-1}, \tag{3.35}$$

Furtheremore,

$$(\beta_m + c_2) \sim c_{10}e_{m-4}e_{w,m-4}e_{m-3}e_{w,m-3}e_{m-2}e_{w,m-2}e_{m-1}e_{w,m-1}. \tag{3.36}$$

Therefore

$$(1 + \gamma_m f'(\xi)) \sim c_9 e_{m-4}e_{m-4}^{\rho_1}e_{m-4}^{\rho}e_{m-4}^{\rho\rho_1}e_{m-4}^{\rho^2}e_{m-4}^{\rho_1\rho^2}e_{m-4}^{\rho^3}e_{m-4}^{\rho_1\rho^3}, \tag{3.37}$$

and

$$(\beta_m + c_2) \sim c_{10}e_{m-4}e_{m-4}^{\rho_1}e_{m-4}^{\rho}e_{m-4}^{\rho\rho_1}e_{m-4}^{\rho^2}e_{m-4}^{\rho_1\rho^2}e_{m-4}^{\rho^3}e_{m-4}^{\rho_1\rho^3}. \tag{3.38}$$

Combining (3.33), (3.34), (3.37) and (3.38), we obtain

$$e_{m,w} \sim e_{m-4}^{1+\rho_1+\rho+\rho_1\rho+\rho^2+\rho_1\rho^2+\rho^3+\rho_1\rho^3+\rho^4}, \tag{3.39}$$

and

$$e_{m+1} \sim e_{m-4}^{2(1+\rho_1+\rho+\rho_1\rho+\rho^2+\rho_1\rho^2+\rho^3+\rho_1\rho^3+\rho^4)}. \tag{3.40}$$

Compare the right and left side of error equations (3.31)), (3.40) and (3.32), (3.39), we have

$$\begin{cases} \rho^4\rho_1 - (1+\rho_1+\rho+\rho\rho_1+\rho^2+\rho^2\rho_1+\rho^3+\rho^3\rho_1+\rho^4) = 0, \\ \rho^5 - 2(1+\rho_1+\rho+\rho\rho_1+\rho^2+\rho^2\rho_1+\rho^3+\rho^3\rho_1+\rho^4) = 0. \end{cases} \tag{3.41}$$

The positive solution of the system of the equations is $\rho_1 \simeq 1.997$ and $\rho \simeq 3.994 \approx 4$. Therefore, it can be concluded that the convergence order of the adaptive method (3.28) is equal to four. □

This method was proposed partially in [10] and [11], but the authors did not earn the order of the convergence.

Remark 3.4. The best method in terms of computational time is to decrease the number of iterations and computational period using TM4. Matching numerical experiments were conducted on the variant of examples that markedly admit the above conclusion. Finally, by numerical experiments, we can conclude the adaptive schemes that support the theoretical results reveal consistent convergence behavior.

4. Numerical results and comparisons

We have used the new methods TM2.6, TM2.7, TM4, Newton, Steffensen, Traub, Džunić-Petković (DPM), Džunić (DM) and Zheng et al. (ZLHM) [37] to solve the following nonlinear equations.

$$\begin{cases} f_1(x) = x\log(1+x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin(\pi x),\ \xi = 0,\ x_0 = 0.6, \\ f_2(x) = 1 + \frac{1}{x^4} - \frac{1}{x} - x^2,\ \xi = 1,\ x_0 = 1.4, \\ f_3(x) = e^{x^3-x} - \cos(x^2-1) + x^3 + 1,\ \xi = -1,\ x_0 = -1.4, \\ f_4(x) = \frac{-5x^2}{2} + x^4 + x^5 + \frac{1}{1+x^2},\ \xi = 1,\ x_0 = 1.4. \end{cases} \tag{4.1}$$

The errors $|x_m - \xi|$ of estimations to the corresponding zeros of nonlinear equations, the number of repetitions (cost of calculations), the efficiency index, and computational convergence order COC are given in Tables 1-2. Meanwhile, $a(-b)$ denotes $a \times 10^b$. A comparison of the convergence improvement of the new methods with the TM4, DWM [14], MWBM [15], RWBM [18], TKM [28], and WKGM [32] memorization methods are given in Table 3. The computational order of convergence (COC) introduced in [33]

$$\rho \approx COC = \frac{\ln|x_{m+1}-\xi| / \ln|x_m - \xi|}{\ln|x_m-\xi| / \ln|x_{m-1}-\xi|}. \tag{4.2}$$

The symbols used in these tables are as follows
1-The number of repetitions (IT).

