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Best Approximation in TVS

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Abstract. In this paper we give new results on the best approximation in the Hausdorff topological vector space and consider relationship with orthogonality. Also we determined under what conditions the map $P_{K,f}$ is upper semicontinous.

Keywords: Upper semi-continuous , f-Boundedly compact set, f-Proximinal, f-Chebyshev,f-Orthogonal.

1. Introduction and Prelimiaries

Many authors have studied the concept of best approximation in normed spaces (see [1-9]) extended some results to semi-normed spaces. Importance of this paper is for continuous of generalizing concept of the best approximation of normed space to vector space.

Let X be a Hausdorff topological vector space over a field F and f is a function on X. An element $k_0 \in K$ is said to be an f-best approximation to x in K if

$$f(x-k_0)=f(x-K)=inf\{f(x-k):\ k\in K\}.$$

We denote by $P_{K,f}(x)$ the collection of all such $k_0 \in K$. The set K is said to be f-proximinal if $P_{K,f}(x)$ is non-empty for each $x \in X$ and f-Chebyshev if $P_{K,f}(x)$ is exactly singleton for each $x \in X$.

For $x, y \in X$, x is said to be f-orthogonal to y, $x \perp_f y$, if

$$f(x) \le f(x+\alpha y)$$

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for every scalar α. Also x is said to be f-orthogonal to K, $x\perp_f K$, if $x\perp_f y$ for all $y \in K$. Let K be a non-empty close subset of X. We define

$$\hat{K_f} = \{x \in X : \ f(x) = f(x - K)\} = {P_{K,f}}^{-1}(\{0\})$$

it is clear that $g_0 \in P_{K,f}(x)$ if and only if $x - g_0 \in \hat{K}$. Suppose $r > 0$ we put

$$S_r = \{y \in X : \ f(x - y) \leq r\}.$$

We say the set K is f-bounded if there is $r > 0$ such that $f(k) \leq r$ for every $k \in K$.

The set K is said to be f-boundedly compact if for each $x \in X$ and for each $r > 0$, $K \bigcap S_r$ is compact.

In this paper, we shall obtain some important results of the f-best -proximinality subsets of X. Also the relations between the upper semi-continuity of metric projection $P_{K,f}$ and f-proximinal subsets of X are discussed.

It is notable that in all of this paper the function f is symmetric function (i.e. $f(-x) = f(x)$ for all $x \in X$), is invariant (i.e. $f(x + y + K) = f(x + K)$ for every non-empty subset K of X and $y \in K$ and $x \in X$) and continuous.

2. Main results

Let X and Y be two topological vector spaces, then a mapping $g : \ X \to 2^Y$ is called **upper-semi-continuous** if the set

$$g^{-1}(A) = \{x \in X : \ g(x) \bigcap A \neq \emptyset\}$$

is close for each close set A in Y.

Proposition 2.1. Proposition 2.1. *Let X be a topological vector space, and K is a f-proximinal subset of X. Then we have the following statements:*

(1) *$g_0 \in P_{K,f}(x)$ if and only if $x - g_0 \in \hat{K_f}$.*

(2) *$x \in \hat{K_f}$ if and only if $-x \in \hat{K_f}$.*

(3) *$x\perp_f K$ then $x \in \hat{K_f}$ and if $x \in \hat{K_f}$ and $\alpha K = K$ then $x\perp_f K$.*

(4) *$P_{K,f}(x)$ is a close also if f fisasublinearfunction then $P_{K,f}(x)$ is f-bounded.*

(5) *If f isaconvexfuncionandKisaconvexset, then $P_{K,f}(x)$ is convex set.*

Proof. (1)

$$\begin{aligned} g_0 \in P_{K,f}(x) &\Leftrightarrow f(x-g_0) = f(x-K) \\ &\Leftrightarrow f(x-g_0) = f(x-g_0-K) \\ &\Leftrightarrow x-g_0 \in \hat{K}_f. \end{aligned}$$

(2)

$$\begin{aligned} x \in \hat{K}_f &\Leftrightarrow f(x) = f(x+K) = \inf\{f(x+k: \ k \in K\} \\ &\Leftrightarrow f(-x) = \inf\{f(-x-k): \ k \in K\} \\ &\Leftrightarrow f(-x) = \inf\{f(-x+k): \ -k \in K\} \\ &\Leftrightarrow -x \in \hat{K}_f. \end{aligned}$$