2- Efficiency index (EI).
3- The exact root of the nonlinear equation ξ.
4- The order of the convergence p.
5- Computational convergence order (COC).

TABLE 1. Comparison of various iterative methods.

Methods	$\lvert x_1-\xi\rvert$	$\lvert x_2-\xi\rvert$	$\lvert x_3-\xi\rvert$	Iter	COC	EI
$f_1(x)=x\log(1+x\sin(x))+e^{-1+x^2+x\cos(x)}\sin(\pi x)$, $\xi=0$, $x_0=0.6$						
Newton (1.2)	0.16(0)	0.25(−1)	0.59(−3)	7	2.00	1.41
Steffensen (2.4)	0.86(0)	0.14(1)	0.83(0)	13	2.00	1.41
ZLHM [37], $\gamma=0.1$	0.48(0)	0.92(−1)	0.88(−2)	7	2.00	1.41
Method(2.7), $\gamma=\beta=0.1$	0.36(0)	0.10(0)	0.12(−1)	6	2.00	1.41
Traub [27], $\gamma=0.1$	0.48(0)	0.56(−1)	0.13(−2)	4	2.48	1.58
TM 2.6, $\gamma=0.1$	0.48(0)	0.54(−1)	0.48(−3)	4	2.62	1.62
DPM [8], $\gamma=0.1$	0.48(0)	0.70(−1)	0.30(−3)	6	3.00	1.73
TM 2.7, $\gamma=\beta=0.1$	0.36(0)	0.13(−1)	0.18(−4)	4	2.75	1.66
DM [9], $\gamma=0.1$	0.36(0)	0.54(−1)	0.19(−4)	4	3.56	1.89
(3.28), $\gamma=\beta=0.1$	0.36(0)	0.54(−1)	0.24(−5)	4	4.00	2.00
$f_2(x)=1+\frac{1}{x^4}-\frac{1}{x}-x^2$, $\xi=1$, $x_0=1.4$						
Newton (1.2)	0.66(−1)	0.76(−2)	0.93(−4)	7	2.00	1.41
Steffensen (2.4)	0.40(0)	0.40(0)	0.40(0)	200	1.00	1.00
ZLHM [37], $\gamma=0.1$	0.61(−1)	0.28(−2)	0.62(−5)	5	2.00	1.41
Method(2.7), $\gamma=\beta=0.1$	0.47(−1)	0.16(−2)	0.19(−5)	6	2.00	1.41
Traub [27], $\gamma=0.1$	0.61(−1)	0.28(−2)	0.13(−5)	4	2.41	1.56
TM 2.6, $\gamma=0.1$	0.61(−1)	0.21(−2)	0.24(−6)	4	2.62	1.62
DPM [8], $\gamma=0.1$	0.61(−1)	0.89(−3)	0.79(−9)	6	3.00	1.73
TM 2.7, $\gamma=\beta=0.1$	0.47(−1)	0.17(−2)	0.19(−7)	4	2.71	1.65
DM [9], $\gamma=0.1$	0.47(−1)	0.13(−3)	0.53(−13)	4	3.56	1.89
(3.28), $\gamma=\beta=0.1$	0.47(−1)	0.13(−3)	0.36(−15)	4	4.00	2.00
$f_3(x)=e^{x^3-x}-\cos(x^2-1)+x^3+1$, $\xi=-1$, $x_0=-1.4$						
Newton (1.2)	0.23(−1)	0.25(−3)	0.24(−7)	6	2.00	1.41
Steffensen (2.4)	0.29(0)	0.18(0)	0.43(−1)	8	2.00	1.41
ZLHM [37], $\gamma=0.1$	0.25(−1)	0.42(−3)	0.11(−6)	6	2.00	1.41
Method(2.7), $\gamma=\beta=0.1$	0.53(−1)	0.19(−2)	0.16(−5)	6	2.00	1.41
Traub [27], $\gamma=0.1$	0.25(−1)	0.92(−5)	0.36(−12)	4	2.47	1.57
TM 2.6, $\gamma=0.1$	0.25(−1)	0.46(−5)	0.15(−14)	4	2.62	1.62
DPM [8], $\gamma=0.1$	0.25(−1)	0.18(−4)	0.64(−14)	6	3.00	1.73
TM 2.7, $\gamma=\beta=0.1$	0.53(−1)	0.12(−4)	0.16(−12)	4	2.75	1.66
DM [9], $\gamma=0.1$	0.53(−1)	0.49(−4)	0.55(−16)	4	3.56	1.89
(3.28), $\gamma=\beta=0.1$	0.53(−1)	0.49(−1)	0.50(−17)	4	4.00	2.00

TABLE 2. Comparison of various iterative methods.

Methods	$\lvert x_1-\xi\rvert$	$\lvert x_2-\xi\rvert$	$\lvert x_3-\xi\rvert$	Iter	COC	EI
$f_4(x)=\frac{-5x^2}{2}+x^4+x^5+\frac{1}{1+x^2}$, $\xi=1$, $x_0=1.4$						
Newton (1.2)	0.20(−1)	0.72(−1)	0.14(−3)	6	2.00	1.41
Steffensen (2.4)	0.40(0)	0.40(0)	0.40(0)	54	2.00	1.41
ZLHM [37], $\gamma=0.1$	0.30(0)	0.20(0)	0.10(0)	54	2.00	1.41
Method(2.7), $\gamma=\beta=0.1$	0.31(0)	0.21(0)	0.11(−1)	6	2.00	1.41
Traub [27], $\gamma=0.1$	0.30(0)	0.82(−1)	0.11(−1)	4	2.26	1.50
TM 2.6, $\gamma=0.1$	0.30(0)	0.11(0)	0.15(−1)	4	2.62	1.62
DPM [8], $\gamma=0.1$	0.30(0)	0.61(−1)	0.18(−2)	6	3.00	1.73
TM 2.7, $\gamma=\beta=0.1$	0.31(0)	0.97(−1)	0.90(−2)	4	2.77	1.66
DM [9], $\gamma=0.1$	0.31(0)	0.33(−1)	0.49(−5)	4	3.57	1.89
(3.28), $\gamma=\beta=0.1$	0.31(0)	0.33(−1)	0.17(−4)	4	4.00	2.00

TABLE 3. Comparison of convergence-order improvement of with-memory methods

With-memory methods	Optimal order	p	Percentage increase
TM [27]	2.00	2.41	%21
TM 2.6	2.00	2.62	%31
DPM [8]	2.00	3.00	%50
TM 2.7	2.00	2.75	%38
WKGM [32]	2.00	2.73	%37
RWBM [18]	2.00	2.41	%21
MWM [14]	2.00	2.41	%21
MWBM [15]	2.00	2.73	%37
DM [9]	2.00	3.56	%78
TKM [28]	2.00	3.56	%78
TKM [28]	4.00	7.00	%75
TKM [28]	8.00	14.00	%75
TKM [28]	16.00	28.00	%75
(3.28)	2.00	4.00	%100

As can be seen from Tables 1-3, the with-memory methods are more efficient than the without-memory methods. Also, the approximation of the accelerator parameter is obtained using the adaptive technique of 100% convergence order improvement and the maximum efficiency index for the iterative method.