(3) If $x \perp_f K$ then $f(x) \leq f(x+\alpha k)$ for every $k \in K$ and so $f(x) \leq f(x+k)$ therefore $x \in \hat{K}_f$. Also clearly if $x \in \hat{K}_f$ and $\alpha K = K$ then $x \perp_f K$.
(4) Suppose (g_α) is a net in $P_{K,f}(x)$ which $g_\alpha \to g_0$. Then $f(x-g_\alpha) = f(x+K)$, and therefore $f(x-g_0) = f(x+K)$ and so $g_0 \in P_{K,f}(x)$ and $P_{K,f}(x)$ is close.
Suppose f is sublinear. For $g_0 \in P_{K,f}(x)$ we have

$$f(g_0) \leq f(-x+g_0) + f(x)) = f(x-g_0) + f(x) = f(x-K) + f(x).$$

If put $r = f(x-K) + f(x)$ then $f(g_0) \leq r$.

(5) Since$K$isconvexthen $\lambda g_1 - (1-\lambda)g_2$ for every $g_1, g_2 \in P_{K,f}(x)$, $0 < \lambda < 1$, then

$$\begin{aligned} f(x-\lambda g_1-(1-\lambda)g_2) &= f(\lambda(x-g_1)+(1-\lambda)(x-g_2)) \\ &\leq \lambda f(x-g_1+(1-\lambda)f(x-g_2) \\ &= \lambda f(x+K)+(1-\lambda)f(x+K) \\ &= f(x+K). \end{aligned}$$

Then $\lambda g_1 + (1-\lambda)g_2 \in P_{K,f}(x)$. □

Theorem 2.2. *Let X be a topological vector space, and K be a f-proximinal subset of X. Then $P_{K,f}$ is upper semi-continuous if and only if $F + \hat{K}_f$ is close for every close set F in K.*

Proof. Suppose $P_{K,f}$ is upper semi-continuous and F is a close set in K. We must prove that $F + \hat{K}_f$ is close. Suppose $\{g_\alpha\}$ is a net in $F + \hat{K}_f$ which $g_\alpha \to g_0$. Then $g_\alpha = u_\alpha + v_\alpha$, such that $u_\alpha \in F$ and $v_\alpha \in \hat{K}_f$. From Proposition 2.1, we have $u_\alpha \in F \cap P_{K,f}(u_\alpha + v_\alpha)$, it follows that

$$g_\alpha \in \{x \in X : F \cap P_{K,f}(x) \neq \emptyset\},$$

therefore $g_0 \in \{x \in X : F \cap P_{K,f}(x) \neq \emptyset\}$, that is $F \cap P_{K,f}(g_0) \neq \emptyset$. Then there is $z \in F \cap P_{K,f}(g_0)$. Also from Proposition 2.1, it follows that $g_0 - z \in \hat{K}_f$ and $z \in F$, that is $g_0 \in F + \hat{K}_f$, and so $F + \hat{K}_f$ is close. For the converse, assume that $F + \hat{K}_f$ is close for every close set F in K. We shall prove that $P_{K,f}$ is upper semi-continuous. For this, suppose F is a close set in K and $\{g_\alpha\}$ is a net in $\{x \in X : F \cap P_{K,f}(x) \neq \emptyset\}$ which is converge to g_0. If choose $z_\alpha \in F \cap P_{K,f}(g_\alpha)$, then $g_\alpha \in F + \hat{K}_f$ and $F + \hat{K}_f$ is close. Since$F + \hat{K}_f$isclose,therefore $g_0 \in F + \hat{K}_f$ and $g_0 = z + x$ for some $z \in F$ and $x \in \hat{K}_f$. It follows that $z \in F \cap P_{K,f}(z + x)$ and therefore $g_0 \in \{x \in X : F \cap P_{K,f}(x) \neq \emptyset\}$. □

Theorem 2.3. *Let K is a f-proximinal subset of a topological vector space X and $\hat{K}_f$ is f-boundedly compact. Then have the following statement:*
(1) *$P_{K,f}$ is upper semi-continuous.*
(2) *If f is sublinear $P_{K,f}(x)$ is compact, for every $x \in X$.*

Proof. (1) Suppose F is a close set in K. Suppose (g_α) is a net in $F + \hat{K}_f$ which $g_\alpha \to g_0$. Since f is continues, it follows that $f(g_\alpha) \to f(g_0)$. Then $\{g_\alpha\}$ is f-bounded and $g_\alpha = u_\alpha + v_\alpha$ for some $u_\alpha \in F$ and $v_\alpha \in \hat{K}_f$. Since $v_\alpha \in \hat{K}_f$, it follows that $f(v_\alpha) \leq f(v_\alpha + u_\alpha) = f(g_\alpha)$, thus $\{v_\alpha\}$ is f-bounded. Since $\hat{K}_f$ is f-boundedly compact, there is a subnet $\{v_{\alpha_\beta}\}$, which is converge to $v_0 \in \hat{K}_f$. Since $u_{\alpha_\beta} \to g_0 - v_0$ and F is close, thus $g_0 - v_0 \in F$ and so $g_0 \in F + \hat{K}_f$.
(2) Suppose $x \in X$ and $\{g_\alpha\}$ is a net in $P_{K,f}(x)$. From Proposition 2.1, we have $x - g_\alpha \in \hat{K}_f$ and since f is sublinear $f(x - g_\alpha) \leq 2f(x)$, therefore there is a subnet $\{x - g_{\alpha_\beta}\}$ such that is converge to $u_0 \in \hat{K}_f$. Put $g_0 = x - u_0$, then from Proposition 2.1, $g_0 \in P_{K,f}(x)$. and so $P_{K,f}(x)$ is compact for every $x \in X$ □

Theorem 2.4. *Let K is a f-proximinal symmetric (K=-K) convex set of a topological vector space X. If $\hat{K}_f$ is a convex set and if $f(x) \leq 0$ then $x = 0$. then K is Chebyshev.*

Proof. Suppose $x \in X$ and $g_1, g_2 \in P_{K,f}(x)$, then from Proposition 2.1, $x - g_1, x - g_2 \in \hat{K}_f$. Put $\hat{g_1} = x - g_1$ and $\hat{g_2} = x - g_2$ and have $x = g_1 + \hat{g_1} = g_2 + \hat{g_2}$ since $-\hat{g_2} \in \hat{K}_f$ and $\hat{K}_f$ is convex then $\frac{1}{2}(\hat{g_1} - \hat{g_2}) \in \hat{K}_f$, also since K is symmetric $-g_2 \in K$ and since K is convex then $\frac{1}{2}(g_1 - g_2) \in K$, it follows that $g_1 - g_2 \in \hat{K}_f \cap K$, then $f(g_1 - g_2) \leq 0$ and $g_1 = g_2$. □

In the following we give good result about f-proximinality. It is notable that a set K is w-compact, if for every net k_α there is convergence subnet.

Theorem 2.5. *Let X be a topological vector space, and K be a close subset of X. If K is w-comact, then K is f-proximinal.*

Proof. Suppose $x \in X$, since $f(x-K) = \inf\{f(x-k) : k \in K\}$ therefore

$$\forall \alpha \; \exists (x_\alpha) : \; f(x-k_\alpha) \leq f(x-K) + \frac{1}{\alpha}.$$

Also since K is w-compact there exist subnet k_{α_β} such that $k_{\alpha_\beta} \to^w k_0$. Therefore $x - k_{\alpha_\beta} \to^w x - k_0$. Since f is continous, then $f(x - k_{\alpha_\beta}) \leq f(x-K) + \frac{1}{\alpha}$ and $f(x-k_0) = liminf f(x-k_{\alpha_\beta}) \leq f(x-K)$. Thus $k_0 \in P_{k,f}(x)$. □

Example 2.6. Let $X = \mathbf{R}^2$ and $K = \{(x,y) : \; y = x\}$. Consider $f(x,y) = x^2 + y^2$, then it is clear that $\hat{K}_f = \{(x,y) : \; y = -x\}$.

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