5. The role of the accelerator parameter in the stability

The dynamical properties of the iterative method give us pivotal data about their numerical qualities as its stability and reliability. Some significant results concerning the dynamic versions of the iterative methods have been obtained in [4, 19, 36]. In what follows, we have compared iterative methods (Newton's method), (Method (ZLH)[37]), and (2.7) using the basins of attraction for three complex polynomials $p_1(z) = z^2 - 1$, $p_2(z) = z^3 - 1$, $p_3(z) = z^4 - 1$. We have used similar method as in [16] and [30, 31] to generate the basins of attraction. To produce the basins of attraction for the zeros of a polynomial and an iterative method, we catch a grid of 500×500 points in a rectangle $D = [-5, 5] \times [-5, 5] \subset C$, and we use these points as z_0. Whenever the sequence developed by the iterative method attaines a zero z^* of polynomial $p_i(x)$, then we take with a tolerance $|z - z^*| < 10^{-6}$ and a maximum of 200 iterations. Therefore, we determine that z_0 is in the basin of attraction of the zero and we paint this point in a color previously selected for this root.

Remark 5.1. Figures 1, 3, and 5 show that the basins of attraction in Newton's method are higher than in Steffensen's method. Figures 2, 4, and 6 show that the accelerator parameter plays an essential role in increasing the absorption domain of an iterative method. Figures 7 and 8 confirm that the convergence region is less if the parameter is smaller. Figure 9 displays the importance of the accelerator parameter in determining the absorption region particularly.

6. Conclusions

In this work, we compared with-memory methods based on Steffensen's method. The efficiency index is higher than the other methods. The self-accelerator parameters have an essential role in increasing the efficiency index, improving the order of convergence, and developing the adsorption region. The largest absorption region corresponds to the smallest amount of self-accelerating parameters.

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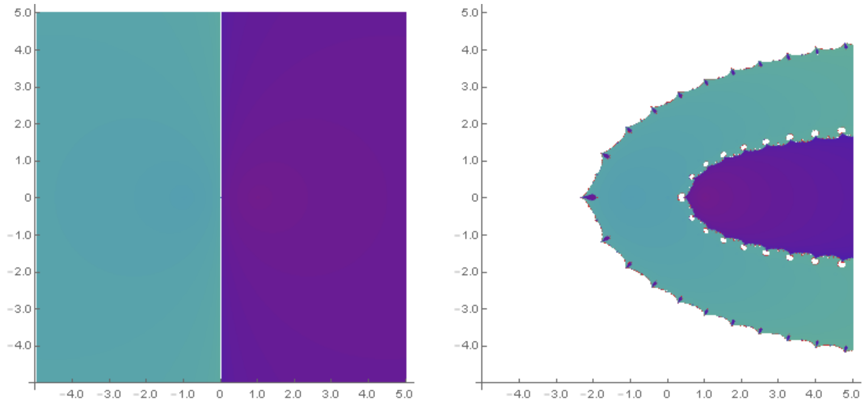
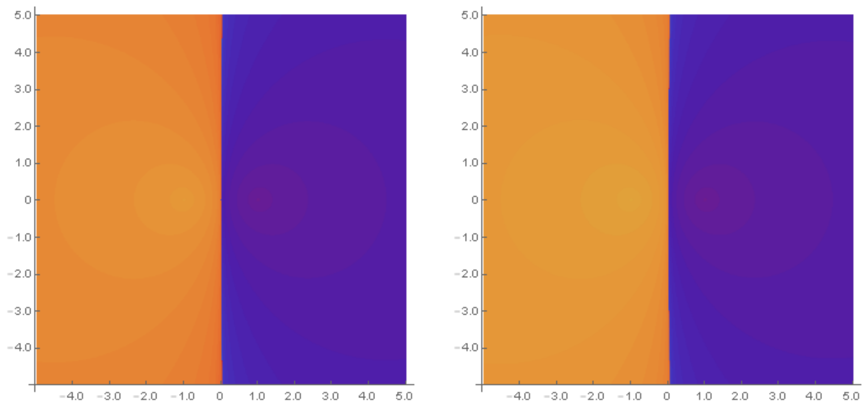


(A) Newton's method

(B) Steffensen's method

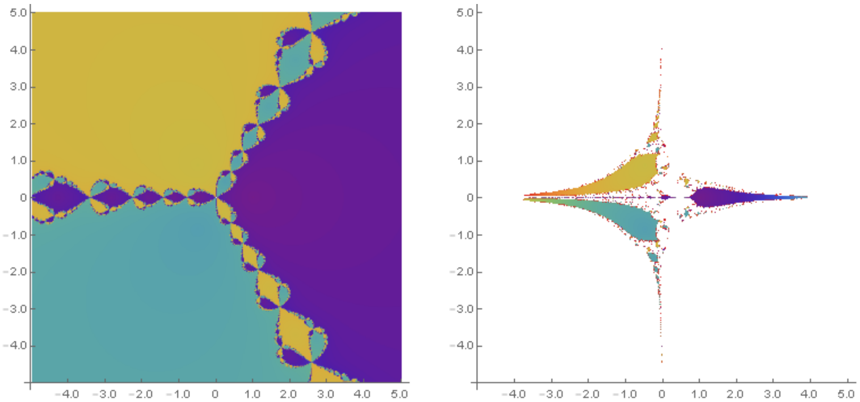
FIGURE 1. Finding the roots of the polynomial $p_1(z) = z^2 - 1$



(A) (ZLH), $\gamma = 0.001$

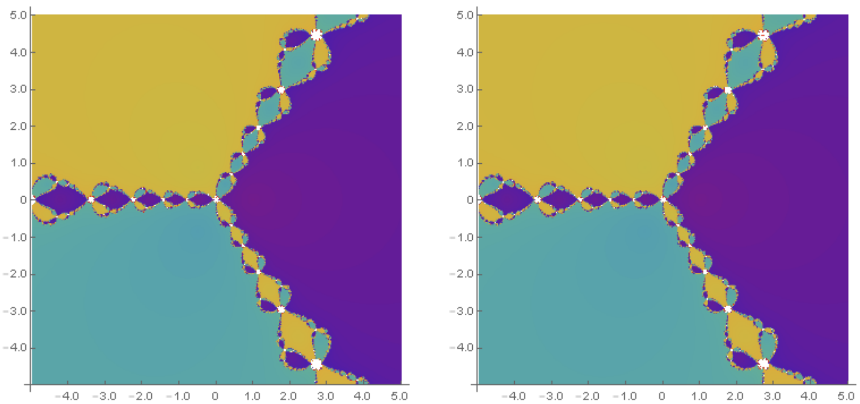
(B) (2.7), $\gamma = \beta = 0.001$

FIGURE 2. Finding the roots of the polynomial $p_1(z) = z^2 - 1$



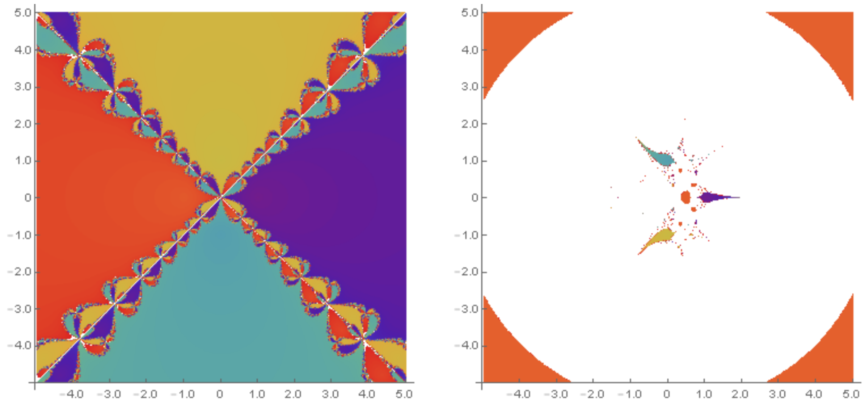
(A) Newton's method (B) Steffensen's method

Figure 3. Finding the roots of the polynomial $p_2(z) = z^3 - 1$



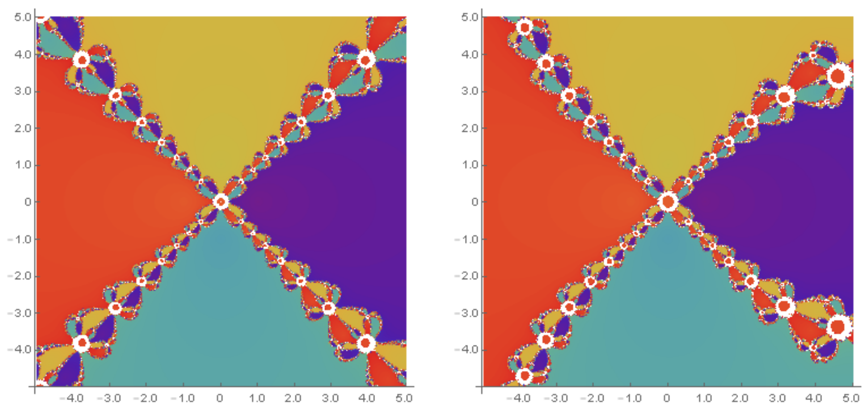
(A) (ZLH), $\gamma = 0.001$ (B) (2.7), $\gamma = \beta = 0.001$

Figure 4. Finding the roots of the polynomial $p_2(z) = z^3 - 1$



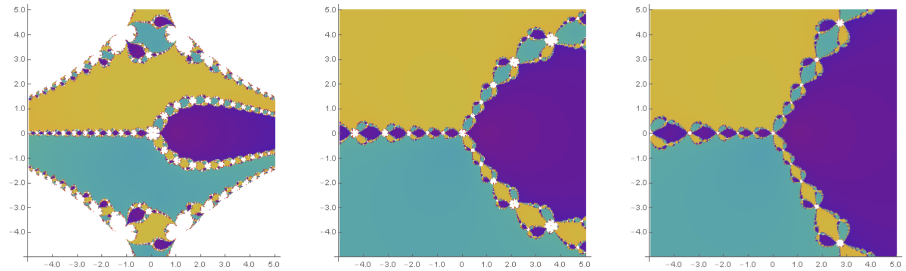
(A) Newton's method (B) Steffensen's method

FIGURE 5. Finding the roots of the polynomial $p_3(z) = z^4 - 1$



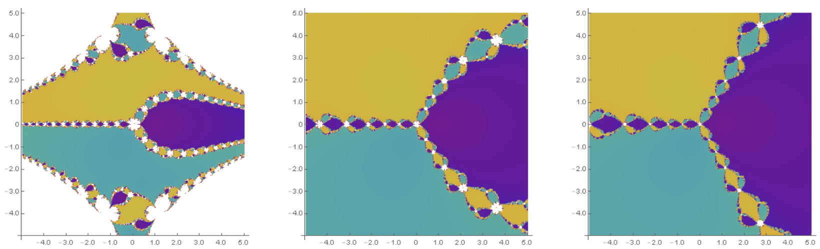
(A) (ZLH), $\gamma = 0.001$ (B) (2.7), $\gamma = \beta = 0.001$

FIGURE 6. Finding the roots of the polynomial $p_3(z) = z^4 - 1$



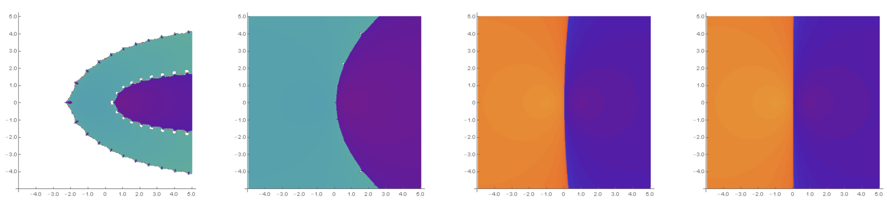
(A) (ZLH), $\gamma = 0.1$ (B) (ZLH), $\gamma = 0.01$ (C) (ZLH), $\gamma = 0.001$

FIGURE 7. Zheng et al.'s method (2.9) for finding the roots of the polynomial $p_2(z)$



(A) $\gamma = \beta = 0.1$ (B) $\gamma = \beta = 0.01$ (C) $\gamma = \beta = 0.001$

FIGURE 8. Method (2.7) for finding the roots of the polynomial $p_2(z)$



(A) $\gamma = 1$ (B) $\gamma = 0.1$ (C) $\gamma = 0.01$ (D) $\gamma = 0.001$

FIGURE 9. Method (3.3) for finding the roots of the polynomial $p_1(z)$